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Integer Sequences Realized by the Subgroup Pattern of the Symmetric Group

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Abstract

The subgroup pattern of a finite group G is the table of marks of G together with a list of representatives of the conjugacy classes of subgroups of G. In this article we describe a collection of sequences realized by the subgroup pattern of the symmetric group.

1 Introduction

The table of marks of a finite group G was introduced by Burnside [2]. It is a matrix whose rows and columns are indexed by a list of representatives of the conjugacy classes of subgroups of G, where, for two subgroups $H, K \leq G$ the (H, K) entry in the table of marks of G is the number of fixed points of K in the transitive action of G on the cosets of H, $(\beta_{G/H}(K))$. If $H_1, \dots, H_r$ is a list of representatives of the conjugacy classes of subgroups of G, the table of marks is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1,\dots,r}.$$

In much the same fashion as the character table of G classifies matrix representations of G up to isomorphism, the table of marks of G classifies permutation representations of G up to equivalence. It also encodes a wealth of information about the subgroup lattice of G in a compact way. The `GAP` [4] library of tables of marks `Tomlib` [11] provides ready access to the tables of marks and conjugacy classes of subgroups of some 400 groups. These tables have been produced using the methods described in [8] and [7]. The data exhibited in later sections has been computed using this library. The purpose of this article is to illustrate how interesting integer sequences related to the subgroup structure of the symmetric group S_n, and the alternating group A_n, can be computed from this data. This paper is organized as follows. In Section 2 we study the conjugacy classes of subgroups of S_n for $n \leq 13$. In Section 3 we examine the tables of marks of S_n for $n \leq 13$ and describe how much more information regarding the subgroup structure of S_n can be obtained. In Section 4 we discuss the Euler Transform and its applications in counting subgroups of S_n.

2 Counting subgroups

Given a list of representatives $\{H_1, \ldots, H_r\}$ of $\mathrm{Sub}(G)/G$, the conjugacy classes of subgroups of G, we can enumerate those subgroups which satisfy particular properties. The numbers of conjugacy classes of subgroups of S_n and A_n are sequences A000638 and A029726 respectively in Sloane's encyclopedia [9]. The `GAP` table of marks library `Tomlib` provides access to the conjugacy classes of subgroups of the symmetric and alternating groups for $n \leq 13$. Table 1 records the number of conjugacy classes of subgroups of S_n which are abelian, cyclic, nilpotent, solvable and supersolvable (SupSol). A similar table for the conjugacy classes of subgroups of the alternating groups can be found in Appendix A.

	A000638	A218909	A000041	A218910	A218911	A218912
n	$\lvert\mathrm{Sub}(S_n)/S_n\rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	4	3	3	3	4	4
4	11	7	5	8	11	9
5	19	9	7	10	17	15
6	56	20	11	25	50	38
7	96	26	15	32	84	65
8	296	61	22	127	268	187
9	554	82	30	156	485	341
10	1593	180	42	531	1418	923
11	3094	236	56	648	2691	1789
12	10723	594	77	3727	9725	6118
13	20832	762	101	4221	18286	11616

Table 1: Sequences in S_n

2.1 Subgroup Orders

A question of historical interest concerns the orders of subgroups of S_n. In [3] Cameron writes: *The Grand Prix question of the Academie des Sciences, Paris, in 1860 asked "How many distinct values can a function of n variables take?" In other words what are the possible indices of subgroups of S_n.* For $n \leq 13$, Table 2 records the numbers of different orders $\mathcal{O}(S_n), \mathcal{O}(A_n)$ of subgroups of S_n and A_n. One might as well also enumerate the number of "missing" subgroup orders, that is, the number, $d(S_n)$, of divisors d such that $d \mid |S_n|$ but S_n has no subgroup of order d. Table 3 records the number of missing subgroup orders of S_n and A_n for $n \leq 13$.

	A218913	A218914
n	$\mathcal{O}(S_n)$	$\mathcal{O}(A_n)$
1	1	1
2	2	1
3	4	2
4	8	5
5	13	9
6	21	15
7	31	22
8	49	38
9	74	59
10	113	89
11	139	115
12	216	180
13	268	226

Table 2: Subgroup Orders

	A218915	A218916
n	$d(S_n)$	$d(A_n)$
1	0	0
2	0	0
3	0	0
4	0	1
5	3	3
6	9	9
7	29	26
8	47	46
9	86	81
10	157	151
11	401	365
12	576	540
13	1316	1214

Table 3: Missing Subgroup Orders

3 Counting using the table of marks

If in addition to a list of conjugacy classes of subgroups of G, the table of marks of G is also available, or can be computed, one can say quite a lot about the structure of the lattice of subgroups of G. We begin this section by giving some basic information about tables of marks and then go on to describe how we can count incidences and edges in the lattice of subgroups.

