



Counting Palindromes According to r -Runs of Ones Using Generating Functions

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Abstract

We derive generating functions for the enumeration of all palindromic binary strings of length n having only runs of 1's of length $\leq r$. We provide asymptotic expressions for fixed r and $n \rightarrow \infty$. Eventually, r is treated as a random variable and an asymptotic equivalent for the largest run of 1's in binary palindromes is derived.

1 Enumeration

In the recent paper [3] the interest was in words over the alphabet $\{0, 1\}$ which are *palindromes* and have runs of 1's of bounded length. We firmly believe that *generating functions* are the most appropriate tool here, and since they were not used in [3], we present this natural approach and show as well how one can deal with the case that the maximal 1-run length is treated as a *random variable*. It is worthwhile to note that all our methods can be found in [1].

Let us start with palindromes of *even length*; they are given as ww^R , with a reversed copy of w attached to w . In unrestricted words, the following factorization is appropriate:

$$(0 + 1)^* = (1^*0)^*1^*.$$

Here, we used the $*$ -operation, common in the study of formal languages, so L^* denotes all words that can be formed from concatenating words taken from L in all possible ways. In

[1], the notation $\text{SEQ}(L)$ is mostly used, describing all SEQUENCES (aka words), formed from L . Now, the mentioned factorization is a very common one for binary words. Each word is (uniquely) decomposed according to each appearance of the letter 0; between them, there are runs (possibly empty) of the letter 1. If a word has s letters 0, then there are $s + 1$ such runs of 1's. In terms of generating functions, since the transition $A \rightarrow A^*$ means $f \rightarrow \frac{1}{1-f}$, the factorization reads as

$$\frac{1}{1-2z} = \frac{1}{1-\frac{z}{1-z}} \frac{1}{1-z}.$$

This factorization can immediately be generalized to the instance when the 1-runs should not exceed the parameter r . Then we first consider the set of restricted runs

$$\mathbf{1}^{\leq r} = \{\varepsilon, 1, 11, \dots, 1^r\},$$

which translates into

$$1 + z + \dots + z^r = \frac{1 - z^{r+1}}{1 - z}.$$

Then we get the formal expression

$$(\mathbf{1}^{\leq r} \mathbf{0})^* \mathbf{1}^{\leq r},$$

which translates into

$$\frac{1}{1 - \frac{z(1 - z^{r+1})}{1 - z}} \frac{1 - z^{r+1}}{1 - z} = \frac{1 - z^{r+1}}{1 - 2z + z^{r+2}}.$$

Now, going to palindromes of even length, the last group of ones must be bounded by $\lfloor \frac{r}{2} \rfloor$. So a syntactic description of palindromes of even length with bounded 1-runs is

$$(\mathbf{1}^{\leq r} \mathbf{0})^* \mathbf{1}^{\leq \lfloor \frac{r}{2} \rfloor};$$

this describes the first half of the word only. From this we go immediately to generating functions, by replacing both letters by a variable z . In this way, we count half of the length of the palindromes of even length. If one wants the full length, one must replace z by z^2 . So we get

$$\frac{1}{1 - z \frac{1 - z^{r+1}}{1 - z}} \frac{1 - z^{\lfloor \frac{r}{2} \rfloor + 1}}{1 - z} = \frac{1 - z^{\lfloor \frac{r}{2} \rfloor + 1}}{1 - 2z + z^{r+2}}. \quad (1)$$

One can read off the coefficient of z^n in the power series expansion of this expression, which leads to a clumsy expression:

Set

$$a_{n,r} = [z^n] \frac{1}{1 - 2z + z^r},$$

then

$$a_{n,r} - 2a_{n-1,r} + a_{n-r,r} = 0,$$

and initial conditions $a_{n,r} = 2^n$ for $n < r$.

