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An Irrationality Measure for Regular Paperfolding Numbers

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In memory of Alf van der Poorten

Abstract

Let $\mathbf{F}(z) = \sum_{n\geqslant 1} f_n z^n$ be the generating series of the regular paperfolding sequence. For a real number α the irrationality exponent $\mu(\alpha)$, of α, is defined as the supremum of the set of real numbers μ such that the inequality $|\alpha - p/q| < q^{-\mu}$ has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. In this paper, using a method introduced by Bugeaud, we prove that

$$\mu(\mathbf{F}(1/b)) \leqslant \frac{275331112987}{137522851840} = 2.002075359\cdots$$

for all integers $b \geqslant 2$. This improves upon the previous bound of $\mu(\mathbf{F}(1/b)) \leqslant 5$ given by Adamczewski and Rivoal.

1 A tale unfolded, and unfolded, and unfolded, and ...

We consider a piece of paper, and choose and edge. We fold the paper in half towards the chosen edge, then we fold it in half again towards the chosen edge, and repeat the process indefinitely. Unfolding the paper, we are left with a sequence of ridges and valleys. Assign to each ridge the value 0, and to each valley the value 1. Such a sequence of 0s and 1s is called a *paperfolding sequence*.

Now, suppose that the paper was horizontal to start out with and consider the two possible folds, say "up" (which we call an up–fold) and "down" (which we call a down–fold), as indicated in Figure 1.

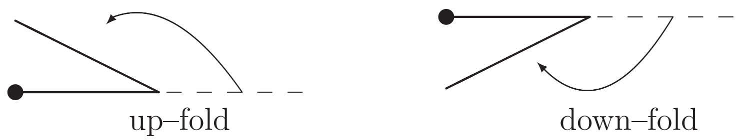

Figure 1: Orientations of up– and down–folds.

In general, there is no reason to prefer one fold choice over the other, but some choices are more traditional than others and have attracted more interest.

The most traditional choice made by paperfolders is to choose the up–fold only; the resulting sequence of 0s and 1s is called the *regular paperfolding sequence*, which we will denote by $\mathbf{f} := \{f_n\}_{n\geqslant 1}$. The sequence starts

$$\mathbf{f} = \{1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 1, 1, 0, 0, \ldots\}.$$

Denote by $\mathbf{F}(z)$ the generating series of the regular paperfolding sequence $\mathbf{f} = \{f_n\}_{n\geqslant 1}$; that is,

$$\mathbf{F}(z) := \sum_{n\geqslant 1} f_n z^n.$$

Note that $\mathbf{F}(z)$ converges in the region $|z| < 1$. Now let $b \geqslant 2$ be an integer. We refer to the numbers $\mathbf{F}(1/b)$ as *regular paperfolding numbers*.

It turns out that the regular paperfolding sequence can be produced by a finite automaton; in particular $\mathbf{f}$ is 2-automatic and is produced by the automaton in Figure 2. Numbers that can be produced by finite automata are very simple from the point of view of computational complexity, and as such the arithmetical properties of automatic numbers are of great interest; see [4] for more details and specific definitions regarding automatic sequences and [1, 2] for Diophantine properties related to automatic numbers.

Let ξ be a real number. The *irrationality exponent* $\mu(\xi)$, of the real number ξ, is defined as the supremum of set of real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{\mu}}$$

has infinitely many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

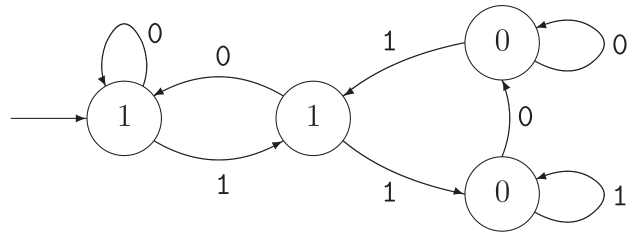

Figure 2: The generating 2–automaton of the regular paperfolding sequence.

