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A Function Related to the Rumor Sequence Conjecture

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Abstract

For an integer $b \geq 2$ and for $x \in [0,1)$, define $\rho_b(x) = \sum_{n=0}^{\infty} \frac{\{\{b^n x\}\}}{b^n}$, where $\{\{t\}\}$ denotes the fractional part of the real number t. A number of properties of ρ_b are derived, and then a connection between ρ_b and the rumor conjecture is established. To form a rumor sequence $\{z_n\}$, first select integers $b \geq 2$ and $k \geq 1$. Then select an integer z_0, and for $n \geq 1$ let $z_n = bz_{n-1} \bmod (n+k)$, where the right side is the least non-negative residue of bz_{n-1} modulo $n+k$. The rumor sequence conjecture asserts that all such rumor sequences are eventually 0. A condition on ρ_b is shown to be equivalent to the rumor conjecture.

1 Introduction

In this note, $b \geq 2$ is a fixed integer. For $x \in [0,1)$, define $\rho_b(x) = \sum_{n=0}^{\infty} \frac{\{\{b^n x\}\}}{b^n}$, where $\{\{t\}\}$ denotes the fractional part of the real number t.

A *b-adic rational* is a rational which can be written as a quotient of an integer and a non-negative power of b. When a b-adic is written in the form $\frac{a}{b^m}$, and $m > 0$, it will be assumed b does not divide a.

The ρ_b function is similar to the well known Takagi function $\tau(x)$ defined by $\tau(x) = \sum_{n=0}^{\infty} \frac{\ll 2^n x \gg}{2^n}$, where $\ll t \gg$ is the distance from t to the nearest integer. Whereas the summand of the Takagi function is a triangle wave, the summand of $\rho_b(x)$ is a sawtooth wave. (See Figures 1, 2.) The function $y = \ll 2^n x \gg$ is continuous and periodic with period 2^{-n}, and it follows that τ is continuous. It turns out τ (see Figure 3) is also nowhere differentiable.

