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# Ramanujan Primes: Bounds, Runs, Twins, and Gaps

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**Abstract**

The $n$th Ramanujan prime is the smallest positive integer $R_n$ such that if $x \geq R_n$, then the interval $\left(\frac{1}{2}x, x\right]$ contains at least $n$ primes. We sharpen Laishram's theorem that $R_n < p_{3n}$ by proving that the maximum of $R_n/p_{3n}$ is $R_5/p_{15} = 41/47$. We give statistics on the length of the longest run of Ramanujan primes among all primes $p < 10^n$, for $n \leq 9$. We prove that if an upper twin prime is Ramanujan, then so is the lower; a table gives the number of twin primes below $10^n$ of three types. Finally, we relate runs of Ramanujan primes to prime gaps. Along the way we state several conjectures and open problems. An appendix explains Noe's fast algorithm for computing $R_1, R_2, \ldots, R_n$.

# 1 Introduction

For $n \geq 1$, the $n$th *Ramanujan prime* is defined as the smallest positive integer $R_n$ with the property that for any $x \geq R_n$, there are at least $n$ primes $p$ with $\frac{1}{2}x < p \leq x$. By its minimality, $R_n$ is indeed a prime, and the interval $\left(\frac{1}{2}R_n, R_n\right]$ contains exactly $n$ primes [10].

In 1919 Ramanujan proved a result which implies that $R_n$ exists, and he gave the first five Ramanujan primes. (We formulate his result as a theorem and quote him.)

**Theorem 1** (Ramanujan). *"Let $\pi(x)$ denote the number of primes not exceeding $x$. Then $\ldots \pi(x) - \pi\left(\frac{1}{2}x\right) \geq 1, 2, 3, 4, 5, \ldots,$ if $x \geq 2, 11, 17, 29, 41, \ldots,$ respectively."*

*Proof.* This follows from properties of the Γ-function. See Ramanujan [4] for details, and Shapiro [8, Section 9.3B] for an exposition of Ramanujan's idea. □

The case $R_1 = 2$ is *Bertrand's Postulate*: for all $x \geq 2$, there exists a prime $p$ with $\frac{1}{2}x < p \leq x$. For $n = 1, 2, 3, \ldots$, the $n$th Ramanujan prime [9, Sequence A104272] is

$$R_n = 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, 101, 107, 127, 149, 151, 167, 179, 181, 227, 229, 233, \ldots.$$

In the present paper, we report progress on three predictions [10, Conjectures 1, 2, 3] about Ramanujan primes: on bounds, runs, and twins.

In the next section, we sharpen Laishram's theorem that $R_n < p_{3n}$, where $p_n$ denotes the $n$th prime. Namely, we prove the optimal bound that the maximum value of $R_n/p_{3n}$ is $R_5/p_{15} = 41/47$. The proof uses another result of Laishram and a computation of the first 169350 Ramanujan primes by Noe's fast algorithm. Our first new conjecture follows.

In Section 3, we present statistics on the length of the longest run of Ramanujan primes among all primes $p < 10^n$, for $n \leq 9$. We pose an open problem on the unexpectedly long runs of non-Ramanujan primes, and make a new conjecture about both types of runs.

In Section 4, we prove that if the larger of two twin primes is Ramanujan, then its smaller twin is also Ramanujan, and we provide a table of data on the number of twins below $10^n$, again for $n \leq 9$. We offer several new conjectures and open problems on twin primes.

In Section 5, we associate runs of odd Ramanujan primes to certain prime gaps.

An appendix explains the algorithm for computing Ramanujan primes and includes a *Mathematica* program.

# 2 Bounds

Here are some estimates for the $n$th Ramanujan prime.

**Theorem 2** (Sondow). *The following inequalities hold:*

$$2n \log 2n < p_{2n} < R_n < 4n \log 4n < p_{4n} \qquad (n > 1). \tag{1}$$

*Moreover, for every $\epsilon > 0$, there exists $N_0(\epsilon) > 0$ such that*

$$R_n < (2 + \epsilon) n \log n \qquad (n \geq N_0(\epsilon)). \tag{2}$$

*In particular, $R_n \sim p_{2n}$ as $n \to \infty$.*

*Proof.* Inequalities of Rosser and Schoenfeld for $\pi(x)$, together with *Rosser's theorem* [6] that $p_n > n \log n$, lead to (1). The bound (2) follows from the Prime Number Theorem. For details, see Sondow [10]. □

A prediction [10, Conjecture 1] that (1) can be improved to $p_{2n} < R_n < p_{3n}$ has been proved by Laishram.

