Journal of Integer Sequences, Vol. 14 (2011),
Article 11.8.5

On the Distribution of Perfect Powers

Rafael Jakimczuk
División Matemática
Universidad Nacional de Luján
Buenos Aires
Argentina
jakimczu@mail.unlu.edu.ar

In memory of my sister Fedra Marina Jakimczuk (1970–2010)

Abstract

Let $N(x)$ be the number of perfect powers that do not exceed x. In this article we obtain asymptotic formulae for the counting function $N(x)$.

1 Introduction

A natural number of the form m^n where m is a positive integer and $n \geq 2$ is called a *perfect power*. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots,$$

and they are sequence A001597 in Sloane's *Encylopedia*. Let $N(x)$ be the number of perfect powers that do not exceed x. M. A. Nyblom [3] proved the following asymptotic formula,

$$N(x) \sim \sqrt{x}.$$

M. A. Nyblom [4] also obtained a formula for the exact value of $N(x)$ using the inclusion-exclusion principle (also called the principle of cross-classification).

In this article we obtain more precise asymptotic formulae for the counting function $N(x)$. For example, we prove

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + \sqrt[7]{x} + o\left(\sqrt[7]{x}\right).$$

Consequently

$$\sqrt{x} + \sqrt[3]{x} < N(x) < \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}.$$

2 Preliminary Results

Let A be a set. The number of elements in A we denote in the form $|A|$.

We need the following results.

Lemma 1. *(Inclusion-exclusion principle) Let us consider a given finite collection of sets $A_1, A_2, \ldots, A_n$. The number of elements in $\cup_{i=1}^n A_i$ is*

$$|\cup_{i=1}^n A_i| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}| \,,$$

where the expression $1 \leq i_1 < \cdots < i_k \leq n$ indicates that the sum is taken over all the k-element subsets $\{i_1, \ldots, i_k\}$ of the set $\{1, 2, \ldots, n\}$.

Proof. See for example either [1, page 233] or [2, page 84]. □

Let $A_n(x)$ $(n \geq 2)$ be the set $\{k^n : k \in N, k^n \leq x\}$, that is, the set of perfect powers whose exponent is n that do not exceed x.

Lemma 2. *We have*

$$|A_n(x)| = \lfloor \sqrt[n]{x} \rfloor \,,$$

where $\lfloor . \rfloor$ denotes the integer-part function.

Proof. We have

$$k^n \leq x \Leftrightarrow k \leq \sqrt[n]{x}.$$

□

M. A. Nyblom [4] proved the following Lemma and the following Theorem.

Lemma 3. *For any set consisting of $m \geq 2$ positive integers $\{n_1, \ldots, n_m\}$ all greater than unity, we have the set equality*

$$\bigcap_{i=1}^{m} A_{n_i}(x) = A_{[n_1, \ldots, n_m]}(x),$$

where $[n_1, \ldots, n_m]$ denotes the least common multiple of the m integers $n_1, \ldots, n_m$.

Let p_n be the n-th prime. Consequently we have,

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, \ldots$$

Theorem 4. *If $x \geq 4$ and $p_1, p_2, \ldots, p_m$ denote the prime numbers that do not exceed $\lfloor \log_2 x \rfloor$, then for $k \geq m$ the number of perfect powers that do not exceed x is*

$$N(x) = \left| \bigcup_{i=1}^{k} A_{p_i}(x) \right|.$$

Besides $A_{p_i}(x) = \{1\}$ for $i \geq m + 1$.

We also need the following two well-known results.

The binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \tag{1}$$

and the following property of the absolute value

$$|a_1 + a_2 + \cdots + a_r| \leq |a_1| + |a_2| + \cdots + |a_r|\,, \tag{2}$$

where $a_1, a_2, \ldots, a_r$ are real numbers.