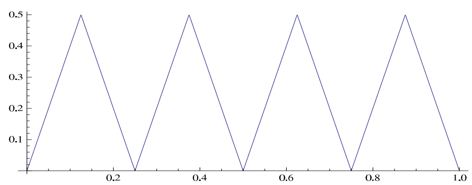

Figure 1: $y = \ll 2^2 x \gg$

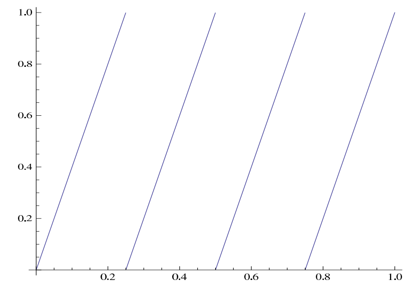

Figure 2: $y = \{\!\{ 2^2 x \}\!\}$

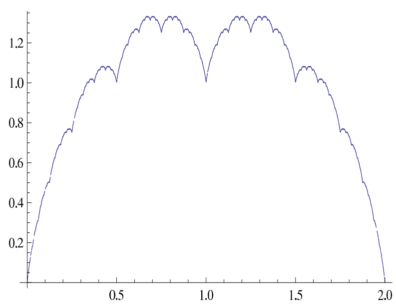

Figure 3: $y = \tau(x)$

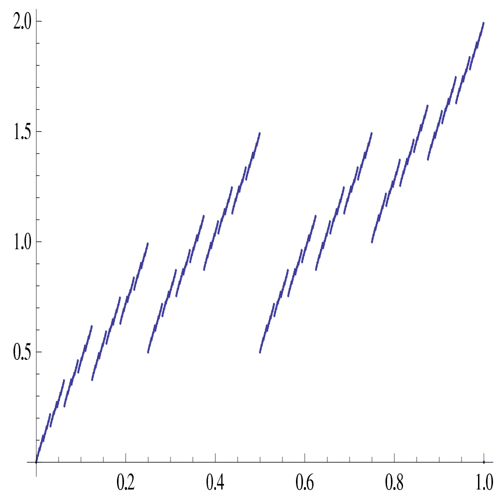

Figure 4: $y = \rho_2(x)$

The Takagi function has interesting analytic properties not shared by ρ_b. For example, while the Takagi function is continuous, ρ_b is easily seen to be continuous except at the b-adics as suggested by Figure 4 for the case $b = 2$. In general, at each b-adic, $\frac{a}{b^m}$, ρ_b is right continuous with a jump discontinuity of $-\frac{1}{b^{m-1}(b-1)}$. On the other hand, the ρ_b function has interesting number theoretic features not shared by τ as we will show in the following section.

In the final section of this note, we show that the ρ_b function is related to the rumor conjecture described in Dearden and Metzger [1], and finally a conjecture concerning ρ_b that is equivalent to the rumor sequence conjecture is stated.

2 Arithmetic Properties of the ρ_b Function

An alternative expression for $\rho_b(x)$ can be given in terms of the base-b expansion of x. In this note we will follow the usual convention that base-b expansions of b-adics terminate rather than end with an infinite string of $(b-1)$'s.

Theorem 1. *If $\sum_{j=1}^{\infty} \frac{d_j}{b^j}$ is the base-b expansion of $x \in [0,1)$, then $\rho_b(x) = \sum_{j=1}^{\infty} \frac{jd_j}{b^j}$.*

Proof. Let $x = \sum_{j=1}^{\infty} \frac{d_j}{b^j} \in [0,1)$. Then

$$\begin{aligned}\rho_b(x) &= \sum_{n=0}^{\infty} \frac{\{\!\{b^n x\}\!\}}{b^n} = \sum_{n=0}^{\infty} \frac{\{\!\{b^n \sum_{j\geq 1} d_j b^{-j}\}\!\}}{b^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{b^n} \left\{\!\!\left\{ \sum_{j=1}^{n} d_j b^{n-j} + \sum_{j>n} d_j b^{n-j} \right\}\!\!\right\} = \sum_{n=0}^{\infty} \frac{1}{b^n} \sum_{j>n} d_j b^{n-j} \\ &= \sum_{n=0}^{\infty} \sum_{j>n} \frac{d_j}{b^j} = \sum_{j=1}^{\infty} \sum_{n=0}^{j-1} \frac{d_j}{b^j} = \sum_{j=1}^{\infty} \frac{jd_j}{b^j}.\end{aligned}$$

□

Theorem 2. *The range of ρ_b is $[0, \frac{b}{b-1})$.*

Proof. Let $y \in [0, \frac{b}{b-1})$ be given. Integers $d_j \in D = \{0, 1, \ldots, b-1\}$ are selected recursively as follows. First let d_1 be the largest integer in D such that $\frac{d_1}{b} \leq y$. Assuming $d_1, d_2, \ldots, d_{i-1}$ have been selected, take d_i to be the largest integer in D such that $\frac{id_i}{b^i} \leq y - \sum_{j=1}^{i-1} \frac{jd_j}{b^j}$. In this way, the base-$b$ expansion of a number $x = \sum_{j=1}^{\infty} \frac{d_j}{b^j}$ is constructed.

We now show that this expansion of x does not end in an infinite sequence of $(b-1)$'s, and consequently $x \in [0,1)$ and $\rho_b(x) = \sum_{j=1}^{\infty} \frac{jd_j}{b^j}$. To this end, by way of contradiction, assume the expansion does end with an infinite sequence of $(b-1)$'s. It can not be that all the digits, d_j, are $b-1$ since, if they were, we would have

$$\sum_{j=1}^{\infty} \frac{jd_j}{b^j} = \sum_{j=1}^{\infty} \frac{j(b-1)}{b^j} = (b-1) \sum_{j=1}^{\infty} \frac{j}{b^j} = (b-1) \frac{b}{(b-1)^2} = \frac{b}{b-1},$$

but $\sum_{j=1}^{\infty} \frac{jd_j}{b^j} \leq y < \frac{b}{b-1}$. So there must be a last digit, d_L, that is less than $b-1$. It follows that for all $m > L$,

$$\frac{m(b-1)}{b^m} \leq y - \sum_{j=1}^{L-1} \frac{jd_j}{b^j} - \frac{Ld_L}{b^L} - \sum_{j=L+1}^{m-1} \frac{j(b-1)}{b^j}.$$