**Theorem 3** (Laishram)**.** *For all $n \geq 1$, we have $R_n < p_{3n}$.*

*Proof.* Dusart's inequalities [2] for Chebychev's function

$$\theta(x) := \sum_{\text{prime } p \leq x} \log p \leq \pi(x) \log x$$

lead to an explicit value of $N_0(\epsilon)$ in (2), for each $\epsilon > 0$. For details, see Laishram [3]. □

Using one of those values and a fast algorithm for computing Ramanujan primes (see the Appendix), we sharpen Theorem 3 by giving an optimal upper bound on $R_n/p_{3n}$, namely, its maximum. (Notice that the rational numbers $R_n/p_{3n}$ are all distinct, because the $p_{3n}$ are distinct primes and $0 < R_n/p_{3n} < 1$. Thus the maximum occurs at only one value of $n$.)

**Theorem 4.** *The maximum value of $R_n/p_{3n}$ is*

$$\max_{n\geq 1} \frac{R_n}{p_{3n}} = \frac{R_5}{p_{15}} = \frac{41}{47} = 0.8723\ldots .$$

*Proof.* Since $41/47 > 0.8666\ldots = 13/15$, it suffices to show $R_n/p_{3n} < 13/15$ for $n \neq 5$.

Set $\epsilon = 3/5$ and substitute $2 + \epsilon = 13/5$ into (2). Using Rosser's theorem with $3n$ in place of $n$, we can write the result as

$$R_n < \frac{13}{15} 3n \log n < \frac{13}{15} p_{3n} \qquad (n \geq N_0(3/5)).$$

According to Laishram [3, Theorem 1], if $0 < \epsilon \leq 1.08$, then $N_0(\epsilon) = (2/\epsilon)^{c/\epsilon}$ in (2), where $c = c(\epsilon) = 6$ at $\epsilon = 0.6$. Hence $N_0(3/5) = (10/3)^{10} = 169350.87\ldots$, and so

$$\frac{R_n}{p_{3n}} < \frac{13}{15} \qquad (n > 169350).$$

To complete the proof, we compute the first 169350 Ramanujan primes and then check that $R_n/p_{3n} < 13/15$ when $5 \neq n \leq 169350$. □

Similarly, one can show that

$$\max_{n\neq 5} \frac{R_n}{p_{3n}} = \frac{R_{10}}{p_{30}} = \frac{97}{113} = 0.8584\ldots ,$$
$$\max_{n\neq 5 \text{ or } 10} \frac{R_n}{p_{3n}} = \frac{R_2}{p_6} = \frac{11}{13} = 0.8461\ldots ,$$

and so on down towards

$$\lim_{n\to\infty} \frac{R_n}{p_{3n}} = \frac{2}{3} = 0.666\ldots .$$

We conclude this section with a related prediction.

**Conjecture 5.** For $m = 1, 2, 3, \ldots$, let $N(m)$ be given by the following table.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | $7, 8, \ldots, 19$ | $20, 21, \ldots$ |
|---|---|---|---|---|---|---|---|---|
| $N(m)$ | 1 | 1245 | 189 | 189 | 85 | 85 | 10 | 2 |

Then we have

$$\pi(R_{mn}) \le m\pi(R_n) \qquad (n \ge N(m)).$$

Equivalently, if we define the function $\rho$ by $\rho(n) := \pi(R_n)$, so that $R_n = p_{\rho(n)}$, then

$$\rho(mn) \le m\rho(n) \qquad (n \ge N(m)).$$

In the cases $m = 2, 3, \ldots, 20$, the statement has been verified for all $n$ with $R_{mn} < 10^9$. The first few values of $\rho(n)$, for $n = 1, 2, 3, \ldots$, are [9, Sequence A179196]

$$\rho(n) = 1, 5, 7, 10, 13, 15, 17, 19, 20, 25, 26, 28, 31, 35, 36, 39, 41, 42, 49, 50, 51, 52, 53, \ldots .$$

Note that Theorems 2 and 3 imply $2n < \rho(n) < 3n$ for all $n > 1$, and $\rho(n) \sim 2n$ as $n \to \infty$. The latter yields $\rho(mn) \sim 2mn \sim m\rho(n)$ as $n \to \infty$, for any fixed $m \ge 1$.