3.1 About Tables of Marks

Let G be a finite group and let $\mathrm{Sub}(G)$ denote the set of subgroups of G. By $\mathrm{Sub}(G)/G$ we denote the set of conjugacy classes of subgroups of G. For $H, K \in \mathrm{Sub}(G)$ let

$$\beta_{G/H}(K) = \#\{Hg \in G/H : (Hg)k = Hg \text{ for all } k \in K\} = \#\{g \in G : K \leq H^g\}/|H|$$

denote the mark of K on H. This number depends only on the G-conjugacy classes of H and K. Note that $\beta_{G/H}(K) \neq 0 \Rightarrow |K| \leq |H|$. If $H_1, \ldots, H_r$ is a list of representatives of the conjugacy classes of subgroups of G, the table of marks of G is then the $(r \times r)$-matrix

$$M(G) = (\beta_{G/H_i}(H_j))_{i,j=1,\ldots,r}.$$

If the subgroups in the transversal are listed by increasing group order the table of marks is a lower triangular matrix. The table of marks $M(S_4)$ of the symmetric group S_4 is shown in Figure 1.

$S_4/1$	24										
$S_4/2$	12	4									
$S_4/2$	12	.	2								
$S_4/3$	8	.	.	2							
$S_4/2^2$	6	6	.	.	6						
$S_4/2^2$	6	2	2	.	.	2					
$S_4/4$	6	2	.	.	.	.	2				
S_4/S_3	4	.	2	1	.	.	.	1			
S_4/D_8	3	3	1	.	3	1	1	.	1		
S_4/A_4	2	2	.	2	2	.	.	.	.	2	
S_4/S_4	1	1	1	1	1	1	1	1	1	1	1
	1	2	2	3	2^2	2^2	4	S_3	D_8	A_4	S_4

Figure 1: Table of Marks $M(S_4)$

As a matrix, we can extract a variety of sequences from the table of marks, the most obvious of which is the sum of the entries. The sum of the entries of $M(S_n)$ for $n \leq 13$ is shown in Figure 4. We can also sum the entries on the diagonal to obtain the sequences in Figure 5.

	A218917	A218918
n	S_n	A_n
1	1	1
2	4	1
3	18	5
4	146	39
5	681	192
6	7518	1717
7	58633	13946
8	952826	243391
9	11168496	2693043
10	232255571	38343715
11	3476965896	545787051
12	108673489373	15787210045
13	1951392769558	268796141406

Table 4: Sum of $M(G)$

	A218919	A218920
n	S_n	A_n
1	1	1
2	3	1
3	10	4
4	47	19
5	165	73
6	950	412
7	5632	2660
8	43772	21449
9	376586	184541
10	3717663	1827841
11	40555909	20043736
12	484838080	240206213
13	6286289685	3119816216

Table 5: Sum of the Diagonal

We will now collect some elementary properties of tables of marks in Lemma 1.

Lemma 1. *Let $H, K \leq G$. Then the following hold:*

(i) The first entry of every row of $M(G)$ is the index of the corresponding subgroup,

$$\beta_{G/H}(1) = |G : H|.$$

(ii) The entry on the diagonal is,

$$\beta_{G/H}(H) = |N_G(H) : H|.$$

(iii) The length of the conjugacy class $[H]$ of H is given by,

$$|[H]| = |G : N_G(H)| = \frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}.$$

(iv) The number of conjugates of H which contain K is given by,

$$|\{H^a : a \in G, K \leq H^a\}| = \frac{\beta_{G/H}(K)}{\beta_{G/H}(H)}.$$

The following formula which follows trivially from Lemma 1 (iv) relates marks to incidences in the subgroup lattice of G.

$$\beta_{G/H}(K) = |N_G(H) : H| \cdot \#\{H^g : K \leq H^g, g \in G\}. \tag{1}$$

As a first application of Formula 1 we obtain the following lemma which enables us to count the total number of subgroups of G.

Lemma 2. *Given a list $\{H_1, \ldots, H_r\}$ of representatives of the conjugacy classes of subgroups of G, the total number of subgroups of G is*

$$|\mathrm{Sub}(G)| = \sum_{i=1}^{r} \frac{\beta_{G/H_i}(1)}{\beta_{G/H_i}(H_i)}.$$

Proof. It follows from Formula 1 that for any subgroup $H \leq G$, $\frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}$ is the length of the conjugacy class of H in G. □

Table 6 lists the total number of subgroups of S_n and A_n for $n \leq 13$.

	A029725	A005432
n	A_n	S_n
1	1	1
2	1	2
3	2	6
4	10	30
5	59	156
6	501	1455
7	3786	11300
8	48337	151221
9	508402	1694723
10	6469142	29594446
11	81711572	404126228
12	2019160542	10594925360
13	31945830446	175238308453

Table 6: Total Number of Subgroups of A_n and S_n

3.2 Counting Incidences

Another immediate consequence of Formula 1 is that by dividing each row of the table of marks of G by its diagonal entry $\beta_{G/H}(H)$ we obtain a matrix $\mathcal{C}(G)$ describing containments in the subgroup lattice of G, where the (H, K)-entry is

$$\mathcal{C}(H, K) = \#\{K^g : H \leq K^g, g \in G\}. \tag{2}$$

Figure 2 illustrates the containment matrix of the symmetric group S_4.