3 Main Results

Theorem 5. *Let p_n be the n-th prime number with $n \geq 2$, where n is an arbitrary but fixed positive integer. Then*

$$N(x) = \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq n-1 \\ p_{i_1} \cdots p_{i_k} < p_n}} x^{\frac{1}{p_{i_1} \cdots p_{i_k}}} + g(x) x^{\frac{1}{p_n}}, \tag{3}$$

where $\lim_{x\to\infty} g(x) = 1$ and the inner sum is taken over the k-element subsets $\{i_1, \ldots, i_k\}$ of the set $\{1, 2, \ldots, n-1\}$ such that the inequality $p_{i_1} \cdots p_{i_k} < p_n$ holds.

Proof. Let $k = \lfloor \log_2 x \rfloor + 1 = \left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1$, where $x \geq 4$. If $p_1, p_2, \ldots, p_m$ denote the prime numbers that do not exceed $\lfloor \log_2 x \rfloor$, then we have

$$p_1 < \cdots < p_m \leq \lfloor \log_2 x \rfloor < \lfloor \log_2 x \rfloor + 1 = k \leq p_{m+1} < p_{m+2} < \cdots \tag{4}$$

Note that if $i \geq m+1$ we have $1 < x^{\frac{1}{p_i}} \leq x^{\frac{1}{k}} < x^{\frac{1}{\frac{\log x}{\log 2}}} = 2$. That is, $1 < x^{\frac{1}{p_i}} < 2$. Consequently if $i \geq m+1$ (see Lemma 2) $|A_{p_i}(x)| = 1$. That is, $A_{p_i}(x) = \{1\}$. Note also that k and m are increasing functions of x. On the other hand $n \geq 2$ is an arbitrary but fixed positive integer.

Equation (4) gives

$$p_m < k. \tag{5}$$

On the other hand

$$m < p_m. \tag{6}$$

Therefore (5) and (6) give

$$m < k, \tag{7}$$

and consequently

$$p_m < p_k. \tag{8}$$

There exist three possible relations between m, k, n and $n+1$ and consequently between p_m, p_k, p_n and p_{n+1}.

First relation.

$$m < k \leq n < n+1$$

and hence

$$p_m < p_k \leq p_n < p_{n+1}.$$

Second relation.

$$m \leq n < n+1 \leq k$$

and hence

$$p_m \leq p_n < p_{n+1} \leq p_k.$$

Third relation.

$$n < n+1 \leq m < k$$

and hence

$$p_n < p_{n+1} \leq p_m < p_k.$$

If we define $S(x) = \max\{n+1, k\}$ then these three relations, Theorem 4 and Lemma 2 give us ($x \geq 4$)

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| \leq N(x) &< \left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| + \sum_{i=n+1}^{S(x)} |A_{p_i}(x)| \\
&= \left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| + \sum_{i=n+1}^{S(x)} \left\lfloor x^{\frac{1}{p_i}} \right\rfloor \\
&\leq \left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| + (S(x)-n) \left\lfloor x^{\frac{1}{p_{n+1}}} \right\rfloor . \qquad (9)
\end{aligned}$$

Note that there exists x_0 such that if $x \geq x_0$ the third relation holds.

Note also that $S(x) - n$ is either equal to 1 or $k - n < k - 1 = \left\lfloor \frac{\log x}{\log 2} \right\rfloor \leq \frac{\log x}{\log 2}$, and so in either case

$$S(x) - n \leq \frac{\log x}{\log 2} \qquad (10)$$

as $x \geq 4$. Consequently (9) and (10) give

$$\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| \leq N(x) < \left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| + \frac{\log x}{\log 2} x^{\frac{1}{p_{n+1}}}. \qquad (11)$$

Lemma 1 gives

$$\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left| A_{p_{i_1}}(x) \cap \cdots \cap A_{p_{i_k}}(x) \right| . \qquad (12)$$

On the other hand, Lemma 3 gives

$$A_{p_{i_1}}(x) \cap \cdots \cap A_{p_{i_k}}(x) = A_{[p_{i_1}, \ldots, p_{i_k}]}(x) = A_{p_{i_1} \cdots p_{i_k}}(x).$$