Hence, for all $m > L$, we have

$$\frac{Ld_L}{b^L} + \sum_{j=L+1}^{m} \frac{j(b-1)}{b^j} \leq y - \sum_{j=1}^{L-1} \frac{jd_j}{b^j}.$$

Consequently

$$\frac{Ld_L}{b^L} + \sum_{j=L+1}^{\infty} \frac{j(b-1)}{b^j} \leq y - \sum_{j=1}^{L-1} \frac{jd_j}{b^j}.$$

Now

$$\sum_{j=L+1}^{\infty} \frac{j(b-1)}{b^j} = \frac{(L+1)b - L}{b^L(b-1)} > \frac{L}{b^L}.$$

Thus

$$\frac{L(d_L+1)}{b^L} = \frac{Ld_L}{b^L} + \frac{L}{b^L} \leq y - \sum_{j=1}^{L-1} \frac{jd_j}{b^j},$$

contradicting the choice of d_L.

For any i with $d_i < b-1$, we have

$$\frac{id_i}{b^i} \leq y - \sum_{j=1}^{i-1} \frac{jd_j}{b^j} < \frac{i(d_i+1)}{b^i}.$$

Since that holds for infinitely many i, and since $\sum_{j=1}^{\infty} \frac{jd_j}{b^j}$ is a positive series, it follows that $\rho_b(x) = \sum_{j=0}^{\infty} \frac{jd_j}{b^j} = y$.

□

The x constructed in the proof above is the largest of the inverses of the given y under ρ_b. Call the x so constructed the *greedy inverse image* of y. In order to construct a valid base-b number x as the greedy inverse of a given y, we explicitly required each d_i to be an element of the set $D = \{0, 1, \ldots, b-1\}$, rather than using a floor function, as in

$$\left\lfloor \frac{b^i \left(y - \sum_{j=1}^{i-1} \frac{jd_j}{b^j} \right)}{i} \right\rfloor.$$

Since this integer may be larger than $b-1$, the restriction on d_i was needed. We now show that d_i is eventually given by this floor function expression.

Corollary 3. *With the notation as in the proof of Theorem 2, for large enough i,*

$$d_i = \left\lfloor \frac{b^i \left(y - \sum_{j=1}^{i-1} \frac{jd_j}{b^j} \right)}{i} \right\rfloor \text{ and } 0 \leq y - \sum_{j=1}^{i} \frac{jd_j}{b^j} < \frac{i}{b^i}.$$

Proof. We show, for i large enough, that the quantity $z = b^i \left(y - \sum_{j=1}^{i-1} \frac{jd_j}{b^j} \right) / i$ is less than b, and, thus, $\lfloor z \rfloor \in D$. From the proof of Theorem 2, there is an integer n with $d_n < b-1$, where d_n is the largest integer in $D = \{0, 1, \ldots, b-1\}$ such that

$$\frac{nd_n}{b^n} \leq y - \sum_{j=1}^{n-1} \frac{jd_j}{b^j}.$$

Thus, we see that

$$\frac{nd_n}{b^n} \leq y - \sum_{j=1}^{n-1} \frac{jd_j}{b^j} < \frac{n(d_n+1)}{b^n}. \tag{1}$$

Equivalently, we have

$$d_n \leq \frac{b^n \left(y - \sum_{j=1}^{n-1} \frac{jd_j}{b^j} \right)}{n} < d_n + 1.$$

Now, since $d_n + 1 < b$, we have that

$$\left\lfloor \frac{b^n \left(y - \sum_{j=1}^{n-1} \frac{jd_j}{b^j} \right)}{n} \right\rfloor \in D.$$