$$
\begin{array}{c|ccccccccccc}
1 & 1 & & & & & & & & & & \\
2 & 3 & 1 & & & & & & & & & \\
2 & 6 & . & 1 & & & & & & & & \\
3 & 4 & . & . & 1 & & & & & & & \\
2^2 & 1 & 1 & . & . & 1 & & & & & & \\
2^2 & 3 & 1 & 1 & . & . & 1 & & & & & \\
4 & 3 & 1 & . & . & . & . & 1 & & & & \\
S_3 & 4 & . & 2 & 1 & . & . & . & 1 & & & \\
D_8 & 3 & 3 & 1 & . & 3 & 1 & 1 & . & 1 & & \\
A_4 & 1 & 1 & . & 1 & 1 & . & . & . & . & 1 & \\
S_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 2 & 2 & 3 & 2^2 & 2^2 & 4 & S_3 & D_8 & A_4 & S_4
\end{array}
$$

Figure 2: Containment Matrix : $\mathcal{C}(S_4)$

The conjugacy classes of subgroups of G are partially ordered by $[H] \leq [K]$ if $H \leq K^g$ for some $g \in G$ i.e. if $\mathcal{C}(H,K) \neq 0$. Therefore we can easily obtain the incidence matrix, $\mathcal{I}(G)$, of the poset of conjugacy classes of subgroups of G by replacing each nonzero entry in $\mathcal{C}(G)$, (or $M(G)$) by an entry 1. Figure 3 shows the incidence matrix $\mathcal{I}(S_4)$ of the poset of conjugacy classes of subgroups of S_4.

$$
\begin{array}{c|ccccccccccc}
1 & 1 & & & & & & & & & & \\
2 & 1 & 1 & & & & & & & & & \\
2 & 1 & . & 1 & & & & & & & & \\
3 & 1 & . & . & 1 & & & & & & & \\
2^2 & 1 & 1 & . & . & 1 & & & & & & \\
2^2 & 1 & 1 & 1 & . & . & 1 & & & & & \\
4 & 1 & 1 & . & . & . & . & 1 & & & & \\
S_3 & 1 & . & 1 & 1 & . & . & . & 1 & & & \\
D_8 & 1 & 1 & 1 & . & 1 & 1 & 1 & . & 1 & & \\
A_4 & 1 & 1 & . & 1 & 1 & . & . & . & . & 1 & \\
S_4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
 & 1 & 2 & 2 & 3 & 2^2 & 2^2 & 4 & S_3 & D_8 & A_4 & S_4
\end{array}
$$

Figure 3: Incidence Matrix : $\mathcal{I}(S_4)$

For comparison with Figure 3 we illustrate the poset of conjugacy classes of subgroups of S_4 in Figure 4.

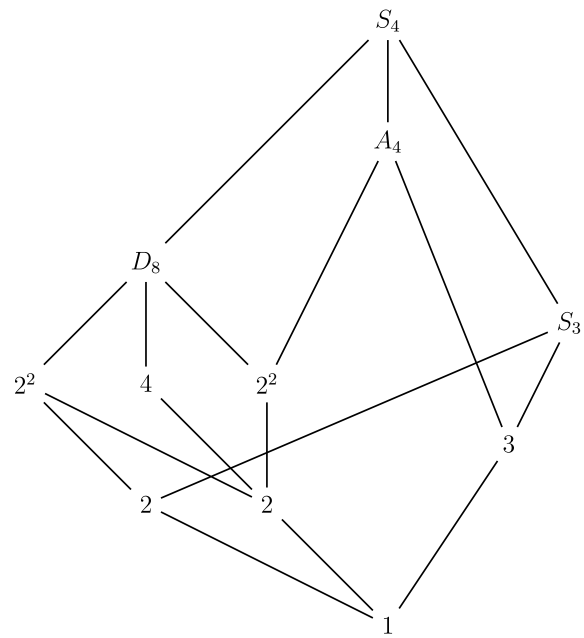

Figure 4: Poset of Conjugacy Classes of Subgroups of S_4

Lemma 3. *The number of incidences in the poset of conjugacy classes of subgroups of G is given by*

$$\sum \mathcal{I}(G).$$

Proof. The incidence matrix $\mathcal{I}(G)$ is obtained by replacing every nonzero entry in the table of marks by an entry 1. By Formula 1 $\mathcal{I}(H, K) = 1$ if and only if K is subconjugate to H in G, i.e. if and only if H and K are incident in the poset of conjugacy classes of subgroups of G. $\square$

Table 7 lists the number of incidences in the poset of conjugacy classes of subgroups of A_n and S_n for $n \leq 13$.

Lemma 4. *The total number of incidences in the entire subgroup lattice of G is given by*

$$\sum \mathcal{C}(G).$$

Proof. For $H, K \in \mathrm{Sub}(G)/G$ the H, K entry in $\mathcal{C}(G)$ is the number of incidences between H, K in the subgroup lattice of G. Thus summing over the entries in $\mathcal{C}(G)$ yields the total number of incidences in the entire subgroup lattice of G. $\square$

Table 8 records the number of incidences in the subgroup lattices of S_n and A_n for $n \leq 13$.

	A218921	A218922
n	S_n	A_n
1	1	1
2	3	1
3	9	3
4	44	13
5	101	32
6	523	128
7	1195	330
8	6751	2309
9	16986	4271
10	87884	12468
11	248635	33329
12	1709781	196182
13	4665651	490137

Table 7: Incidences in Poset

	A218924	A218923
n	A_n	S_n
1	1	1
2	1	3
3	3	11
4	18	68
5	85	262
6	657	2261
7	4374	14032
8	55711	176245
9	530502	1821103
10	6603007	30883491
11	82736601	415843982
12	2032940127	10779423937
13	32102236563	177718085432

Table 8: Incidences in Subgroup Lattice

3.3 Counting Edges in Hasse Diagrams

The table of marks also allows us to count the number of edges in both the Hasse diagrams of the poset of conjugacy classes of subgroups and the subgroup lattice of G. Computing such data requires careful analysis of maximal subgroups in the subgroup lattice.