In 2009, Adamczewski and Rivoal [3] showed that $\mu(\mathbf{F}(1/b)) \leqslant 5$ for each integer $b \geqslant 2$. The main result of this paper is to improve upon their result. In particular, we prove that

Theorem 1. *For all integers $b \geqslant 2$ we have*

$$\mu\left(\mathbf{F}\left(1/b\right)\right) \leqslant \frac{275331112987}{137522851840} = 2.002075359\cdots.$$

Our proof of Theorem 1 will mostly follow the method outlined by Bugeaud in [6], but with the use of a computation. The steps are as follows: get a useful rational function approximation to $\mathbf{F}(z)$, exploit a functional equation satisfied by $\mathbf{F}(z)$ to make this approximation even more useful, and then turn these functional rational approximations into true rational approximations to the number $\mathbf{F}(1/b)$. This method can be thought of as using ideas of Mahler's method in transcendence theory piggy–backed onto Padé's approximation theory. Combining this with a computation, we are able to yield the desired result.

2 Useful observations

In this section, we highlight some useful facts about the regular paperfolding sequence; see [7, 8, 9, 11, 12, 13] for details and further information on paperfolding sequences.

The regular paperfolding sequence $\mathbf{f} = \{f_n\}_{n\geqslant 1}$ satisfies for all $n \geqslant 0$ the recurrences

$$f_{4n+1} = 1, \qquad f_{4n+3} = 0, \qquad \text{and} \qquad f_{2(n+1)} = f_{(n+1)}.$$

Viewing the regular paperfolding sequence in the light of these recurrences proves extremely useful for our purposes as they are equivalent to the generating series $\mathbf{F}(z)$ satisfying the Mahler–type functional equation

$$\mathbf{F}(z^2) = \mathbf{F}(z) - \frac{z}{1-z^4}. \tag{1}$$

By applying (1) multiple times, for all $m \geqslant 1$ we have that

$$\mathbf{F}(z^{2^m}) = \mathbf{F}(z) - \sum_{j=0}^{m-1} \frac{z^{2^j}}{1-z^{2^{j+2}}}. \tag{2}$$

3 Hankel determinants and rational approximations

To apply Bugeaud's method, we require several definitions and lemmas.

Let $\mathbf{a} = \{a_n\}_{n\geqslant 1}$. The n–th *Hankel matrix* $H_n^1(\mathbf{a})$ of the sequence $\mathbf{a}$ is the $n \times n$ matrix

$$H_n^1(\mathbf{a}) := (a_{i+j-1})_{1\leqslant i,j\leqslant n}.$$

The n–th *Hankel determinant* is denoted $\det H_n^1(\mathbf{a})$.

Given an analytic function $\mathbf{G}(z)$, the rational function $R(x)$, whose numerator has degree bounded by m and denominator has degree bounded by n, is the $[m/n]_{\mathbf{G}}$ *Padé approximant* to $\mathbf{G}(z)$ provided

$$\mathbf{G}(z) - R(z) = O(z^{m+n+1}).$$

Connecting Hankel determinants to Padé approximants, we will use the following lemma; see [5, Page 35] (as stated this is not verbatim the result cited; it is the version of the result for sequences starting at $n = 1$ as opposed to sequences starting at $n = 0$).

Lemma 2 (Brezinski [5])**.** *Let* $\mathbf{c} = \{c_n\}_{n\geqslant 1}$ *and* $\mathbf{C}(z) = \sum_{n\geqslant 1} c_n z^n \in \mathbb{Z}[[z]]$ *with* $|z| < 1$. *If* $\det H_k^1(\mathbf{c}) \neq 0$, *then the* $[k/k]_{\mathbf{C}}$ *Padé approximant exists and satisfies*

$$\mathbf{C}(z) - [k/k]_{\mathbf{C}} = \frac{\det H_{k+1}^1(\mathbf{c})}{\det H_k^1(\mathbf{c})} z^{2k+1} + O(z^{2k+2}).$$

We require the following two lemmas, the first is a result of Adamczewski and Rivoal [3, Lemma 4.1] and the second is a slight, though extremely necessary, modification of Lemma 2 of [6] (we have included its proof for completeness).