Therefore, d_n may be expressed as

$$d_n = \left\lfloor \frac{b^n \left(y - \sum_{j=1}^{n-1} \frac{jd_j}{b^j} \right)}{n} \right\rfloor.$$

And, rearranging (1) we see that

$$0 \leq y - \sum_{j=1}^{n} \frac{jd_j}{b^j} < \frac{n}{b^n}.$$

Inductively, consider any $i \geq n$ where

$$0 \leq y - \sum_{j=1}^{i} \frac{jd_j}{b^j} < \frac{i}{b^i}. \tag{2}$$

By definition, d_{i+1} is the greatest integer in D such that

$$d_{i+1} \leq \frac{b^{i+1} \left(y - \sum_{j=1}^{i} \frac{jd_j}{b^j} \right)}{i+1}. \tag{3}$$

Moreover, using (2), we have

$$\frac{b^{i+1} \left(y - \sum_{j=1}^{i} \frac{jd_j}{b^j} \right)}{i+1} < \frac{i}{i+1} b < b.$$

Hence, as before, we have that

$$d_{i+1} = \left\lfloor \frac{b^{i+1} \left(y - \sum_{j=1}^{i} \frac{jd_j}{b^j} \right)}{i+1} \right\rfloor,$$

since the value of the floor expression is an element of the set D. Finally, from (3), we have

$$\frac{(i+1)d_{i+1}}{b^{i+1}} \le y - \sum_{j=1}^{i} \frac{jd_j}{b^j} < \frac{(i+1)(d_{i+1}+1)}{b^{i+1}}.$$

From which we have

$$0 \le y - \sum_{j=1}^{i+1} \frac{jd_j}{b^j} < \frac{i+1}{b^{i+1}},$$

completing the induction.

□

There are several easily verified functional identities satisfied by ρ_b stated in the next theorem.

Theorem 4. *The following identities hold for ρ_b:*

(a) For the b-adic $x = \frac{a}{b^m} \in [0,1)$, $\rho_b(x) + \rho_b(1-x) = \frac{b}{b-1} - \frac{1}{b^{m-1}(b-1)}$.

(b) For any non-b-adic $x \in [0,1)$, $\rho_b(x) + \rho_b(1-x) = \frac{b}{b-1}$.

(c) For $x \in [0,1)$ and integer $m \ge 1$, $\rho_b\left(\frac{x}{b^m}\right) = \frac{m}{b^m}x + \frac{1}{b^m}\rho_b(x)$.

(d) If $b^m x \in [0,1)$, then $\rho_b(b^m x) = b^m \rho_b(x) - mb^m x$.

Theorem 5. *Suppose $\frac{s}{t}$ is a rational number in lowest terms with $\gcd(t,b) = 1$. If $\rho_b(\frac{s}{t}) = \frac{u}{v}$, a rational in lowest terms, then (1) there is a divisor $t' > 1$ of t such that $(t')^2$ divides v, and (2) b divides u.*

Proof. Since t is relatively prime to b, the base-b expansion of $\frac{s}{t}$ is purely periodic. Let r be the order of b modulo t, so that r is the period of that expansion. That means there is an integer c so that $ct = b^r - 1$. Then

$$\frac{s}{t} = \frac{cs}{ct} = \frac{cs}{b^r - 1} = \sum_{m \ge 1} \frac{cs}{b^{mr}} = \sum_{m \ge 1} \frac{\sum_{i=1}^{r} b^{r-i} d_i}{b^{mr}},$$

where

$$\frac{s}{t} = \sum_{j \ge 1} \frac{d_j}{b^j} = \sum_{m \ge 0} \sum_{i=1}^{r} \frac{d_i}{b^{mr+i}} \quad \text{has period } r.$$

