Journal of Integer Sequences, Vol. 14 (2011),
Article 11.4.7

Greatest Common Divisors in Shifted Fibonacci Sequences

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Abstract

It is well known that successive members of the Fibonacci sequence are relatively prime. Let

$$f_n(a) = \gcd(F_n + a, F_{n+1} + a).$$

Therefore $(f_n(0))$ is the constant sequence $1, 1, 1, \ldots$, but Hoggatt in 1971 noted that $(f_n(\pm 1))$ is unbounded. In this note we prove that $(f_n(a))$ is bounded if $a \neq \pm 1$.

1 Introduction

Let the generalized Fibonacci sequence be defined by

$$G_n = G_{n-1} + G_{n-2}, \quad \text{for } n \geq 3,$$

and $G_1 = \alpha$, $G_2 = \beta$. It is well known that [3, p. 109]

$$G_n = \alpha F_{n-2} + \beta F_{n-1}.$$

If $\alpha = \beta = 1$, then the generalized Fibonacci sequence G_n is the Fibonacci sequence F_n, A000045, and if $\alpha = 1$ and $\beta = 3$, G_n is the Lucas sequence L_n, A000032. It is well known that successive members of the Fibonacci sequence are relatively prime. Consider a slightly different sequence,

$$(F_n + a),$$

which we call a shifted Fibonacci sequence by a, e.g., A000071, A001611, and A157725. In 1971 Hoggatt [1] noted that

$$\begin{aligned}\gcd(F_{4n+1}+1, F_{4n+2}+1) &= L_{2n},\\ \gcd(F_{4n+1}-1, F_{4n+2}-1) &= F_{2n},\\ \gcd(F_{4n+3}-1, F_{4n+4}-1) &= L_{2n+1}.\end{aligned}$$

That is to say, the successive members of the shifted Fibonacci sequence by ± 1 are not always relatively prime. Let

$$f_n(a) = \gcd(F_n + a, F_{n+1} + a).$$

Therefore $(f_n(0))$ is the constant sequence $1, 1, 1, \ldots$, but $(f_n(\pm 1))$ is unbounded above.

In 2003 Hernández and Luca [2] proved that there exists a constant c such that

$$\gcd(F_m + a, F_n + a) > \exp(cm),$$

holds for infinitely many pairs of positive integers $m > n$.

In this note we prove that $(f_n(a))$ is bounded above if $a \neq \pm 1$. In fact we prove the following two theorems in this note.

Theorem 1. *For any integers α, β, n and a with $\alpha^2 + \alpha\beta - \beta^2 - a^2 \neq 0$, we have*

$$\gcd(G_{2n-1} + a, G_{2n} + a) \le |\alpha^2 + \alpha\beta - \beta^2 - a^2|. \tag{1}$$

Theorem 2. *For any integers α, β, n and a with $\alpha^2 + \alpha\beta - \beta^2 + a^2 \neq 0$, we have*

$$\gcd(G_{2n} + a, G_{2n+1} + a) \le |\alpha^2 + \alpha\beta - \beta^2 + a^2|. \tag{2}$$

Let $\alpha = \beta = 1$ in Theorem 1 and Theorem 2. We can get the corollary.

Corollary 1. *For integers n and a,*

$$\begin{aligned}\gcd(F_{2n-1} + a, F_{2n} + a) &\le |a^2 - 1|, \quad \textit{if } a \neq \pm 1,\\ \gcd(F_{2n} + a, F_{2n+1} + a) &\le a^2 + 1.\end{aligned}$$

Hence we conclude that $(f_n(a))$ is bounded above if $a \neq \pm 1$. Another easy corollary is that

$$\ell_n(a) = \gcd(L_n + a, L_{n+1} + a)$$

has only finitely many values.

Corollary 2. *For integers n and a,*

$$\begin{aligned}\gcd(L_{2n-1} + a, L_{2n} + a) &\le a^2 + 5,\\ \gcd(L_{2n} + a, L_{2n+1} + a) &\le |a^2 - 5|.\end{aligned}$$

Similarly, let $\alpha = 1$ and $\beta = 3$ in Theorem 1 and Theorem 2. We conclude that $\ell_n(a)$ is bounded above for any integers a.

In the next section we will derive two basic lemmas. From them, we determine $f_n(1)$, $f_n(2)$, $f_n(-1)$, $f_n(-2)$, and $\ell_n(1)$, in Section 3, 4, and 5. In the last section we prove Theorems 1 and 2.