Formula 1 describes containments in the poset of conjugacy classes of subgroups looking upward through the subgroup lattice of G. But we can also view marks as containments looking downward through the subgroup lattice of G.

Lemma 5. *Let $H, K \in \mathrm{Sub}(G)/G$. Then the number of conjugates of H contained in K is given by*

$$E^{\uparrow}(H, K) = |\{H^g, g \in G : H^g \leq K\}| = \frac{\beta_{G/K}(H)\beta_{G/H}(1)}{\beta_{G/H}(H)\beta_{G/K}(1)}$$

Proof. The total number of edges between the classes $[H]_G$ and $[K]_G$ can be counted in two different ways, as the length of the class times the number of edges leaving one member of the class. Thus

$$|[H_G| \cdot |\{H^g, g \in G : H^g \leq K\}| = |[K]_G| \cdot |\{K^g, g \in G : K^g \geq H\}|.$$

By Formula 1 $|[H]_G| = \frac{\beta_{G/H}(1)}{\beta_{G/H}(H)}$ and $|[K]_G| = \frac{\beta_{G/K}(1)}{\beta_{G/K}(K)}$. Thus $E^{\uparrow}(H, K)$ can be expressed in terms of marks. □

3.3.1 Identifying Maximal Subgroups

It will be necessary, for the sections that follow, to identify for $H_i \in \mathrm{Sub}(G)/G$ which classes $H_j \in \mathrm{Sub}(G)/G$ are maximal in H_i.

Lemma 6. *Let $H_i \in \mathrm{Sub}(G)/G = \{H_1, \ldots, H_r\}$. Denote by $\rho_i = \{j : H_j <_G H_i\}$ the set of indices in $\{1, \ldots, r\}$ of proper subgroups of H_i up to conjugacy in G. Then the positions of all maximal subgroups of H_i are given by*

$$\mathrm{Max}(H_i) = \rho_i \setminus \bigcup_{j \in \rho_i} \rho_j \tag{3}$$

The set of values ρ_i are easily read off the table of marks of G by simply identifying the nonzero entries in the row corresponding to G/H_i. Formula 3 is implemented in `GAP` via the function `MaximalSubgroupsTom`.

Lemma 7. *Let $\mathrm{Sub}(G)/G = \{H_1, \ldots, H_r\}$ be a list of representatives of the conjugacy classes of subgroups of G. The number of edges in the Hasse diagram of the poset of conjugacy classes of subgroups of G is given by*

$$|E(\mathrm{Sub}(G)/G)| = \sum_{i=1}^{r} |\mathrm{Max}(H_i)|.$$

Proof. By Lemma 6, $\mathrm{Max}(H_i)$ is a list of the positions of the maximal subgroups of H_i up to conjugacy in G. In the Hasse diagram of the poset $\mathrm{Sub}(G)/G$ each edge corresponds to a maximal subgroup. □

Table 9 records the number of edges in the Hasse diagram of the poset of conjugacy classes of subgroups of S_n and A_n for $n \leq 13$. In order to count the number of edges in the Hasse diagram of the entire subgroup lattice of G we appeal to Formula 1 and Lemma 5.

Lemma 8. *Let $\mathrm{Sub}(G)/G = \{H_1, \ldots, H_r\}$ be as above. The total number of edges $E(L(G))$ in the Hasse diagram of the subgroup lattice of G is given by*

$$E(L(G)) = \sum_{i=1}^{r} \sum_{j \in \mathrm{Max}(H_i)} E^{\uparrow}(H_i, H_j).$$

Proof. By restricting $E^{\uparrow}(H_i, H_j)$ to those classes H_i, H_j which are maximal we obtain the number of edges connecting maximal subgroups of G. □

Table 10 records the total number of edges in the Hasse diagram of the subgroup lattice of S_n and A_n for $n \leq 13$.

	A218925	A218926
n	S_n	A_n
1	0	0
2	1	0
3	4	1
4	17	5
5	37	13
6	149	44
7	290	98
8	1080	419
9	2267	722
10	8023	1592
11	17249	3304
12	72390	12645
13	153419	24792

Table 9: Edges in Poset

	A218928	A218927
n	A_n	S_n
1	0	0
2	0	1
3	1	8
4	15	66
5	168	501
6	2051	6469
7	19305	60428
8	283258	926743
9	3255913	11902600
10	46464854	240066343
11	670282962	3677270225
12	18723796793	108748156239
13	321480817412	1980478458627

Table 10: Edges in Subgroup Lattice

3.4 Maximal Property-P Subgroups

For any property P which is inherited by subgroups of G we can use the table of marks of G to enumerate the maximal property P subgroups of G.

Lemma 9. *Let* $\mathrm{Sub}(G)/G = \{H_1, \dots, H_r\}$ *and let* $\rho = \{i \in [1, \dots, r] : H_i$ *is a property* P *subgroup*$\}$. *Then the positions of the maximal property* P *subgroups of* G *are given by*

$$P(G) = \rho \setminus \bigcup_{j \in \rho} \mathrm{Max}(H_j) \tag{4}$$

In Figure 5 the classes of maximal abelian subgroups of S_4 are boxed.