# 3 Runs

Since $p_{2n} < R_n \sim p_{2n}$ as $n \to \infty$, the probability of a randomly chosen prime being Ramanujan is slightly less than 1/2, roughly speaking. More precisely, column 2 in Table 1 gives the probability $P_n$ (rounded to 3 decimal places) that a prime $p < 10^n$ is a Ramanujan prime, for $n = 1, 2, \ldots, 9$.

Let us consider a coin-tossing model. Suppose that a biased coin has probability $P$ of heads. According to Schilling [7], the expected length $\mathrm{E}L_N = \mathrm{E}L_N(P)$ of the longest run of heads in a sequence of $N$ coin tosses is approximately equal to

$$\mathrm{E}L_N \approx \frac{\log N}{\log(1/P)} - \left(\frac{1}{2} - \frac{\log(1-P)+\gamma}{\log(1/P)}\right),$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. The variance $\mathrm{Var}L_N = \mathrm{Var}L_N(P)$ is close [7] to

$$\mathrm{Var}L_N \approx \frac{\pi^2}{6\log(1/P)^2} + \frac{1}{12}$$

"and is quite remarkable for the property that it is essentially constant with respect to" $N$.

For example, with a fair coin,

$$\mathrm{E}L_N \approx \frac{\log N}{\log 2} - \left(\frac{3}{2} - \frac{\gamma}{\log 2}\right) = \frac{\log N}{\log 2} - 0.667\ldots \qquad \left(P = \frac{1}{2}\right) \tag{3}$$

and

$$\mathrm{Var}L_N \approx \frac{\pi^2}{6(\log 2)^2} + \frac{1}{12} = 3.507\ldots \qquad \left(P = \frac{1}{2}\right). \tag{4}$$

Schilling points out that by (4) "the standard deviation of the longest run is approximately $(\mathrm{Var} L_N)^{1/2} \approx 1.873$, an amazingly small value. This implies that the length of the longest run is quite predictable indeed; normally it is within about two of its expectation."

This is nearly true of the longest run of Ramanujan primes in the sequence of prime numbers below $10^n$ (where $P = P_n \lessapprox 1/2$), at least for $n \leq 9$. But for non-Ramanujan primes (where $P = 1 - P_n \gtrapprox 1/2$), the actual length of the longest run exceeds the expected length by much more than two, at least for $n = 6, 7, 8, 9$. (See Table 1, in which the two columns marked "Actual" are [9, Sequences A189993 and A189994].)

| $n$ | Probability $P_n$ of a prime $p < 10^n$ being Ramanujan | Length of the longest run below $10^n$ of Ramanujan primes | | non-Ramanujan primes | |
|---|---|---|---|---|---|
| | | Expected | Actual | Expected | Actual |
| 1 | .250 | 1 | 1 | 5 | 3 |
| 2 | .400 | 5 | 2 | 8 | 4 |
| 3 | .429 | 8 | 5 | 11 | 7 |
| 4 | .455 | 11 | 13 | 14 | 13 |
| 5 | .465 | 14 | 13 | 18 | 20 |
| 6 | .471 | 18 | 20 | 21 | 36 |
| 7 | .476 | 21 | 21 | 24 | 47 |
| 8 | .479 | 24 | 26 | 28 | 47 |
| 9 | .482 | 28 | 31 | 31 | 65 |

Table 1: Length of the longest run of (non-)Ramanujan primes below $10^n$.

**Problem 6.** Explain the unexpectedly long runs of non-Ramanujan primes among primes $p < 10^n$, for $n \geq 6$.

Formula (3) suggests the following predictions supported by Table 1. They strengthen an earlier prediction [10, Conjecture 2] that arbitrarily long runs of both types exist.

**Conjecture 7.** We have

$$\limsup_{N\to\infty} \frac{\text{length of the longest run of Ramanujan primes among primes} \leq p_N}{\log N/\log 2} \geq 1$$

and the same holds true if "Ramanujan" is replaced with "non-Ramanujan".