2 Preliminaries

Lemma 1. *For integers n, k, and a,*

$$\gcd(G_n + aF_k, G_{n-1} - aF_{k+1}) = \gcd(G_{n-2} + aF_{k+2}, G_{n-3} - aF_{k+3}). \tag{3}$$

Proof. Since $\gcd(a, b) = \gcd(a + bc, b)$ for any integers a, b, and c, we have

$$\begin{aligned} \gcd(G_n + aF_k, G_{n-1} - aF_{k+1}) &= \gcd(G_n + aF_k - (G_{n-1} - aF_{k+1}), G_{n-1} - aF_{k+1}) \\ &= \gcd(G_{n-2} + aF_{k+2}, G_{n-1} - aF_{k+1}) \\ &= \gcd(G_{n-2} + aF_{k+2}, G_{n-1} - aF_{k+1} - (G_{n-2} + aF_{k+2})) \\ &= \gcd(G_{n-2} + aF_{k+2}, G_{n-3} - aF_{k+3}). \end{aligned}$$

□

Lemma 2. *For integers m, k, and a,*

$$\gcd(G_m + a, G_{m+1} + a) = \gcd(G_{m-(2k)} + aF_{2k-1}, G_{m-(2k+1)} - aF_{2k}). \tag{4}$$

Proof. We simplify $\gcd(G_m + a, G_{m+1} + a)$,

$$\begin{aligned} \gcd(G_m + a, G_{m+1} + a) &= \gcd(G_m + a, G_{m+1} + a - (G_m + a)) \\ &= \gcd(G_m + a, G_{m-1}). \end{aligned}$$

Because $F_{-1} = 1$ and $F_0 = 0$ we can write

$$\gcd(G_m + a, G_{m+1} + a) = \gcd(G_m + aF_{-1}, G_{m-1} + aF_0),$$

and applying (3) k times gives the result. □

3 The sequence $(f_n(1))$

Theorem 3. *For any integer n, we have*

$$\gcd(F_{4n-1} + 1, F_{4n} + 1) = F_{2n-1}, \tag{5}$$

$$\gcd(F_{4n} + 1, F_{4n+1} + 1) = \begin{cases} 2, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{otherwise,} \end{cases} \tag{6}$$

$$\gcd(F_{4n+1} + 1, F_{4n+2} + 1) = L_{2n}, \tag{7}$$

$$\gcd(F_{4n+2} + 1, F_{4n+3} + 1) = \begin{cases} 2, & \text{if } n \equiv 2 \pmod 3, \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Proof. Let $m = 4n - 1$, $k = n$, and $a = 1$ in (4):

$$\begin{aligned} \gcd(F_{4n-1} + 1, F_{4n} + 1) &= \gcd(F_{2n-1} + F_{2n-1}, F_{2n-2} - F_{2n}) \\ &= \gcd(2F_{2n-1}, -F_{2n-1}) \\ &= F_{2n-1}, \end{aligned}$$

giving (5). Let $m = 4n+1$, $k = n$, and $a = 1$ in (4):

$$\begin{aligned}\gcd(F_{4n+1}+1, F_{4n+2}+1) &= \gcd(F_{2n+1}+F_{2n-1}, F_{2n}-F_{2n})\\ &= F_{2n+1}+F_{2n-1}\\ &= L_{2n},\end{aligned}$$

giving (7). Let $m = 4n$, $k = n$, and $a = 1$ in (4):

$$\begin{aligned}\gcd(F_{4n}+1, F_{4n+1}+1) &= \gcd(F_{2n}+F_{2n-1}, F_{2n-1}-F_{2n})\\ &= \gcd(F_{2n+1}, -F_{2n-2}).\end{aligned}$$