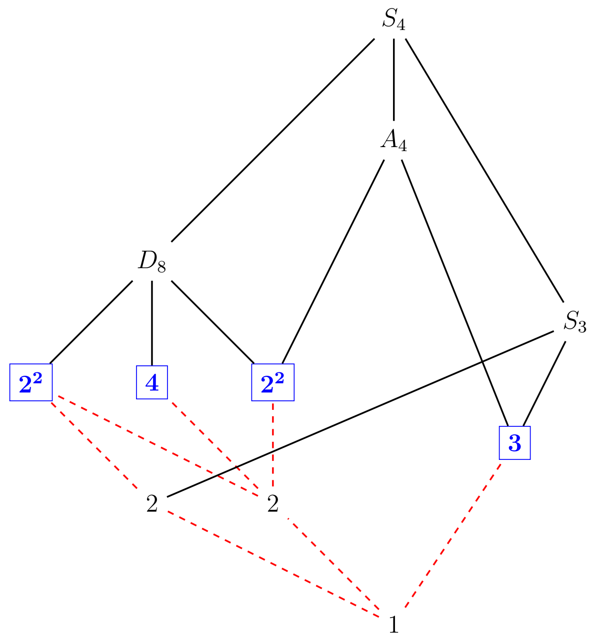

Figure 5: Maximal Abelian Subgroups of S_4

Table 11 records, for each of the properties listed across the first row of the table, the numbers of maximal property P classes of subgroups of S_n. A similar table for the alternating groups can be found in the Appendix.

	A218929	A218930	A218931	A218932	A218933
n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	2	2	2
4	1	2	4	3	2
5	3	3	5	3	3
6	4	4	7	5	5
7	5	5	10	6	6
8	6	6	17	11	7
9	9	8	23	15	9
10	12	11	30	20	12
11	14	14	41	24	15
12	17	19	61	34	20
13	24	23	80	43	25

Table 11: Maximal Property-P Subgroups of S_n

4 Connected subgroups and the Euler transform

The conjugacy classes of subgroups of the symmetric group play an important role in the theory of combinatorial species as described in [6]. Permutation groups have been used to answer many questions about species. Every species is the sum of its molecular subspecies. These molecular species correspond to conjugacy classes of subgroups of $\mathsf{Sym}(n)$. Molecular species decompose as products of atomic species which in turn correspond to connected subgroups of $\mathsf{Sym}(n)$ in the following sense. It will be convenient to denote the symmetric group on a finite set X by $\mathsf{Sym}(X)$.

Definition 10. For each $H \leq \mathsf{Sym}(X)$ there is a finest partition of $X = \bigsqcup Y_i$ such that $H = \prod H_i$ with $H_i \leq \mathsf{Sym}(Y_i)$. We allow $H_i = 1$ when $|Y_i| = 1$. We say that H is a connected subgroup of $\mathsf{Sym}(X)$ if the finest partition is X.

Example 11. Let $X = \{1, 2, 3, 4\}$ and consider $H = \langle(1, 2), (3, 4)\rangle$ and $H' = \langle(1, 2)(3, 4)\rangle$. Partitioning X into $Y_1 = \{1, 2\}, Y_2 = \{3.4\}$ gives $H = H_1 \times H_2$ where $H_1 = \langle(1, 2)\rangle \leq \mathsf{Sym}(\{1, 2\}), H_2 = \langle(3, 4)\rangle \leq \mathsf{Sym}(\{3, 4\})$, hence H is not connected. On the other hand, H' is connected since there is no finer partition of X which permits us to write H' as a product of connected H_i.

An algorithm to test a group H acting on a set X for connectedness checks each non-trivial H-stable subset Y of X. If H is the direct product of its action on Y and its action on $X \setminus Y$ then H is not connected.

In general a subgroup $H \leq \mathsf{Sym}(X)$ is a product of connected subgroups $H_i \leq \mathsf{Sym}(Y_i)$. Sequence A000638 records the number of molecular species of degree n or equivalently the number of conjugacy classes of subgroups of $\mathsf{Sym}(n)$. Sequence A005226 records the number of atomic species of degree n or equivalently the number of conjugacy classes of connected subgroups of $\mathsf{Sym}(n)$. These sequences are related by the Euler Transform.

4.1 The Euler Transform

If two sequences of integers $\{c_k\} = (c_1, c_2, c_3, \ldots)$ and $\{m_n\} = (m_1, m_2, m_3, \ldots)$ are related by

$$1 + \sum_{n\geq 1} m_n x^n = \prod_{k\geq 1} \left(\frac{1}{1 - x^k}\right)^{c_k}. \tag{5}$$

Then we say that $\{m_n\}$ is the Euler transform of $\{c_k\}$ and that $\{c_k\}$ is the inverse Euler transform of $\{m_n\}$ (see [1]). One sequence can be computed from the other by introducing the intermediate sequence $\{b_n\}$ defined by

$$b_n = \sum_{d|n} d c_d = n m_n - \sum_{k=1}^{n-1} b_k m_{n-k}. \tag{6}$$

Then

$$m_n = \frac{1}{n}\Big(b_n + \sum_{k=1}^{n-1} b_k m_{n-k}\Big), \qquad\qquad c_n = \frac{1}{n}\sum_{d|n} \mu(n/d) b_d, \tag{7}$$

where μ is the number-theoretic Möbius function.