For $n = 1, 2, \ldots$, the first run of $n$ Ramanujan primes begins at

$$2, 67, 227, 227, 227, 2657, 2657, 2657, 2657, 2657, 2657, 2657, 2657, 562871, 793487, \ldots,$$

and the first run of $n$ non-Ramanujan primes at

$$3, 3, 3, 73, 191, 191, 509, 2539, 2539, 5279, 9901, 9901, 9901, 11593, 11593, 55343, 55343, \ldots,$$

respectively [9, Sequences A174602 and A174641].

## 4 Twins

If $p_n+2 = p_{n+1}$, then $p_n$ and $p_{n+1}$ are *twin primes*; the smallest are 3 and 5. If $R_n+2 = R_{n+1}$, then $R_n$ and $R_{n+1}$ are *twin Ramanujan primes*; the smallest are 149 and 151.

Given primes $p$ and $q > p$, a necessary condition for them to be twin Ramanujan primes is evidently that

$$\pi(p) - \pi\left(\frac{1}{2}p\right) + 1 = \pi(q) - \pi\left(\frac{1}{2}q\right). \tag{5}$$

To see that the condition is not sufficient, even when $p$ and $q$ are *consecutive primes* $p_k$ and $p_{k+1}$, verify (5) for any one of the pairs

$$(p,q) = (p_k, p_{k+1}) = (\mathbf{11}, 13),\ (\mathbf{47}, 53),\ (\mathbf{67}, \mathbf{71}),\ (109, 113),\ (137, 139), \tag{6}$$

where Ramanujan primes are in **bold**.

It is less evident that (5) is a necessary condition for $p$ and $q$ even to be (ordinary) twin primes, but that is not hard to prove [10, Proposition 1].

**Proposition 8.** *If $p$ and $q = p+2$ are twin primes with $p > 5$, then (5) holds.*

The converse is false, even when $p$ and $q$ are consecutive primes both of which are Ramanujan, as the example $(p_{19}, p_{20}) = (\mathbf{67}, \mathbf{71}) = (R_8, R_9)$ shows.

As mentioned, each pair in (6) consists of consecutive primes $p < q$ satisfying (5). However, in no pair is $q$ a Ramanujan prime but not $p$; in fact, such a pair cannot exist.

**Proposition 9.** (i). *If the larger of two twin primes is Ramanujan, then the smaller is also Ramanujan: they are twin Ramanujan primes.*
(ii). *More generally, given consecutive primes $(p,q) = (p_k, p_{k+1})$ satisfying (5), if $q = R_{n+1}$, then $p = R_n$.*

*Proof.* Part (i) is (vacuously) true for twin primes $p$ and $q = p+2$ with $p \le 5$. For $p > 5$ it suffices, by Proposition 8, to prove part (ii).

Since $q = R_{n+1}$, we have $\pi(x) - \pi\left(\frac{1}{2}x\right) \ge n+1$ when $x \ge q$, and (5) implies that $\pi(p) - \pi\left(\frac{1}{2}p\right) = n$. To prove that $p = R_n$, we have to show that $\pi(p-1) - \pi\left(\frac{1}{2}(p-1)\right) < n$, and that $\pi(x) - \pi(\frac{1}{2}x) \ge n$ for $x = p+1, p+2, \dots, q-1$.

If $\ell$ is any prime, then $\pi(\ell-1)+1 = \pi(\ell)$ and $\pi\left(\frac{1}{2}(\ell-1)\right) = \pi\left(\frac{1}{2}\ell\right)$, so that the quantity $\pi(y) - \pi\left(\frac{1}{2}y\right)$ increases by 1 from $y = \ell-1$ to $y = \ell$. Taking $\ell = p$ or $\ell = q$, we infer that $\pi(\ell-1) - \pi\left(\frac{1}{2}(\ell-1)\right) = n-1$ or $n$, respectively. As $p$ and $q$ are consecutive primes, it follows that $\pi(x) = \pi(q-1)$ and $\pi\left(\frac{1}{2}x\right) \le \pi\left(\frac{1}{2}(q-1)\right)$, for $x = p+1, p+2, \dots, q-1$, implying $\pi(x) - \pi(\frac{1}{2}x) \ge n$. This proves the required inequalities. □

Part (i) was conjectured by Noe [9, Sequence A173081].