First, calculate $\rho_b(\frac{s}{t})$ as follows:

$$\begin{aligned}
\frac{u}{v} = \rho_b\left(\frac{s}{t}\right) &= \sum_{j\geq 1} \frac{jd_j}{b^j} = \sum_{m\geq 0}\sum_{i=1}^{r} \frac{(mr+i)d_{mr+i}}{b^{mr+i}} = \sum_{m\geq 0}\sum_{i=1}^{r} \frac{(mr+i)d_i}{b^{mr+i}} \\
&= \sum_{m\geq 0} \frac{1}{b^{mr}} \left(mr\sum_{i=1}^{r} \frac{d_i}{b^i} + \sum_{i=1}^{r} \frac{id_i}{b^i} \right) \\
&= \sum_{m\geq 0} \frac{1}{(b^r)^{m+1}} \left(mr\sum_{i=1}^{r} b^{r-i}d_i + \sum_{i=1}^{r} ib^{r-i}d_i \right) \\
&= \sum_{m\geq 0} \frac{1}{(b^r)^{m+1}} \left(mrcs + w\right), \text{where } w = \sum_{1\leq i\leq r} ib^{r-i}d_i \\
&= \frac{rcs}{(b^r-1)^2} + \frac{w}{b^r-1} = \frac{rcs + (b^r-1)w}{(b^r-1)^2} = \frac{rcs+ctw}{c^2t^2} \\
&= \frac{rs+tw}{ct^2}.
\end{aligned}$$

Let $d = \gcd(t, r)$ and define t' and r' by $t = t'd$ and $r = r'd$. Then, we have

$$\rho_b\left(\frac{s}{t}\right) = \frac{r's+t'w}{ctt'} = \frac{r's+t'w}{cd(t')^2}.$$

Since r divides $\varphi(t)$ we have

$$r \leq \varphi(t) < t, \text{ hence, } 1 \leq r' < t'.$$

In particular, we have that $t' \neq 1$.

Since t' is relatively prime to both s and r', we have that $(t')^2$ does not cancel when the fraction is reduced to lowest terms. That completes the proof of (1).

For the proof of (2), calculate $\rho_b(\frac{s}{t})$ as

$$\begin{aligned}
\frac{u}{v} = \rho_b\left(\frac{s}{t}\right) &= \sum_{m\geq 0}\sum_{i=1}^{r} \frac{(mr+i)d_i}{b^{mr+i}} = \sum_{i=1}^{r}\sum_{m\geq 0} \frac{(mr+i)d_i}{b^{mr+i}} \\
&= \frac{1}{(b^r-1)^2}\sum_{i=1}^{r} d_i b^{r-i}(r-i+b^r i).
\end{aligned}$$

Note that b is a factor of each term in the sum, including the term when $i = r$. Since b is relatively prime to $b^r - 1$, it follows that b divides u.

□

Corollary 6. *There are rationals in the range $[0, \frac{b}{b-1})$ of ρ_b that are not images of any rationals in its domain.*

Example 7. For $b \neq 3$, the rational $\frac{1}{3}$ cannot be the image of a rational under ρ_b.

The conditions given in Theorem 5 apparently do not completely characterize the rationals that are images of rationals. In particular, for $b = 2$ we suspect that among $\frac{2k}{9}$, $k = 1, 2, 4, 5, 7, 8$, only $\frac{8}{9} = \rho_2(\frac{1}{3})$ and $\frac{10}{9} = \rho_2(\frac{2}{3})$ have rational inverse images.

In Theorem 8 we derive an expression for $\rho_b(\frac{a}{b^r})$ analogous to one for the Takagi function given by Maddock [2]. If the base-b expansion of the positive integer a is given by $a = \sum_{i=0}^{m-1} e_i b^i$, define $\sigma_b(a)$ by

$$\sigma_b(a) = \sum_{i=0}^{m-1} i e_i b^i.$$

It is easy to check that $\sigma_b(a)$ can be written in a way that does not specifically involve the base-b expansion:

$$\sigma_b(a) = \sum_{j \geq 1} b^j \left\lfloor \frac{a}{b^j} \right\rfloor = \sum_{1 \leq b^j \leq a} (a - (a \bmod b^j)).$$

The σ_b functions are related to several sequences in Sloane's OEIS database. Specifically, sequence A080277 is $a + \sigma_2(a) = \sum_{j \geq 0} 2^j \lfloor \frac{a}{2^j} \rfloor$, while A080333 is $a + \sigma_3(a) = \sum_{j \geq 0} 3^j \lfloor \frac{a}{3^j} \rfloor$. Also, the sums $s_a = \sum_{1 \leq b^j \leq a} (a \bmod b^j)$ appear in OEIS for $b = 2$ and $b = 3$ as A049802 and A049803 respectively.