Lemma 3 (Adamczewski and Rivoal [3])**.** *Let* ξ, δ, ρ *and* ϑ *be real numbers such that* $0 < \delta \leqslant \rho$ *and* $\vartheta \geqslant 1$. *Let us assume that there exists a sequence* $\{p_n/q_n\}_{n\geqslant 1}$ *of rational numbers and some positive constants* c_0, c_1 *and* c_2 *such that both*

$$q_n < q_{n+1} \leqslant c_0 q_n^{\vartheta},$$

and

$$\frac{c_1}{q_n^{1+\rho}} \leqslant \left|\xi - \frac{p_n}{q_n}\right| \leqslant \frac{c_2}{q_n^{1+\delta}}.$$

Then we have that

$$\mu(\xi) \leqslant (1+\rho)\frac{\vartheta}{\delta}.$$

Lemma 4. *Let* $K \geqslant 1$ *and* n_0 *be positive integers. Let* $\{a_j\}_{j\geqslant 1}$ *be the increasing sequence of integers composed of all the numbers of the form* $k2^n$, *where* $n \geqslant n_0$ *and* k *ranges over all the odd integers in* $[2^{K-1}+1, 2^K+1]$. *Then*

$$a_{j+1} \leqslant \left(\frac{2^{K-1}+3}{2^{K-1}+1}\right) a_j.$$

Proof. Let n be large enough and consider the increasing sequence $\{a_j\}_{j\geqslant 1}$ of all integers of the form $k2^n$ where k is an odd number in $[2^{K-1}+1, 2^K+1]$. Note that for a given j we have that for some m and some odd number a with $1 \leqslant a \leqslant 2^{K-1}+1$ we have $a_j = 2^m(2^{K-1}+a)$. We consider two cases.

If $a < 2^{K-1}+1$, then $a_{j+1} \leqslant 2^m(2^{K-1}+a+2)$, so that

$$\frac{a_{j+1}}{a_j} \leqslant \frac{2^{K-1}+a+2}{2^{K-1}+a} \leqslant \frac{2^{K-1}+3}{2^{K-1}+1}.$$

If $a = 2^{K-1}+1$, then $a_{j+1} \leqslant 2^{m+1}(2^{K-1}+1) = 2^m(2^K+2)$, so that

$$\frac{a_{j+1}}{a_j} \leqslant \frac{2^K+2}{2^K+1} \leqslant \frac{2^{K-1}+3}{2^{K-1}+1}.$$

This proves the lemma. □

In order to use Lemma 2, we give the following computational result.

Lemma 5. *For all* $k \leqslant 2^{13}+3$ *we have that* $\det H_k^1(\mathbf{f}) \neq 0$.

Lemma 5 is provided by a computation; see Appendix A for the C++ program written for this lemma. The computation was parallelized in a trivial manner: node $i \in [1, P]$ calculates the determinants of $H^1_{P\cdot j+i}(\mathbf{f})$ for $j = 1, \ldots, N$ and N as large as possible. The implementation uses the GNU Scientific Library [10], particularly those commands that do LU-decomposition and determinants on matrices of massive size. (Doing an LU-decomposition—in place—*before* the determinant calculation improves the calculation.)

This result allows us apply Lemma 4 with $K = 13$. One would like to show that all such determinants are nonzero, but this seems a daunting task at the moment, though in the future one may be more determined; see the section on Concluding Remarks for further discussion on this topic.

4 Proof of the main result

Proof of Theorem 1. Throughout this proof we will assume that $|z| < 1$ so that our big-O terms have meaning.

By Lemma 2 and Lemma 5, for all $k \leqslant 2^{13}+3$ there exist polynomials $P_{k,0}(z), Q_{k,0}(z) \in \mathbb{Z}[z]$, with

$$\deg P_{k,0}(z) \leqslant k \qquad \text{and} \qquad \deg Q_{k,0}(z) \leqslant k,$$

and a nonzero $h_k \in \mathbb{Q}$ such that

$$\mathbf{F}(z) - \frac{P_{k,0}(z)}{Q_{k,0}(z)} = h_k z^{2k+1} + O(z^{2k+2}).$$

Thus, sending $z \mapsto z^{2^m}$ we have that

$$\mathbf{F}(z^{2^m}) - \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})} = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}),$$

and using the functional equation (2) for $\mathbf{F}(z)$ it follows that

$$\mathbf{F}(z) - \sum_{j=0}^{m-1} \frac{z^{2^j}}{1 - z^{2^{j+2}}} - \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})} = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}).$$