**Corollary 10.** *If we denote*

$$\pi_{2,1}(x) := \#\{\textit{pairs of twin primes} \le x : \textit{one or both are Ramanujan}\},$$
$$\pi_{2,2}(x) := \#\{\textit{pairs of twin primes} \le x : \textit{both are Ramanujan}\},$$

*then for all $x \geq 0$ we have the equalities*

$$\pi_{2,1}(x) = \#\{\textit{pairs of twin primes} \leq x \,:\, \textit{the smaller is Ramanujan}\},$$
$$\pi_{2,2}(x) = \#\{\textit{pairs of twin primes} \leq x \,:\, \textit{the larger is Ramanujan}\}.$$

*Proof.* By Proposition 9 part (i), given twin primes $p$ and $p+2$, if $p+2 = R_{n+1}$, then $p = R_n$. The corollary follows. □

Table 2 gives some figures (see [9, Sequences A007508, A173081, A181678]) on

$$\pi_2(x) := \#\{\text{pairs of twin primes} \leq x\},$$

$\pi_{2,1}(x), \pi_{2,2}(x)$, and their ratios. Proposition 8 and Corollary 10 will help to explain why many values of the ratios are greater than might be expected a priori.

| | $\pi_* = \pi_*(10^n)$ | | | | | |
|---|---|---|---|---|---|---|
| $n$ | $\pi_2$ | $\pi_{2,1}$ | $\pi_{2,2}$ | $\pi_{2,1}/\pi_2$ | $\pi_{2,2}/\pi_2$ | $\pi_{2,2}/\pi_{2,1}$ |
| 1 | 2 | 0 | 0 | 0 | 0 | – |
| 2 | 8 | 6 | 0 | .750 | 0 | 0 |
| 3 | 35 | 28 | 10 | .800 | .286 | .357 |
| 4 | 205 | 167 | 73 | .815 | .356 | .437 |
| 5 | 1224 | 694 | 508 | .788 | .415 | .527 |
| 6 | 8169 | 6305 | 3468 | .772 | .425 | .550 |
| 7 | 58980 | 45082 | 25629 | .764 | .434 | .568 |
| 8 | 440312 | 335919 | 194614 | .763 | .442 | .579 |
| 9 | 3424506 | 2605867 | 1537504 | .761 | .449 | .590 |

Table 2: Counting three types of pairs of twin primes below $10^n$.

The probability that two randomly chosen primes $p$ and $q$ are both Ramanujan is slightly less than $1/2 \times 1/2 = 1/4$, roughly speaking. The probability increases if $p$ and $q$ are twin primes, because then Proposition 8 guarantees that the necessary condition (5) holds.

For that reason, and based on the first 1000 Ramanujan primes, it was predicted [10, Conjecture 3] that more than 1/4 of the twin primes up to $x$ are twin Ramanujan primes, if $x \geq 571$. This is borne out for $x = 10^n$, with $3 \leq n \leq 9$, by Table 2. It shows that the prediction can be improved to $\pi_{2,2}(x)/\pi_2(x) > 2/5$, for $x \geq 10^5$.

Corollary 10 implies that whether a twin prime pair is counted in $\pi_{2,1}(x)$ or $\pi_{2,2}(x)$ depends on only one of the two primes being Ramanujan. This suggests that the ratios $\pi_{2,1}(x)/\pi_2(x)$ and $\pi_{2,2}(x)/\pi_2(x)$ should approach 1/2 as $x$ tends to infinity.

We conclude this section with these and other conjectures based on our results and on Table 2, as well as with two more open problems.

**Conjecture 11.** For all $x \geq 10^5$, we have

$$\frac{\pi_{2,1}(x)}{\pi_2(x)} < \frac{4}{5}, \qquad \frac{\pi_{2,2}(x)}{\pi_2(x)} > \frac{2}{5}, \qquad \frac{\pi_{2,2}(x)}{\pi_{2,1}(x)} > \frac{2/5}{4/5} = \frac{1}{2}.$$

**Conjecture 12.** If $\pi_2(x) \to \infty$ as $x \to \infty$, then $\pi_{2,1}(x) \sim \pi_{2,2}(x) \sim \frac{1}{2}\pi_2(x)$.