Since $\gcd(F_{qn+r}, F_n) = \gcd(F_n, F_r)$ for integers q, r, and n. This gives

$$\gcd(F_{4n}+1, F_{4n+1}+1) = \gcd(F_{2n-2}, F_3).$$

Because $\gcd(F_k, F_r) = F_{\gcd(k,r)}$ for integers k and r,

$$\begin{aligned}\gcd(F_{4n}+1, F_{4n+1}+1) &= \gcd(F_{2n-2}, F_3)\\ &= F_{\gcd(2n-2,3)}\\ &= \begin{cases} F_3 = 2, & n \equiv 1 \pmod 3,\\ F_1 = 1, & \text{otherwise,}\end{cases}\end{aligned}$$

which is (6). Let $m = 4n+2$, $k = n+1$, and $a = 1$ in (4):

$$\begin{aligned}\gcd(F_{4n+2}+1, F_{4n+3}+1) &= \gcd(F_{2n}+F_{2n+1}, F_{2n-1}-F_{2n+2})\\ &= \gcd(F_{2n+2}, F_{2n-1}-F_{2n+2})\\ &= \gcd(F_{2n+2}, F_{2n-1})\\ &= \gcd(F_3, F_{2n-1})\\ &= F_{\gcd(3,2n-1)}\\ &= \begin{cases} F_3 = 2, & n \equiv 2 \pmod 3,\\ F_1 = 1, & \text{otherwise,}\end{cases}\end{aligned}$$

which is (8). □

4 The sequence $(f_n(2))$

Theorem 4. *For any integer n, we have*

$$\gcd(F_{4n-1}+2, F_{4n}+2) = 1, \tag{9}$$

$$\gcd(F_{4n}+2, F_{4n+1}+2) = 1, \tag{10}$$

$$\gcd(F_{4n+1}+2, F_{4n+2}+2) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 2,\\ 1, & \text{if } n \equiv 1 \pmod 2,\end{cases} \tag{11}$$

$$\gcd(F_{4n+2}+2, F_{4n+3}+2) = \begin{cases} 5, & \text{if } n \equiv 1 \pmod 5,\\ 1, & \text{otherwise.}\end{cases} \tag{12}$$

Proof. Let $m = 4n - 1$, $k = n$, and $a = 2$ in (4):

$$\begin{aligned}\gcd(F_{4n-1}+2, F_{4n}+2) &= \gcd(F_{2n-1}+2F_{2n-1}, F_{2n-2}-2F_{2n})\\ &= \gcd(3F_{2n-1}, F_{2n-1}+F_{2n})\\ &= \gcd(3F_{2n-1}, F_{2n+1}).\end{aligned}$$

Since $\gcd(a, bc) = \gcd(a, \gcd(a, b)c)$ and $\gcd(F_{2n-1}, F_{2n+1}) = \gcd(F_{2n-1}, F_2) = 1$, we have

$$\begin{aligned}\gcd(F_{4n-1}+2, F_{4n}+2) &= \gcd(3\gcd(F_{2n-1}, F_{2n+1}), F_{2n+1})\\ &= \gcd(3, F_{2n+1}) = \gcd(F_4, F_{2n+1})\\ &= F_{\gcd(4,2n+1)} = F_1 = 1,\end{aligned}$$

which is (9). Let $m = 4n$, $k = n$, and $a = 2$ in (4):

$$\begin{aligned}\gcd(F_{4n}+2, F_{4n+1}+2) &= \gcd(F_{2n}+2F_{2n-1}, F_{2n-1}-2F_{2n})\\ &= \gcd(F_{2n-1}+F_{2n+1}, -F_{2n}-F_{2n-2})\\ &= \gcd(L_{2n}, L_{2n-1})\\ &= 1,\end{aligned}$$

which is (10). Let $m = 4n + 1$, $k = n$, and $a = 2$ in (4):

$$\begin{aligned}\gcd(F_{4n+1}+2, F_{4n+2}+2) &= \gcd(F_{2n+1}+2F_{2n-1}, F_{2n}-2F_{2n})\\ &= \gcd(F_{2n+1}+2F_{2n-1}+2F_{2n}, F_{2n})\\ &= \gcd(3F_{2n+1}, F_{2n})\\ &= \gcd(3, F_{2n}) = \gcd(F_4, F_{2n})\\ &= F_{\gcd(4,2n)}\\ &= \begin{cases} F_4 = 3, & \text{if } n \equiv 0 \pmod 2,\\ F_1 = 1, & \text{if } n \equiv 1 \pmod 2,\end{cases}\end{aligned}$$

which is (11). Let $m = 4n + 2$, $k = n$, and $a = 2$ in (4):

$$\begin{aligned}\gcd(F_{4n+2}+2, F_{4n+3}+2) &= \gcd(F_{2n+2}+2F_{2n-1}, F_{2n+1}-2F_{2n})\\ &= \gcd(F_{2n+2}+2F_{2n-1}, -F_{2n}+F_{2n-1})\\ &= \gcd(F_{2n+2}+2F_{2n-1}, -F_{2n-2})\\ &= \gcd(F_{2n-2}, F_{2n+2}+2F_{2n}).\end{aligned}$$