There are many applications of this pair of transforms (see [10]). For example, the inverse Euler transform applied to the sequence of numbers of unlabeled graphs on n nodes (A000088) yields the sequence of numbers of connected graphs on n nodes (A001349). To understand how Formula 5 can be used to count connected graphs we note that the coefficient of x^n in the expansion of the product on the right hand side of Formula 5 is

$$m_n = \sum_{1^{a_1},2^{a_2},\ldots,n^{a_n} \vdash n} \prod_i \left(\!\!\binom{c_i}{a_i}\!\!\right) \tag{8}$$

where $\left(\!\binom{c_i}{a_i}\!\right)$ denotes the number of a_i-element multisets chosen from a set of c_i objects. On the other hand, an unlabeled graph on n nodes, as a collection of connected components, can be characterized by a pair $(\lambda, (C_1, \ldots, C_n))$ where $\lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$ is a partition of n and C_i is a multiset of a_i connected unlabeled graphs on i nodes, for $1 \leq i \leq n$. If c_i is the number of connected unlabeled graphs on i nodes then, by Formula 8, m_n is the total number of unlabeled graphs on n nodes.

In the same way, the inverse Euler transform of A000638 (the number of conjugacy classes of subgroups of S_n) is A005226, (the number of connected conjugacy classes of subgroups of S_n) as formalized in the following Lemma.

Lemma 12. *There is a bijection between the conjugacy classes of subgroups of S_n and the set of pairs of the form $(\lambda, (C_1, \ldots, C_n))$ where $\lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$ is a partition of n and C_i is a multiset of a_i conjugacy classes of connected subgroups of S_i for $i = 1, \ldots, n$.*

Proof. Given a representative H of the conjugacy class of subgroups $[H] \in \mathsf{Sub}(S_n)/S_n$ we associate a pair $(\lambda, (C_1, \ldots, C_n))$ to H as follows. Write $H = \prod H_k$ where each H_k is a connected subgroup of $\mathsf{Sym}(Y_k)$. Then $X = \{1, \ldots, n\} = \bigsqcup Y_k$. Recording the size of each Y_k yields a partition $\lambda = 1^{a_1}, 2^{a_2}, \ldots, n^{a_n}$. For $1 \leq i \leq n, C_i$ is the multiset of S_i-classes of subgroups H_k with $|Y_k| = i$. Bijectivity follows from the fact that conjugate subgroups yield the same λ and since $H^g = \prod H_k^g$, conjugate subgroups yield conjugate C_i. □

4.2 Counting Connected Subgroups of the Alternating Group

In Section 4 we noted that molecular species correspond to conjugacy classes of subgroups of $\mathsf{Sym}(n)$ and that atomic species correspond to conjugacy classes of connected subgroups of $\mathsf{Sym}(n)$ in the sense of Definition 10. In this Section we will count connected conjugacy classes of subgroups of the alternating group, up to S_n conjugacy and A_n conjugacy.

4.2.1 S_n-Orbits of Subgroups of the Alternating Group

In order to count the number of S_n-conjugacy classes of subgroups of the alternating group we introduce the following notation. Let

$$\mathcal{B} = \{H \leq S_n : H \leq A_n \text{ and } \mathcal{R} = \{H \leq S_n : H \nleq A_n\}.$$

Then $\mathsf{Sub}(S_n)/S_n = \mathcal{B}/S_n \sqcup \mathcal{R}/S_n$ and $\mathcal{B}/S_n$ is the set of S_n conjugacy classes of subgroups of the alternating group. The set $\mathcal{R}/S_n$ is the set of conjugacy classes of subgroups of S_n which are not contained in A_n. Table 12 illustrates both of these sequences together with the numbers of conjugacy classes of subgroups of S_n and A_n. In order to count the number of connected S_n-conjugacy classes of subgroups of A_n we apply the inverse Euler transform to the sequence $|\mathcal{B}/S_n|$ in Table 12, to obtain

$$\text{A218968}: \; 1, 0, 1, 3, 4, 12, 12, 65, 58, 167, 198, 1207, 1178.$$

We can also count the number of connected conjugacy classes of subgroups of S_n not contained in A_n (i.e. corresponding to $\mathcal{R}/S_n$) by subtracting the sequence above from A005226 to obtain

$$\text{A218969}: \; 0, 1, 1, 3, 2, 15, 8, 65, 66, 431, 443, 3643, 3594.$$

	A000638	A029726	A218966	A218965
n	$\lvert\mathsf{Sub}(S_n)/S_n\rvert$	$\lvert\mathsf{Sub}(A_n)/A_n\rvert$	$\lvert\mathcal{B}/S_n\rvert$	$\lvert\mathcal{R}/S_n\rvert$
1	1	1	1	0
2	2	1	1	1
3	4	2	2	2
4	11	5	5	6
5	19	9	9	10
6	56	22	22	34
7	96	40	37	59
8	296	137	112	184
9	554	223	195	359
10	1593	430	423	1170
11	3094	788	780	2314
12	10723	2537	2401	8322
13	20832	4558	4409	16423