Recall Brun's famous theorem [1] that the series of reciprocals of the twin primes converges or is finite (unlike the series of reciprocals of all the primes, which Euler showed diverges). Its sum [9, Sequence A065421] is *Brun's constant* $B_2$,

$$B_2 := \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots \stackrel{?}{=} 1.9021605\ldots.$$

Here $\stackrel{?}{=}$ means that the value of $B_2$ is conditional "on heuristic considerations about the distribution of twin primes" (Ribenboim [5, p. 201]).

**Problem 13.** Compute the analogous constant $B_{2,1}$ for twin primes at least one of which is Ramanujan,

$$B_{2,1} := \left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\left(\frac{1}{29}+\frac{1}{31}\right)+\left(\frac{1}{41}+\frac{1}{43}\right)+\cdots.$$

The numbers $11, 17, 29, 41, \ldots$ [9, Sequence A178128] are the lesser of twin primes if at least one is Ramanujan. By Corollary 10, that is the same as the lesser of twin primes if it is Ramanujan.

**Problem 14.** Compute the analogous constant $B_{2,2}$ for twin Ramanujan primes,

$$B_{2,2} := \left(\frac{1}{149}+\frac{1}{151}\right)+\left(\frac{1}{179}+\frac{1}{181}\right)+\left(\frac{1}{227}+\frac{1}{229}\right)+\left(\frac{1}{239}+\frac{1}{241}\right)+\cdots.$$

The numbers $149, 179, 227, 239, \ldots$ [9, Sequence A178127] are the lesser of twin Ramanujan primes.

# 5 Prime gaps

Let us say that there is a *prime gap from $a$ to $b \geq a$* if none of the numbers $a, a+1, a+2, \ldots, b$ is prime. Given a run of $r$ odd Ramanujan primes starting at $p$, we can associate to it a prime gap of length at least $r$ starting at $\frac{1}{2}(p+1)$.

**Proposition 15.** (i). *If $p = R_n$ is odd, then the integer $\frac{1}{2}(p+1)$ is not prime.*
(ii). *More generally, given a run of $r \geq 1$ odd Ramanujan primes from $p = R_n = p_k$ to $q = R_{n+r-1} = p_{k+r-1}$, there is a prime gap from $\frac{1}{2}(p+1)$ to $\frac{1}{2}(q+1)$.*
(iii). *Parts* (i) *and* (ii) *are sharp in the sense that, for* certain *runs of Ramanujan primes $p$ to $q$, both $\frac{1}{2}(p+1)-1$ and $\frac{1}{2}(q+1)+1$ are prime numbers.*
(iv). *But in the case $r = 2$, if $p$ and $q$ are* twin *Ramanujan primes, then the prime gap from $\frac{1}{2}(p+1)$ to $\frac{1}{2}(q+1)$ always lies in a longer prime gap of length 5 or more.*

*Proof.* (i). Since $p = R_n$ is odd, $\pi(p) = \pi(p+1)$, and the quantity $\pi(x) - \pi\left(\frac{1}{2}x\right)$ does not decrease from $x = p$ to $x = p+1$. Hence $\pi\left(\frac{1}{2}p\right) \geq \pi\left(\frac{1}{2}(p+1)\right)$, and so $\frac{1}{2}(p+1)$ is not prime.
(ii). By (i), the case $r = 1$ holds. Taking $r = 2$, let $p = R_n = p_k$ and $q = R_{n+1} = p_{k+1}$ be odd. By (i), neither $\frac{1}{2}(p+1)$ nor $\frac{1}{2}(q+1)$ is prime. If an integer $i$ lies strictly between them, then the oddness of $p$ and $q$ implies $p + 1 < j := 2i - 1 < q - 1$. Since $p = p_k$ and $q = p_{k+1}$, we have $\pi(p) = k = \pi(j+1)$. As $p = R_n$ and $q = R_{n+1}$, it follows that $\pi(x) - \pi\left(\frac{1}{2}x\right)$ does not decrease from $x = p$ to $x = j+1$. Hence $\pi\left(\frac{1}{2}p\right) \geq \pi\left(\frac{1}{2}(j+1)\right) = \pi(i)$, and so $i$ is also not prime. This proves (ii) for runs of length 2.