Theorem 8. *For the b-adic rational $\frac{a}{b^r}$, where $0 \leq a < b^r$, we have*

$$\rho_b\left(\frac{a}{b^r}\right) = \frac{ra - \sigma_b(a)}{b^r}.$$

Proof. Let the base-b expansion of a be $a = \sum_{i=0}^{r-1} e_i b^i$. We then have

$$\begin{aligned} \rho_b\left(\frac{a}{b^r}\right) &= \rho_b\left(\sum_{i=0}^{r-1} \frac{e_i}{b^{r-i}}\right) = \sum_{i=0}^{r-1} \frac{(r-i)e_i}{b^{r-i}}, \\ &= \frac{1}{b^r}\left[\sum_{i=0}^{r-1} r e_i b^i - \sum_{i=0}^{r-1} i e_i b^i\right], \\ &= \frac{1}{b^r}\left[ra - \sigma_b(a)\right]. \end{aligned}$$

□

Theorem 9. *Consider the rational number s/t in reduced form with t relatively prime to b. Let $r = \operatorname{ord}_t(b)$, $ct = b^r - 1$, and $a = cs$. Then,*

$$\rho_b\left(\frac{s}{t}\right) = \rho_b\left(\frac{a}{b^r - 1}\right) = \frac{r b^r a}{(b^r - 1)^2} - \frac{\sigma_b(a)}{b^r - 1}.$$

Proof. Given the base-b expansion $a = \sum_{i=0}^{r-1} e_i b^i$, we have

$$\frac{a}{b^r - 1} = \sum_{k \geq 1} \sum_{i=0}^{r-1} \frac{e_i}{b^{rk-i}}.$$

Hence, we calculate

$$\begin{aligned}
\rho_b\left(\frac{s}{t}\right) = \rho_b\left(\frac{a}{b^r-1}\right) &= \sum_{k\geq 1}\sum_{i=0}^{r-1}(rk-i)\frac{e_i}{b^{rk-i}}, \\
&= \sum_{k\geq 1}\frac{1}{b^{rk}}\left[\sum_{i=0}^{r-1} rke_ib^i - \sum_{i=0}^{r-1} ie_ib^i\right], \\
&= \sum_{k\geq 1}\frac{1}{b^{rk}}\left[kra - \sigma_b(a)\right], \\
&= \frac{rb^r a}{(b^r-1)^2} - \frac{\sigma_b(a)}{b^r-1}.
\end{aligned}$$

□

Theorem 9 leads to a relation between two values of ρ_b. With s, t, r, a as in the proof of that theorem, we see

$$\begin{aligned}
\rho_b\left(\frac{s}{t}\right) = \rho_b\left(\frac{a}{b^r-1}\right) &= \frac{rb^r a}{(b^r-1)^2} - \frac{\sigma_b(a)}{b^r-1} \\
&= \frac{rb^r a - (b^r-1)\sigma_b(a)}{(b^r-1)^2} \\
&= \frac{ra + (b^r-1)(ra - \sigma_b(a))}{(b^r-1)^2} \\
&= \frac{ra}{(b^r-1)^2} + \frac{b^r}{b^r-1}\rho_b\left(\frac{a}{b^r}\right).
\end{aligned}$$

3 The Connection Between ρ_b and Rumor Sequences

In Dearden and Metzger [1], rumor sequences (**ru**nning **mo**dulus **r**ecursive sequences) were introduced as follows:

Let $b \geq 2$ and $k \geq 1$ be integers. To construct an *(integer) rumor sequence* select an integer z_0, and for $n \geq 1$ let $z_n = bz_{n-1} \bmod (n+k)$, where the right side is the least non-negative residue of bz_{n-1} modulo $n+k$. The rumor sequence conjecture asserts that all such integer rumor sequences are eventually 0. Since the conjecture concerns only the eventual behavior of such sequences and since $0 \leq z_1 < k+1$, nothing is lost by restricting z_0 to the interval $[0, k)$.