Table 12: Red and Blue Subgroups of S_n

4.2.2 Connected Subgroups of the Alternating Group

Every subgroup of A_n is either connected or not connected with respect to the set $\{1, \ldots, n\}$ and shares this property with all subgroups in its A_n-conjugacy class. So we wish to count the number of A_n-conjugacy classes of connected subgroups of A_n. Unfortunately, the Euler transform does not apply to A_n-orbits. We test for connectedness a list of representatives of $\mathsf{Sub}(A_n)/A_n$ in GAP and obtain

$$\text{A218967}: \ 1, 0, 1, 3, 4, 12, 15, 87, 64, 168, 205, 1336, 1198.$$

Remark 13. There is a sequence in the encyclopedia, A116653, which currently claims to count both the number of atomic species based on conjugacy classes of subgroups of the alternating group (i.e. the number of S_n-conjugacy classes of connected subgroups of A_n) and the number of A_n-conjugacy classes of connected subgroups of A_n. However this sequence is merely the inverse Euler transform of sequence A029726, the number of conjugacy classes of subgroups of the alternating group. The number of S_n-conjugacy classes of connected subgroups of A_n is sequence A218968 and the number of A_n-conjugacy classes of connected subgroups of A_n is A216967.

4.3 Connected Subgroups with Additional Properties

Appealing to Definition 10 we can count the connected subgroups of S_n which possess additional group theoretic properties. If the property of interest is compatible with taking direct products we can apply the inverse Euler transform to the sequence of numbers of all conjugacy classes of subgroups of S_n with this property to obtain the sequence of numbers of conjugacy classes of connected subgroups of S_n with this property. Table 13 records the number of connected subgroups of S_n which additionally possess the properties listed in the first row of the table. Each of the sequences in Table 13 is the inverse Euler transform of the corresponding sequence in Table 1.

	A000638	A218971	A218972	A218973	A218974
n	$\lvert\mathrm{Sub}(S_n)/S_n\rvert$	Abelian	Nilpotent	Solvable	SupSol
1	1	1	1	1	1
2	2	1	1	1	1
3	4	1	1	2	2
4	11	3	4	6	4
5	19	1	1	4	4
6	56	6	9	23	15
7	96	1	1	16	13
8	296	17	69	122	81
9	554	5	8	109	77
10	1593	40	238	551	352
11	3094	2	2	570	406
12	10723	162	2339	4633	2995
13	20832	5	8	4224	2866

Table 13: Connected Subgroups of S_n

4.4 Connected Partitions

The number of conjugacy classes of cyclic subgroups of S_n equals the number of partitions of n. The inverse Euler transform of this sequence yields the all ones sequence. This does not count the number of conjugacy classes of connected cyclic subgroups of S_n since the direct product of cyclic groups is not necessarily cyclic.

Definition 14. Let $\lambda = [x_1, \ldots, x_l]$ be a partition of n. Let G_λ be the simple graph with l vertices labeled by $x_1, \ldots, x_l$ where two vertices are connected by an edge if and only if their labels x_i, x_j are not coprime. We call the partition λ connected if the graph G_λ is connected.

Proposition 15. *The number of conjugacy classes of connected cyclic subgroups of S_n equals the number of connected partitions of n.*

Proof. Let $C = \langle c \rangle \leq S_n$ be cyclic. Then the cycle lengths of the permutation c form a partition λ of n. For simplicity, we assume $\lambda = [x_1, x_2]$. Then $c = ab$ where a is an x_1 cycle and b is an x_2 cycle. Let $d = \gcd(x_1, x_2)$. If $d = 1$ then G_λ is not connected and $1 = yx_1 + zx_2$, for some $y, z \in \mathbb{Z}$. Then $c^{yx_1} = a, c^{zx_2} = b$ and $C = \langle a \rangle \times \langle b \rangle$ is not connected. If $d > 1$ then C does not contain a generator of $\langle a \rangle$ or of $\langle b \rangle$. The case for general λ follows by a similar argument. □

Example 16. There are 3 connected partitions λ of 13. Their graphs G_λ are shown in Figure 6.

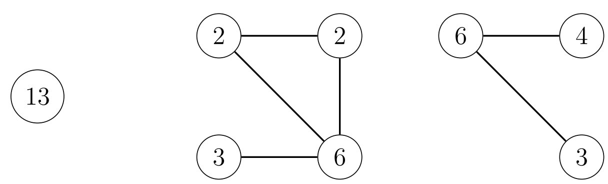

Figure 6: The Graphs of the Connected Partitions of 13

Using Proposition 15 we obtain the sequence of numbers of conjugacy classes of connected cyclic subgroups of S_n

$$\text{A218970}: \ 1, 1, 1, 2, 1, 4, 1, 5, 3, 8, 2, 14, 3.$$

Remark 17. There are two sequences in the encyclopedia which are quite similar to this sequence. Sequence A018783 counts the number of partitions of n into parts all of which have a common factor greater than 1. Sequence A200976 counts the number of partitions of n such that each pair of parts (if any) has a common factor greater than 1. For $n \leq 13$, sequence A218970 above differs from both of these sequences when $n = 1, 11, 13$.