The general case follows easily by induction on $r$. Namely, given a run of length $r > 2$ from $R_n = p_k$ to $R_{n+r-1} = p_{k+r-1}$, break it into a run of length 2 from $R_n = p_k$ to $R_{n+1} = p_{k+1}$, concatenated with a run of length $r-1$ from $R_{n+1} = p_{k+1}$ to $R_{n+r-1} = p_{k+r-1}$.
(iii). For $r = 1$, the composite number $\frac{1}{2}(R_2 + 1) = \frac{1}{2}(11+1) = 6$ lies between the primes 5 and 7. For an example with $r > 1$, take the run $(R_{293}, R_{294}) = (4919, 4931) = (p_{657}, p_{658})$ of length $r = 2$. It is associated to the prime gap from $\frac{1}{2}(R_{293}+1) = 2460$ to $\frac{1}{2}(R_{294}+1) = 2466$, which is bounded by the primes 2459 and 2467.
(iv). Since $p > 3$ and $q$ are twin primes, $(p, q) = (6k-1, 6k+1)$ for some $k$. If $k = 2i$ is even, then $\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right) = (6i, 6i+1)$ lies in the prime gap from $6i$ to $6i+4$.

Now assume that $k = 2i+1$ is odd. Then $\left(\frac{1}{2}(p+1), \frac{1}{2}(q+1)\right) = (6i+3, 6i+4)$ will lie in a prime gap from $6i+2$ to $6i+6$, unless $6i+5$ is prime. But if $6i+5 = \frac{1}{2}(q+3)$ were prime, then, since $q + 2 = 6k+3$ is not prime, $\pi(x) - \pi\left(\frac{1}{2}x\right)$ would decrease from $x = q$ to $x = q+3$, contradicting the fact that $q$ is a Ramanujan prime. This completes the proof. □

For part (iii), the first "sharp" example of a run of length $r = 1, 2, \ldots, 11$ begins at the Ramanujan prime

$$11, 4919, 1439, 7187, 37547, 210143, 3376943, 663563, 4429739, 17939627, 12034427,$$

respectively [9, Sequence A177804]. An example of part (iv) is the prime gap associated to the twin Ramanujan primes $R_{14} = 149$ and $R_{15} = 151$, which lies in the larger prime gap

$$74, \frac{1}{2}(R_{14}+1) = 75, \frac{1}{2}(R_{15}+1) = 76, 77, 78.$$

# 6 Appendix on the algorithm

To compute a range of Ramanujan primes $R_i$ for $1 \leq i \leq n$, we perform simple calculations in each interval $(k/2, k]$ for $k = 1, 2, \ldots, p_{3n} - 1$. To facilitate the calculation, we use a counter $s$ and a list $L$ with $n$ elements $L_i$. Initially, $s$ and all $L_i$ are set to zero. They are updated as each interval is processed.

After processing an interval, $s$ will be equal to the number of primes in that interval, and each $L_i$ will be equal either to the greatest index of the intervals so far processed that contain exactly $i$ primes, or to zero if no interval having exactly $i$ primes has yet been processed.

Having processed interval $k-1$, to find the number of primes in interval $k$ we perform two operations: add 1 to $s$ if $k$ is prime, and subtract 1 from $s$ if $k/2$ is prime. We then update the $s$th element of the list to $L_s = k$, because now $k$ is the largest index of all intervals processed that contain exactly $s$ primes.

After all intervals have been processed, the list $R$ of Ramanujan primes is obtained by adding 1 to each element of the list $L$.

These ideas are captured in the following *Mathematica* program for finding the first 169350 Ramanujan primes.

```
nn = 169350;
L = Table[0, {nn}];
s = 0;
Do[
   If[PrimeQ[k], s++];
   If[PrimeQ[k/2], s--];
   If[s < nn, L[[s+1]] = k],
   {k, Prime[3*nn]-1}];
R = L + 1
```

Although it is adequate for computing a modest number of them, to compute many more requires a speedup of several orders of magnitude. That can be achieved by using a lower-level programming language and generating prime numbers via a sieve. With this speedup we computed all Ramanujan primes below $10^9$ in less than three minutes on a 2.8 GHz Pentium 4 computer.

# 7 Acknowledgment

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