To establish a connection between the rumor sequence conjecture and the ρ_b function, it is convenient to generalize the notion of integer rumor sequences to *real* rumor sequences.

Let $b \geq 2$ and $k \geq 1$ be integers. To construct a *(real) rumor sequence*, select any real number x_0 and for $n \geq 1$ let $x_n = bx_{n-1} \bmod (n+k)$ where the right hand side is taken to be

$$bx_{n-1} - (n+k)\left\lfloor \frac{bx_{n-1}}{n+k} \right\rfloor. \tag{4}$$

As with integer rumors, there is no loss if x_0 is restricted to the interval $[0, k)$. The real and integer rumors are identical when $x_0 = z_0$ is an integer.

It will be shown that the rumor conjecture for integer rumor sequences is true if and only if the greedy inverse image under ρ_b of every b-adic rational is a b-adic rational. It is worth noting that, in general, not all inverse images of a b-adic under ρ_b need be b-adic.

Example 10. Consider the 3-adic rational $y = \frac{2}{3}$ in the range of ρ_3. With $b = 3$, let the greedy ρ_3 inverse image of $\frac{5}{6}$ be x. Since 6 is not divisible by a square greater than 1, x must be irrational. It follows that $1 - x$ is irrational and, by Theorem 4(b), we see

$$\rho_3(1-x) = \frac{3}{2} - \frac{5}{6} = \frac{2}{3}.$$

Theorem 11. *For $b \geq 2$, all integer rumor sequences are eventually 0 if and only if the greedy inverse image under ρ_b of every b-adic is b-adic.*

Proof. Suppose that all integer rumor sequences are eventually zero, and let $y = a/b^m$ be a b-adic rational in $[0, b/(b-1))$. By Corollary 3, there is an integer n so that for $k \geq n$ we have

$$d_k = \left\lfloor \frac{b^k \left(y - \sum_{j=1}^{k-1} j d_j / b^j\right)}{k} \right\rfloor \text{ and } 0 \leq y - \sum_{j=1}^{k} \frac{j d_j}{b^j} < \frac{k}{b^k}.$$

Now, consider the real rumor sequence with initial value $x_0 \in [0, n)$ given by

$$x_0 = b^n \left(y - \sum_{j=1}^{n} \frac{j d_j}{b^j}\right).$$

Applying the rumor recursion (4), we have

$$\begin{aligned}
x_1 &= bx_0 - (n+1)\left\lfloor \frac{bx_0}{n+1} \right\rfloor \\
&= b^{n+1}\left(y - \sum_{j=1}^{n} \frac{j d_j}{b^j}\right) - (n+1)\left\lfloor \frac{b^{n+1}\left(y - \sum_{j=1}^{n} j d_j / b^j\right)}{n+1} \right\rfloor \\
&= b^{n+1}\left(y - \sum_{j=1}^{n} \frac{j d_j}{b^j}\right) - (n+1) d_{n+1}, \text{ by Corollary 3} \\
&= b^{n+1}\left(y - \sum_{j=1}^{n+1} \frac{j d_j}{b^j}\right).
\end{aligned}$$

More generally, induction shows that, for all $i \geq 0$, we have

$$x_i = b^{n+i}\left(y - \sum_{j=1}^{n+i} \frac{jd_j}{b^j}\right) = b^{n+i}\left(\frac{a}{b^m} - \sum_{j=1}^{n+i} \frac{jd_j}{b^j}\right).$$