Therefore (Lemma 2) we obtain

$$\left|A_{p_{i_1}}(x) \cap \cdots \cap A_{p_{i_k}}(x)\right| = \left|A_{p_{i_1}\cdots p_{i_k}}(x)\right| = \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor. \tag{13}$$

Substituting (13) into (12) we find that

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor \\ &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \\ &- \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} \left(x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} - \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor\right). \end{aligned} \tag{14}$$

Now

$$0 \le x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} - \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor < 1. \tag{15}$$

Consequently (see (1) and (2))

$$\begin{aligned}&\left|\sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} \left(x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} - \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor\right)\right| \le \sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_k\le n} 1 \\ &= \sum_{k=1}^{n}\binom{n}{k} \le \sum_{k=0}^{n}\binom{n}{k} = (1+1)^n = 2^n. \end{aligned} \tag{16}$$

That is,

$$\sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} \left(x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} - \left\lfloor x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \right\rfloor\right) = O(1). \tag{17}$$

Equations (14) and (17) give

$$\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\cdots<i_k\le n} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} + O(1) \tag{18}$$

If

$$B(x) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{1\le i_1<\cdots<i_k\le n \\ p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \tag{19}$$

and

$$C(x) = \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{1\le i_1<\cdots<i_k\le n \\ p_{i_1}\cdots p_{i_k}>p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} = o\left(x^{\frac{1}{p_{n+1}}}\right) \tag{20}$$

then

$$\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\leq i_1<\cdots<i_k\leq n} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} = B(x)+C(x). \tag{21}$$

Equations (18) and (21) give

$$\left|\bigcup_{i=1}^{n} A_{p_i}(x)\right| = B(x)+C(x)+O(1). \tag{22}$$

Equations (22) and (11) give

$$B(x)+C(x)+O(1)\leq N(x)<B(x)+C(x)+O(1)+\frac{\log x}{\log 2}x^{\frac{1}{p_{n+1}}}.$$

Therefore,

$$C(x)+O(1)\leq N(x)-B(x)<C(x)+O(1)+\frac{\log x}{\log 2}x^{\frac{1}{p_{n+1}}}. \tag{23}$$

Equations (23) and (20) give

$$-\epsilon<\frac{N(x)-B(x)}{\log x\ x^{\frac{1}{p_{n+1}}}}<\frac{1}{\log 2}+\epsilon \qquad (\epsilon>0).$$

That is,

$$N(x)=B(x)+O\left(\log x\ x^{\frac{1}{p_{n+1}}}\right). \tag{24}$$

Note that (see (19)) if $k=1,\ldots,n$ then,

$$\sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} = \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_{i_1}\cdots p_{i_k}<p_{n}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} + \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_n\leq p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}}. \tag{25}$$

Now, if $k=1,\ldots,n-1$ then

$$\sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_{i_1}\cdots p_{i_k}<p_{n}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} = \sum_{\substack{1\leq i_1<\cdots<i_k\leq n-1\\ p_{i_1}\cdots p_{i_k}<p_{n}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}}, \tag{26}$$

and if $k=2,\ldots,n$ then

$$\sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_n\leq p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} = \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_n<p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \tag{27}$$

On the other hand, if $k=1$ then

$$\sum_{\substack{1\leq i_1\leq n\\ p_n\leq p_{i_1}<p_{n+1}}} x^{\frac{1}{p_{i_1}}} = x^{\frac{1}{p_n}} \tag{28}$$

Equations (19), (25), (26), (27) and (28) give

$$\begin{aligned} B(x) &= \sum_{k=1}^{n-1}(-1)^{k+1} \sum_{\substack{1\leq i_1<\cdots<i_k\leq n-1\\ p_{i_1}\cdots p_{i_k}<p_n}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \\ &+ x^{\frac{1}{p_n}} + \sum_{k=2}^{n}(-1)^{k+1} \sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ p_n<p_{i_1}\cdots p_{i_k}<p_{n+1}}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \end{aligned}$$