Define $P_{k,m}(z)$ and $Q_{k,m}(z)$ by

$$\frac{P_{k,m}(z)}{Q_{k,m}(z)} := \sum_{j=0}^{m-1} \frac{z^{2^j}}{1 - z^{2^{j+2}}} + \frac{P_{k,0}(z^{2^m})}{Q_{k,0}(z^{2^m})},$$

so that

$$\mathbf{F}(z) - \frac{P_{k,m}(z)}{Q_{k,m}(z)} = h_k z^{2^m(2k+1)} + O(z^{2^m(2k+2)}).$$

Let $b \geqslant 2$ be an integer and $z = 1/b$; for $\varepsilon > 0$, we have for large enough m, say $m \geqslant m_0(k)$, that

$$(1-\varepsilon) h_k b^{-2^m(2k+1)} \leqslant \left| \mathbf{F}(1/b) - \frac{P_{k,m}(1/b)}{Q_{k,m}(1/b)} \right| \leqslant (1+\varepsilon) h_k b^{-2^m(2k+1)}.$$

It remains to determine the degree of $P_{k,m}(z)$ and $Q_{k,m}(z)$. To this end, note that writing

$$\frac{P_m(z)}{Q_m(z)} = \sum_{j=0}^{m-1} \frac{z^{2^j}}{1 - z^{2^{j+2}}},$$

we have that

$$\deg Q_m(z) \leqslant \deg(1 - z^{2^{m+1}}) = 2^{m+1}$$

and that

$$\begin{aligned} \deg P_m(z) &\leqslant \max_{0 \leqslant j \leqslant m-1} \left\{ \frac{z^{2^j}(1 - z^{2^{m+1}})}{1 - z^{2^{j+2}}} \right\} \\ &= \max_{0 \leqslant j \leqslant m-1} \{2^{m+1} - 2^{j+2} - 2^j\} \\ &= 2^{m+1} - 5 \\ &\leqslant 2^{m+1}. \end{aligned}$$

Using the definitions of $P_m(z)$ and $Q_m(z)$,

$$\deg Q_{k,m}(z) = \deg Q_m(z) Q_{k,0}(z^{2^m}) = \deg Q_m(z) + \deg Q_{k,0}(z^{2^m}) < 2^m(k+2),$$

and

$$\deg P_{k,m}(z) = \max\{\deg P_m(z) Q_{k,0}(z^{2^m}), P_{k,0}(z^{2^m})\} < 2^m(k+2).$$

We continue following Bugeaud [6] by defining the integers

$$p_{k,m} := b^{2^m(k+2)} P_{k,m}(1/b)$$

and

$$q_{k,m} := b^{2^m(k+2)} Q_{k,m}(1/b).$$

Since h_k is nonzero there exist positive real constants $c_i(k)$ ($i = 1, \ldots, 4$) depending only on k so that

$$c_1(k) b^{2^m(k+2)} \leqslant q_{k,m} \leqslant c_2(k) b^{2^m(k+2)}, \tag{3}$$

and

$$\frac{c_3(k)}{b^{2^{m+1}(k+1/2)}} \leqslant \left| \mathbf{F}(1/b) - \frac{p_{k,m}}{q_{k,m}} \right| \leqslant \frac{c_4(k)}{b^{2^{m+1}(k+1/2)}}. \tag{4}$$

Note that

$$2^m(k+2) = 2^m(k+1/2)\left(\frac{k+2}{k+1/2}\right),$$

so that by (3) there are positive constants $c_5(k)$ and $c_6(k)$ such that

$$\frac{c_5(k)}{q_{k,m}^{2 \cdot \frac{k+1/2}{k+2}}} \leqslant \frac{1}{b^{2^{m+1}(k+1/2)}} \leqslant \frac{c_6(k)}{q_{k,m}^{2 \cdot \frac{k+1/2}{k+2}}}.$$

Applying this to (4) yields

$$\frac{c_3(k)c_5(k)}{q_{k,m}^{1+\frac{k-1}{k+2}}} \leqslant \left| \mathbf{F}(1/b) - \frac{p_{k,m}}{q_{k,m}} \right| \leqslant \frac{c_4(k)c_6(k)}{q_{k,m}^{1+\frac{k-1}{k+2}}}. \tag{5}$$