Now, for $i \geq m-n$, the sequence x_i is obtained from an integer rumor recursion, and by our assumption that integer rumor sequence is eventually zero, say from term i_0 on. That means the greedy inverse image under ρ_b of the b-adic rational $a/b^m = \sum_{j=1}^{n+i_0} jd_j/b^j$ is the b-adic rational

$$v = \sum_{j=1}^{n+i_0} \frac{d_j}{b^j} = \frac{\sum_{j=1}^{n+i_0} d_j b^{n+i_0-j}}{b^{n+i_0}}.$$

Conversely, suppose that the greedy inverse image of b-adic rationals in $[0, b/(b-1))$ are b-adic rationals. Consider an integer rumor recursion with initial value z_0 in $[0,k)$. By our assumption the greedy inverse of the b-adic rational $y = z_0/b^k$ is a b-adic rational $\sum_{j=1}^{n} j/b^j$, where

$$y = \sum_{j=1}^{n} \frac{jd_j}{b^j}, \text{ with } d_j \in \{0, 1, \ldots, b-1\}.$$

Since $f(x) = x/b^x$ is a nondecreasing function on positive integers for all integers $b \geq 2$, we have $z_0/b^k < k/b^k \leq m/b^m$ for all $m = 1, 2, 3, \ldots, k$. Therefore, it follows that

$$0 \leq \frac{z_0}{b^k} - \sum_{j=1}^{m-1} \frac{jd_j}{b^j} < \frac{m}{b^m}, \text{ for } m = 1, 2, \ldots, k\ .$$

Hence,

$$d_j = 0 \text{ for } j = 1, 2, \ldots, k.$$

It follows that

$$\frac{z_0}{b^k} = y = \sum_{j=k+1}^{n-k} \frac{(k+i)d_{k+i}}{b^{k+i}}. \tag{5}$$

Moreover, for all $m = 1, 2, \ldots, n-k$, we have

$$\frac{(k+m)d_{k+m}}{b^{k+m}} \leq \frac{z_0}{b^k} - \sum_{i=1}^{m-1} \frac{(k+i)d_{k+i}}{b^{k+i}} < \frac{(k+m)(d_{k+m}+1)}{b^{k+m}}.$$

Multiplying through by b^k gives

$$\frac{(k+m)d_{k+m}}{b^m} \leq z_0 - \sum_{i=1}^{m-1} \frac{(k+i)d_{k+i}}{b^i} < \frac{(k+m)(d_{k+m}+1)}{b^m}.$$

In particular, for $m = 1$ we have

$$\frac{(k+1)d_{k+1}}{b} \leq z_0 < \frac{(k+1)d_{k+1}}{b}$$

or

$$d_{k+1} \leq \left\lfloor \frac{bz_0}{k+1} \right\rfloor < d_{k+1} + 1.$$

It follows that

$$z_1 = bz_0 - (k+1) \left\lfloor \frac{bz_0}{k+1} \right\rfloor = bz_0 - (k+1)d_{k+1}.$$

Hence,

$$\frac{z_1}{b} = z_0 - \frac{(k+1)d_{k+1}}{b}.$$

In general, induction shows that, for all $m \geq 1$,

$$\frac{z_m}{b^m} = z_0 - \sum_{i=1}^{m} \frac{(k+i)d_{k+i}}{b^i}.$$

Therefore, by equation (5), we have

$$\frac{z_{n-k}}{b^{n-k}} = z_0 - \sum_{i=1}^{n-k} \frac{(k+i)d_{k+i}}{b^i} = 0.$$

Thus, any integer rumor sequence is eventually zero.

□

The following corollary follows immediately from the proof of Theorem 11.

Corollary 12. *Let $b \geq 2$ be an integer. The integer rumor sequence with initial term z_0, where $0 \leq z_0 < k$, is eventually 0 if and only if the greedy inverse image of $\frac{z_0}{b^k}$ under ρ_b is b-adic.*

Conjecture 13. The greedy inverse image of every b-adic under ρ_b is b-adic.

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