That is,

$$\begin{aligned} B(x) &= \sum_{k=1}^{n-1}(-1)^{k+1} \sum_{\substack{1\leq i_1<\cdots<i_k\leq n-1\\ p_{i_1}\cdots p_{i_k}<p_n}} x^{\frac{1}{p_{i_1}\cdots p_{i_k}}} \\ &+ x^{\frac{1}{p_n}} + o\left(x^{\frac{1}{p_n}}\right) \end{aligned} \tag{29}$$

Finally, equations (29) and (24) give (3). □

4 Examples

If $n = 2$ then Theorem 5 becomes

$$N(x) = \sqrt{x} + g(x)\sqrt[3]{x}, \tag{30}$$

where $\lim_{x\to\infty} g(x) = 1$.

If $n = 3$ then Theorem 5 becomes

$$N(x) = \sqrt{x} + \sqrt[3]{x} + g(x)\sqrt[5]{x},$$

where $\lim_{x\to\infty} g(x) = 1$.

If $n = 4$ then Theorem 5 becomes

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + g(x)\sqrt[7]{x},$$

where $\lim_{x\to\infty} g(x) = 1$. Consequently

$$\sqrt{x} + \sqrt[3]{x} < N(x) < \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x}$$

If $n = 5$ then Theorem 5 becomes

$$N(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[5]{x} - \sqrt[6]{x} + \sqrt[7]{x} - \sqrt[10]{x} + g(x)\sqrt[11]{x},$$

where $\lim_{x\to\infty} g(x) = 1$.

To finish, we shall establish two simple theorems.

Theorem 6. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \dots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of these n open intervals that contain some perfect power. Then*

$$\lim_{n\to\infty} \frac{S(n)}{n} = 0.$$

Therefore, almost all the open intervals are empty.

Proof. We have (Nyblom's asymptotic formula)

$$N(x) = \sqrt{x} + f(x)\sqrt{x},$$

where $\lim_{x\to\infty} f(x) = 0$. Consequently

$$N(n^2) = n + f(n^2)n,$$

where n are the n squares $1^2, 2^2, \dots, n^2$. Therefore

$$0 \leq S(n) \leq N(n^2) - n = f(n^2)n.$$

That is

$$0 \leq \frac{S(n)}{n} \leq f(n^2).$$

□

Using equation (30) we can obtain a more strong result.

Theorem 7. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \dots, ((n-1)^2, n^2)$. Let $F(n)$ be the number of perfect powers in these n open intervals. Then $F(n) \sim n^{\frac{2}{3}}$.*

Proof. Equation (30) gives

$$N(n^2) = n + g(n^2)n^{\frac{2}{3}},$$

where n are the n squares $1^2, 2^2, \dots, n^2$. Therefore

$$F(n) = N(n^2) - n = g(n^2)n^{\frac{2}{3}} \sim n^{\frac{2}{3}}.$$

□

5 Acknowledgements

The author would like to thank the anonymous referee for his/her valuable comments and suggestions for improving the original version of this article. The author is also very grateful to Universidad Nacional de Luján.

References

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1960.

[2] W. J. LeVeque, *Topics in Number Theory*, Addison-Wesley, 1958.

[3] M. A. Nyblom, A counting function for the sequence of perfect powers, *Austral. Math. Soc. Gaz.* **33** (2006), 338–343.

[4] M. A. Nyblom, Counting the perfect powers, *Math. Spectrum* **41** (2008), 27–31.

2000 *Mathematics Subject Classification*: Primary 11A99; Secondary 11B99.
Keywords: Perfect powers, counting function, asymptotic formulae.

(Concerned with sequence A001597.)

Received May 28 2011; revised version received August 16 2011. Published in *Journal of Integer Sequences*, September 25 2011.

Return to Journal of Integer Sequences home page.