Since $F_{2n+2} + 2F_{2n} = F_{2n+1} + 3F_{2n} = 4F_{2n} + F_{2n-1}$, we have

$$\begin{aligned}\gcd(F_{4n+2}+2, F_{4n+3}+2) &= \gcd(F_{2n-2}, 4F_{2n}+F_{2n-1})\\ &= \gcd(F_{2n-2}, 5F_{2n})\\ &= \gcd(F_{2n-2}, 5\gcd(F_{2n-2}, F_{2n}))\\ &= \gcd(F_{2n-2}, 5) = \gcd(F_{2n-2}, F_5)\\ &= F_{\gcd(2n-2,5)}\\ &= \begin{cases} F_5 = 5, & \text{if } n \equiv 1 \pmod 5\\ F_1 = 1, & \text{otherwise,}\end{cases}\end{aligned}$$

which is (12). □

5 The sequences $(f_n(-1))$, $(f_n(-2))$, and $(\ell_n(1))$

Applying the same methods we get

Theorem 5. *For any integer n, we have*

$$\begin{aligned}
\gcd(F_{4n-1}-1, F_{4n}-1) &= L_{2n-1}, \\
\gcd(F_{4n}-1, F_{4n+1}-1) &= \begin{cases} 2, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{otherwise}, \end{cases} \\
\gcd(F_{4n+1}-1, F_{4n+2}-1) &= F_{2n}, \\
\gcd(F_{4n+2}-1, F_{4n+3}-1) &= \begin{cases} 2, & \text{if } n \equiv 2 \pmod 3, \\ 1, & \text{otherwise}. \end{cases}
\end{aligned}$$

Theorem 6. *For any integer n, we have*

$$\begin{aligned}
\gcd(F_{4n-1}-2, F_{4n}-2) &= 1, \\
\gcd(F_{4n}-2, F_{4n+1}-2) &= \begin{cases} 5, & \text{if } n \equiv 4 \pmod 5, \\ 1, & \text{otherwise},. \end{cases} \\
\gcd(F_{4n+1}-2, F_{4n+2}-2) &= \begin{cases} 1, & \text{if } n \equiv 0 \pmod 2, \\ 3, & \text{if } n \equiv 1 \pmod 2, \end{cases} \\
\gcd(F_{4n+2}-2, F_{4n+3}-2) &= 1.
\end{aligned}$$

Theorem 7. *For any integer n, we have*

$$\begin{aligned}
\gcd(L_{4n-1}+1, L_{4n}+1) &= \begin{cases} 3, & \text{if } n \equiv 0 \pmod 6, \\ 1, & \text{if } n \equiv 1 \pmod 6, \\ 6, & \text{if } n \equiv 2 \pmod 6, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ 3, & \text{if } n \equiv 4 \pmod 6, \\ 2, & \text{if } n \equiv 5 \pmod 6. \end{cases} \\
\gcd(L_{4n}+1, L_{4n+1}+1) &= \begin{cases} 4, & \text{if } n \equiv 1 \pmod 3, \\ 1, & \text{otherwise}, \end{cases} \\
\gcd(L_{4n+1}+1, L_{4n+2}+1) &= \begin{cases} 2, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{otherwise}, \end{cases} \\
\gcd(L_{4n+2}+1, L_{4n+3}+1) &= \begin{cases} 4, & \text{if } n \equiv 2 \pmod 3, \\ 1, & \text{otherwise}. \end{cases}
\end{aligned}$$

6 The proofs of Theorems 1 and 2

First we give the proof of Theorem 1. Let $m = 4n - 1$ and $k = n$ in (4):

$$\begin{aligned}\gcd(G_{4n-1} + a, G_{4n} + a) &= \gcd(G_{2n-1} + aF_{2n-1}, G_{2n-2} - aF_{2n}) \\ &= \gcd(\alpha F_{2n-3} + \beta F_{2n-2} + aF_{2n-1}, \alpha F_{2n-4} + \beta F_{2n-3} - aF_{2n}).\end{aligned}$$