Let $K \geqslant 3$ be an integer and denote by $m_1(k)$ an integer such that the sequence $\{q_{k-2,m}\}_{m \geqslant m_1(k)}$ is increasing. We define the sequence of positive integers $\{Q_{K,n}\}_{n\geqslant 1}$ as the sequence of all the integers $q_{k-2,m}$ with k odd, $2^{K-1}-1 \leqslant k \leqslant 2^K - 1$, $m \geqslant m_1(k)$, put in increasing order. Thus by (3), (5), and Lemma 4, there exist positive constants $C_1(K)$, $C_2(K)$ and $C_3(K)$ such that both

$$Q_{K,n} < Q_{K,n+1} \leqslant C_1(K) \cdot Q_{K,n}^{\frac{2^{K-1}+3}{2^{K-1}+1}}$$

and

$$\frac{C_2(K)}{Q_{K,n}^{1+\frac{2^K}{2^K+3}}} \leqslant \left| \mathbf{F}(1/b) - \frac{P_{K,n}}{Q_{K,n}} \right| \leqslant \frac{C_3(K)}{Q_{K,n}^{1+\frac{2^{K-1}}{2^{K-1}+3}}},$$

where we have chosen the $P_{K,n}$ as the $p_{k,m}$ that corresponds with the identity $Q_{K,n} = q_{k,m}$.

Applying Lemma 3 with $\rho = \frac{2^K}{2^K+3}$, $\delta = \frac{2^{K-1}}{2^{K-1}+3}$ and $\vartheta(K) = \frac{2^{K-1}+3}{2^{K-1}+1}$, we have that

$$\mu(\mathbf{F}(1/b)) \leqslant \frac{(2^{K+1}+3)(2^{K-1}+3)^2}{(2^{K-1}+1)(2^K+3)2^{K-1}}. \tag{6}$$

Since we may take K up to 13, we have that

$$\mu(\mathbf{F}(1/b)) \leqslant \frac{275331112987}{137522851840} = 2.002075359\cdots$$

for all $b \geqslant 2$. This proves the theorem. $\square$

5 Concluding remarks

Note that using (6), the larger the available K the better your bound. Thus (6) provides the following immediate corollary.

Corollary 6. *If* $\det H_k^1(\mathbf{f}) \neq 0$ *for all* k*, then* $\mu(\mathbf{F}(1/b)) = 2$ *for all integers* $b \geqslant 2$*.*

While the proof of the assumption in the previous corollary is not in hand, computational evidence lends some credence. Indeed, using some simple computations we are led to believe that the sequence $\{\det H_n^1(\mathbf{f})\}_{n\geqslant 1}$ satisfies some nice properties. One such property that could be conjectured is that if one writes $|H_n^1(\mathbf{f})|$ for the determinant modulo 2, we have

$$\sum_{n\geqslant 1} |H_n^1(\mathbf{f})| z^n = \frac{z + z^2 + z^5 + z^8 + z^9 + z^{10}}{1 - z^{10}};$$

that is, the sequence $\{|H_n^1(\mathbf{f})\}_{n\geqslant 1}$ is periodic with period 10 and 6 out of every 10 values is odd.

6 Acknowledgements

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A C++ program used for Lemma 5

```
#include <iomanip>
#include <fstream>
#include <iostream>
#include <sys/time.h>
#include <gsl/gsl_matrix_float.h>
#include <gsl/gsl_permutation.h>
#include <gsl/gsl_linalg.h>

using namespace std;

int as (int n) {
    //generate paper folding sequence
    if (n % 4 == 1) {
        return 1;
    } else if (n % 4 == 3){
        return 0;
    } else {
        return as(n/2);
    }
}
```

```
int main(int argc, char *argv[]) {
    int n;

    ofstream outf(argv[1]);

    for (n=8555; n>0; n+=2) {
        //create an n x n matrix
        gsl_matrix *gslMatrix = gsl_matrix_alloc(n, n);

        //fill matrix
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < n; j++) {
                gsl_matrix_set(gslMatrix, i, j, as(i+1+j) );
            }
        }

        int sign = 0;
        gsl_permutation *perm = gsl_permutation_alloc(n);
        gsl_linalg_LU_decomp(gslMatrix, perm, &sign);

        int res;

        //test for zero-determinant
        if (gsl_linalg_LU_det(gslMatrix, sign)==0) {
            res = 1;
        } else {
            res = 0;
        };

        gsl_matrix_free(gslMatrix);
        gsl_permutation_free(perm);
    }

    return 0;
}
```

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