Then the number of palindromes of even length $2n$ with all 1-runs $\leq r$ is given by

$$[z^n] \frac{1 - z^{\lfloor \frac{r}{2} \rfloor + 1}}{1 - 2z + z^{r+2}} = a_{n,r+2} - a_{n - \lfloor \frac{r}{2} \rfloor - 1, r+2}.$$

We can alternatively express the coefficients in (1) using the *higher order Fibonacci numbers*, as it was done in [3]: Consider $U_{n,r} = U_{n-1,r} + \dots + U_{n-r,r}$ for $n \geq r$, with initial values $U_{0,r} = \dots = U_{r-2,r} = 0$, $U_{r-1,r} = 1$. Then

$$\sum_{n \geq 0} U_{n,r} z^n = \frac{z^{r-1}}{1 - (z^r + \dots + z)} = \frac{z^{r-1}}{1 - z \frac{1-z^r}{1-z}} = \frac{z^{r-1}(1-z)}{1 - 2z + z^{r+1}}.$$

Further,

$$\sum_{n \geq 0} (U_{0,r} + \dots + U_{n,r}) z^n = \frac{z^{r-1}}{1 - 2z + z^{r+1}},$$

or

$$\sum_{k=0}^{n+r} U_{k,r+1} = [z^n] \frac{1}{1 - 2z + z^{r+2}}.$$

Consequently

$$\begin{aligned} [z^n] \frac{1 - z^{\lfloor \frac{r}{2} \rfloor + 1}}{1 - 2z + z^{r+2}} &= [z^n] \frac{1}{1 - 2z + z^{r+2}} - [z^{n - \lfloor \frac{r}{2} \rfloor - 1}] \frac{1}{1 - 2z + z^{r+2}} \\ &= \sum_{k=0}^{n+r} U_{k,r+1} - \sum_{k=0}^{n - \lfloor \frac{r}{2} \rfloor - 1 + r} U_{k,r+1} \\ &= \sum_{k=n - \lfloor \frac{r}{2} \rfloor + r}^{n+r} U_{k,r+1}. \end{aligned}$$

This is the expression given in [3] once one changes the index of summation. Note that $r - \lfloor \frac{r}{2} \rfloor = \lceil \frac{r}{2} \rceil$.

Now we move to palindromes of odd length with middle letter 1: $w1w^R$. Then w is described by

$$(\mathbf{1}^{\leq r} \mathbf{0})^* \mathbf{1}^{\leq \lfloor \frac{r-1}{2} \rfloor}.$$

In this way, the last group of ones plus the middle 1 plus the first group of ones of the reversed word is still $\leq r$ as it should.

The corresponding generating function is

$$\frac{1 - z^{\lfloor \frac{r-1}{2} \rfloor + 1}}{1 - 2z + z^{r+2}}$$

and the coefficient of z^n (counting palindromes of odd length $2n + 1$ with middle letter 1) is

$$a_{n,r+2} - a_{n-\lfloor \frac{r-1}{2} \rfloor - 1, r+2}.$$

Again, we can alternatively express the corresponding number by higher order Fibonacci numbers:

$$\begin{aligned} [z^n] \frac{1 - z^{\lfloor \frac{r-1}{2} \rfloor + 1}}{1 - 2z + z^{r+2}} &= [z^n] \frac{1}{1 - 2z + z^{r+2}} - [z^{n-\lfloor \frac{r-1}{2} \rfloor - 1}] \frac{1}{1 - 2z + z^{r+2}} \\ &= \sum_{k=0}^{n+r} U_{k,r+1} - \sum_{k=0}^{n-\lfloor \frac{r-1}{2} \rfloor - 1 + r} U_{k,r+1} \\ &= \sum_{k=n-\lfloor \frac{r-1}{2} \rfloor + r}^{n+r} U_{k,r+1}. \end{aligned}$$

Note that $r - \lfloor \frac{r-1}{2} \rfloor = \lfloor \frac{r}{2} \rfloor + 1$.

Finally we move to palindromes of odd length with middle letter 0: $w0w^R$. Then we have

$$(\mathbf{1}^{\leq r} \mathbf{0})^* \mathbf{1}^{\leq r},$$

since the middle 0 interrupts the last run of ones of the first group. The corresponding generating function is

$$\frac{1 - z^{r+1}}{1 - 2z + z^{r+2}},$$

where again the coefficient of z^n refers to a palindrome of length $2n + 1$ with middle 0.