Using the recursion relation for F_n, let

$$a_n = \alpha F_{2n-3} + \beta F_{2n-2} + aF_{2n-1} = (\alpha + a)F_{2n-3} + (\beta + a)F_{2n-2}$$

and

$$b_n = \alpha F_{2n-4} + \beta F_{2n-3} - aF_{2n} = (-\alpha + \beta - a)F_{2n-3} + (\alpha - 2a)F_{2n-2}.$$

Since $\gcd(a_n, b_n)$ divides $\gamma a_n + \theta b_n$ for any integers γ and θ, and

$$\begin{aligned}(\alpha + a)b_n - (-\alpha + \beta - a)a_n &= (\alpha^2 + \alpha\beta - \beta^2 - a^2)F_{2n-2} \\ (\alpha - 2a)a_n - (\beta + a)b_n &= (\alpha^2 + \alpha\beta - \beta^2 - a^2)F_{2n-3},\end{aligned}$$

we see that if $\alpha^2 + \alpha\beta - \beta^2 - a^2 \neq 0$, then the greatest common divisor of the two numbers is $|\alpha^2 + \alpha\beta - \beta^2 - a^2|$. Therefore $\gcd(a_n, b_n)$ divides $\alpha^2 + \alpha\beta - \beta^2 - a^2$. That is to say,

$$\gcd(G_{4n-1} + a, G_{4n} + a) \leq |\alpha^2 + \alpha\beta - \beta^2 - a^2|.$$

If we let $m = 4n + 1$ and $k = n$ in (4) we have, in exactly the same way, that

$$\gcd(G_{4n+1} + a, G_{4n+2} + a) \leq |\alpha^2 + \alpha\beta - \beta^2 - a^2|.$$

□

In the following we give the proof of Theorem 2. Let $m = 4n$ and $k = n$ in (4):

$$\begin{aligned}\gcd(G_{4n} + a, G_{4n+1} + a) &= \gcd(G_{2n} + aF_{2n-1}, G_{2n-1} - aF_{2n}) \\ &= \gcd(\alpha F_{2n-2} + \beta F_{2n-1} + aF_{2n-1}, \alpha F_{2n-3} + \beta F_{2n-2} - aF_{2n}).\end{aligned}$$

Using the recursion relation for F_n, let

$$a_n = \alpha F_{2n-2} + \beta F_{2n-1} + aF_{2n-1} = \alpha F_{2n-2} + (\beta + a)F_{2n-1}$$

and

$$b_n = \alpha F_{2n-3} + \beta F_{2n-2} - aF_{2n} = (-\alpha + \beta - a)F_{2n-2} + (\alpha - a)F_{2n-1}.$$

Since $\gcd(a_n, b_n)$ divides $\gamma a_n + \theta b_n$ for any integers γ and θ, and

$$\begin{aligned}(\alpha - a)a_n - (a + \beta)b_n &= (\alpha^2 + \alpha\beta - \beta^2 + a^2)F_{2n-2} \\ \alpha b_n - (\beta - \alpha - a)a_n &= (\alpha^2 + \alpha\beta - \beta^2 + a^2)F_{2n-1},\end{aligned}$$

we see that if $\alpha^2 + \alpha\beta - \beta^2 + a^2 \neq 0$, then the greatest common divisor of the two numbers is $|\alpha^2 + \alpha\beta - \beta^2 + a^2|$. Therefore $\gcd(a_n, b_n)$ divides $\alpha^2 + \alpha\beta - \beta^2 + a^2$. That is to say,

$$\gcd(G_{4n} + a, G_{4n+1} + a) \leq |\alpha^2 + \alpha\beta - \beta^2 + a^2|.$$

If we let $m = 4n + 2$ and $k = n$ in (4) we have, in exactly the same way, that

$$\gcd(G_{4n+2} + a, G_{4n+3} + a) \leq |\alpha^2 + \alpha\beta - \beta^2 + a^2|.$$

□

7 Acknowledgement

The author would like to thank the referee for some useful comments and suggestions.

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2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11A05.
Keywords: Fibonacci numbers, Lucas numbers, generalized Fibonacci numbers.

(Concerned with sequences A000032, A000045, A000071, A001611, and A157725.)

Received January 31 2011; revised version received March 26 2011. Published in *Journal of Integer Sequences*, March 26 2011.

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