5 Concluding remarks

The sequences presented in this article have been computed using GAP. A GAP file containing the programs can be found at www.maths.nuigalway.ie/~liam/CountingSubgroups.g. The GAP table of marks library Tomlib can be found here [11] and is a requirement for computing many of the sequences presented. It is worth pointing out that Holt has determined all conjugacy classes of subgroups of S_n for values of n up to and including $n = 18$, (see [5]). The majority of the sequences presented in this article rely on the availability of the table of marks of S_n and so we restrict our attention to $n \leq 13$. We are grateful to Des MacHale for suggesting many of the sequences that we compute in this article. We thank the anonymous referees for helpful suggestions.

Appendix A Additional Sequences

Using the methods described in this article the following additional sequences have been computed.

	A029726	A218934	A218935	A218936	A218937	A218938
n	$\lvert \mathrm{Sub}(A_n)/A_n \rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	2	2	2	2	2	2
4	5	4	3	4	5	4
5	9	5	4	5	8	7
6	22	9	6	10	19	14
7	40	12	8	13	33	22
8	137	30	12	53	122	70
9	223	41	17	69	192	122
10	430	60	23	122	364	225
11	788	81	29	160	650	395
12	2537	193	40	734	2194	1240
13	4558	243	52	848	3845	2185

Table 14: Conjugacy classes of subgroups of A_n

	A005432	A062297	A051625	A218939	A218940	A218941
n	$\|\mathrm{Sub}(S_n)\|$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	6	5	5	5	6	6
4	30	21	17	24	30	28
5	156	87	67	102	154	144
6	1455	612	362	837	1429	1259
7	11300	3649	2039	5119	11065	9560
8	151221	35515	14170	78670	148817	123102
9	1694723	289927	109694	664658	1667697	1371022
10	29594446	3771118	976412	13514453	29103894	23449585
11	404126228	36947363	8921002	137227213	396571224	317178020
12	10594925360	657510251	101134244	4919721831	10450152905	8296640115
13	175238308453	7736272845	1104940280	60598902665	172658168937	136245390535

Table 15: Total number of subgroups of S_n

	A218955	A218956	A218957	A218958	A218959
n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	4	4	4
4	1	7	11	13	7
5	21	31	51	31	31
6	76	101	241	246	211
7	456	491	1506	1296	1156
8	1956	3011	9649	10774	5419
9	12136	18467	80281	83238	40027
10	80836	114983	640741	788820	348331
11	807676	1283723	6196576	6835170	3204796
12	8779816	13380643	66883411	81364944	38422891
13	104127596	148321603	775421219	848378532	467645179

Table 16: Total number of maximal property-P subgroups of S_n

	A029725	A218942	A051636	A218943	A218944	A218945
n	$\lvert\mathrm{Sub}(A_n)\rvert$	Abelian	Cyclic	Nilpotent	Solvable	SupSol
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	2	2	2	2	2	2
4	10	9	8	9	10	9
5	59	37	32	37	58	53
6	501	207	167	252	488	418
7	3786	1192	947	1507	3664	2894
8	48337	11449	6974	21739	47210	33675
9	508402	93673	53426	186983	498102	369763
10	6469142	892783	454682	2369258	6293475	4769542
11	81711572	8534308	4303532	22872863	78805290	58853842
12	2019160542	148561283	50366912	746597568	1960342409	1395051100
13	31945830446	1740198891	553031624	9157758326	31130243721	21847262156

Table 17: Total no of subgroups of A_n

	A218946	A218947	A218948	A218949	A218950
n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	1	1	1
4	1	2	2	2	2
5	3	3	3	3	3
6	4	3	5	4	3
7	5	4	6	5	5
8	6	6	13	6	6
9	10	8	19	8	7
10	12	10	22	10	9
11	14	13	27	14	12
12	17	18	40	20	17
13	24	22	54	24	20

Table 18: Maximal property-P subgroups of A_n

	A029726	A218951	A218952	A218953	A218954
n	$\lvert\mathrm{Sub}(A_n)/A_n\rvert$	Abelian	Nilpotent	Solvable	SupSol
1	1	1	1	1	1
2	1	0	0	0	0
3	2	1	1	1	1
4	5	2	2	3	2
5	9	1	1	3	3
6	22	3	4	10	6
7	40	1	1	11	6
8	137	14	36	80	42
9	223	5	9	52	39
10	430	12	49	145	85
11	788	2	2	165	104
12	2537	69	489	1208	686
13	4558	3	4	1033	617

Table 19: Connected subgroups of A_n

The number of connected even partitions of n

A218975 : $1, 0, 1, 1, 1, 2, 1, 3, 3, 4, 2, 8, 2.$

	A218960	A218961	A218962	A218963	A218964
n	Solvable	SupSol	Abelian	Cyclic	Nilpotent
1	1	1	1	1	1
2	1	1	1	1	1
3	3	3	3	3	3
4	1	10	10	9	10
5	36	40	30	30	30
6	225	110	115	100	110
7	686	645	861	665	1001
8	4655	5670	10536	3885	4005
9	28728	47754	78474	33093	45696
10	397005	311850	1008000	371700	379155
11	2210890	3014550	9302964	3790875	4913040
12	26975025	24022845	73024380	37839285	36701280
13	26121667	46950904	563291872	350984414	158538380

Table 20: Total number of maximal property-P subgroups of A_n

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