Explicitly we get

$$[z^n] \frac{1 - z^{r+1}}{1 - 2z + z^{r+2}} = a_{n,r+2} - a_{n-r-1, r+2}.$$

In terms of higher order Fibonacci numbers, this reads

$$\begin{aligned} [z^n] \frac{1 - z^{r+1}}{1 - 2z + z^{r+2}} &= [z^n] \frac{1}{1 - 2z + z^{r+2}} - [z^{n-r-1}] \frac{1}{1 - 2z + z^{r+2}} \\ &= \sum_{k=0}^{n+r} U_{k,r+1} - \sum_{k=0}^{n-1} U_{k,r+1} \\ &= \sum_{k=n}^{n+r} U_{k,r+1}. \end{aligned}$$

2 Asymptotics

We refer to the paper [2] which might be the first to consider asymptotics for words of restricted runs. The recent paper [4] has many examples of this type. Here, we only consider the key steps and refer for error bounds to the cited literature.

One has to study the *dominant* zero of the denominator, denoted by ρ , which is close to $\frac{1}{2}$ when r gets large (no restriction). From

$$1 - 2\rho + \rho^{r+2} = 0$$

we infer

$$\rho = \frac{1}{2} + \frac{1}{2}\rho^{r+2} \approx \frac{1}{2} + \frac{1}{2^{r+3}}.$$

This procedure is called *bootstrapping*. We also need the constant A in

$$\frac{1}{1 - 2z + z^{r+2}} \sim \frac{A}{1 - z/\rho} \quad \text{as } z \rightarrow \rho,$$

which we get by L'Hopital's rule as

$$A = \frac{-1/\rho}{-2 + (r+2)z^{r+1}} \Big|_{z=\rho} = \frac{-1/\rho}{-2 + (r+2)\rho^{r+1}} = \frac{1}{2\rho - (r+2)(2\rho - 1)}.$$

So we get the following asymptotic formulæ, valid for $n \rightarrow \infty$ and fixed r :

$$\begin{aligned} [z^n] \frac{1 - z^{\lfloor \frac{r}{2} \rfloor + 1}}{1 - 2z + z^{r+2}} &\sim (1 - \rho^{\lfloor \frac{r}{2} \rfloor + 1}) A \rho^{-n}, \\ [z^n] \frac{1 - z^{\lfloor \frac{r-1}{2} \rfloor + 1}}{1 - 2z + z^{r+2}} &\sim (1 - \rho^{\lfloor \frac{r-1}{2} \rfloor + 1}) A \rho^{-n}, \\ [z^n] \frac{1 - z^{r+1}}{1 - 2z + z^{r+2}} &\sim (1 - \rho^{r+1}) A \rho^{-n}. \end{aligned}$$

And now we turn to the instance where r is a random variable X , and compute, as a showcase, the expected value, so we answer the question about the average value of the longest 1-run in palindromes, in the 3 respective models. As mentioned, this was basically done already by Knuth. When r gets large, the constant A may be replaced by 1, terms of the form ρ^r may be dropped, and in ρ^{-n} , it is enough to use the approximation

$$\rho^{-n} \sim 2^n (1 - 2^{-r-2})^n \sim 2^n \exp(-n/2^{r+2}).$$

Furthermore, to get a probability distribution, we have to divide by 2^n , which is the number of binary words of length n . So the probability that the parameter X is $\leq r$ is in all 3 instances approximated by

$$\exp(-n/2^{r+2}).$$

For an expected value, one has to compute

$$\sum_{r \geq 0} [1 - \exp(-n/2^{r+2})].$$

This evaluation can be found in many texts [2, 1, 4]; it is done with the *Mellin transform*, and the result is

$$\log_2 n + \frac{\gamma}{\log 2} - \frac{3}{2} - \frac{1}{\log 2} \sum_{k \neq 0} \Gamma\left(\frac{2k\pi i}{\log 2}\right) e^{-2k\pi i \cdot \log_2 n}.$$

Observe that the series in this expression represents a periodic function with small amplitude.

Asymptotically, thus, palindromes with middle letter 0 resp. 1 resp. no middle letter all lead to the same result.

References

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