Journal of Integer Sequences, Vol. 20 (2017),
Article 17.7.4

The rth Moment of the Divisor Function: An Elementary Approach

Florian Luca
School of Mathematics
University of the Witwatersrand
Private Bag 3, Wits 2050
Johannesburg
South Africa
and
Max Planck Institute for Mathematics
Vivatsgasse 7
53111 Bonn
Germany
and
Department of Mathematics
Faculty of Sciences
University of Ostrava
30 dubna 22
701 03 Ostrava 1
Czech Republic
florian.luca@wits.ac.za

László Tóth
Department of Mathematics
University of Pécs
Ifjúság útja 6
H-7624 Pécs
Hungary
ltoth@gamma.ttk.pte.hu

Abstract

For integer $r \geq 1$ we give an elementary proof for the main term of the asymptotic behavior of the rth moment of the number of divisors of n for positive integers $n \leq x$.

1 Introduction

Let $\tau(n)$ be the number of divisors of n. Ramanujan [2] stated without proof that, given any real number $\varepsilon > 0$, the estimate

$$\sum_{n\le x} \tau(n)^2 = x(A(\log x)^3 + B(\log x)^2 + C\log x + D) + O(x^{3/5+\varepsilon})$$

holds with $A = \pi^{-2}$. An elementary proof of the asymptotic formula

$$\sum_{n\le x} \tau(n)^2 \sim Ax(\log x)^3,$$

as $x \to \infty$, appears in several places (see, for example, [1, Thm. 7.8]). Wilson [3] proved Ramanujan's claim and generalized it by showing that for any integer $r \ge 2$ one has

$$\sum_{n\le x} \tau(n)^r = x(C_{r,1}(\log x)^{2^r-1} + C_{r,2}(\log x)^{2^r-2} + \cdots + C_{r,2^r}) + O(x^{\frac{2^r-1}{2^r+2}+\varepsilon}).$$

Note that when $r = 2$, Wilson's error term is better than the one claimed by Ramanujan. We are not aware even of elementary proofs for the asymptotic formula

$$\sum_{n\le x} \tau(n)^r \sim C_r x(\log x)^{2^r-1}$$

as $x \to \infty$ for any $r \ge 2$. In this note, we give an elementary proof of the following more general result.

Theorem 1. *Let k be a positive integer and $f(n)$ be a multiplicative function which on prime powers p^α satisfies*

$$f(p) = k \quad \text{and} \quad f(p^\alpha) = \alpha^{O(1)} \quad \textit{for all primes } p \textit{ and integers } \alpha \ge 2,$$

where the constant implied by the above O is uniform in p. Then

$$\sum_{n\le x} f(n) = xC_f(\log x)^{k-1} + O(x(\log x)^{k-2})$$

where

$$C_f = \frac{1}{(k-1)!}\left(\prod_{p\ge 2}\left(1-\frac{1}{p}\right)^k\left(\sum_{\alpha\ge 0}\frac{f(p^\alpha)}{p^\alpha}\right)\right).$$

In the case $f(n) = \tau(n)^r$ for integer $r \ge 1$, Theorem 1 applies with $k = 2^r$.

The only facts that we use are Abel's summation formula, the Möbius inversion formula, the elementary estimate

$$\sum_{n\le t}\frac{1}{n} = \log t + \gamma + O(1/t) \tag{1}$$

valid for all real $t \ge 1$, and the fact that the counting function of the *squarefull* numbers $s \le t$ is $O(t^{1/2})$, where s is squarefull if and only if $p^2 \mid s$ for all prime factors p of s, all provable by elementary means.

2 A lemma

Lemma 2. *Assume that r is a positive integer and $f(n)$ is some arithmetic function such that*

$$\sum_{n\le x} f(n) = \sum_{j=0}^{r} c_j(\log x)^j + O(x^{-1/2+o(1)}), \tag{2}$$

for some constants c_j, $j = 0, \ldots, r$. Then

$$\sum_{n\le x} f(n)(\log(x/n))^k = \sum_{\ell=0}^{k+r} C_\ell(\log x)^\ell + O(x^{-1/2+o(1)}), \tag{3}$$

holds for all positive integers k with some constants $C_0, \ldots, C_{k+r}$. Here, if $\ell \in \{k, k+1, \ldots, k+r\}$, then

$$C_\ell := c_{\ell-k}\left(1 + (\ell-k)\sum_{i=1}^{k}\frac{(-1)^i}{\ell-k+i}\binom{k}{i}\right). \tag{4}$$

Furthermore, if $r \ge t \ge 1$ are positive integers and

$$\sum_{n\le x} f(n) = \sum_{j=t}^{r} c_j(\log x)^j + O((\log x)^{t-1}), \tag{5}$$

then

$$\sum_{n\le x} f(n)(\log(x/n))^k = \sum_{j=k+t}^{k+r} C_j(\log x)^j + O((\log x)^{t+k-1}). \tag{6}$$

Proof. We show how to deduce (3) out of (2) with the leading coefficients given by (4). Let

$$A(x) = \sum_{n\le x} f(n).$$

Then

$$A(x) = \sum_{j=0}^{r} c_j(\log x)^j + R(x),$$

where $|R(x)| = x^{-1/2+o(1)}$ as $x \to \infty$. Let $i \ge 1$. Put

$$B_i(x) := \sum_{n\le x} f(n)(\log n)^i.$$

Then, by the Abel summation formula and by interchanging the order between the summation and the integration, we get

$$
\begin{aligned}
B_i(x) &= A(x)(\log x)^i - i\int_1^x A(t)\left(\frac{(\log t)^{i-1}}{t}\right)dt \\
&= \sum_{j=0}^r \left(c_j(\log x)^{j+i} - i\int_1^x \left(\frac{c_j(\log t)^{j+i-1}}{t}\right)dt\right) \\
&- i\int_1^x \frac{(\log t)^{i-1}R(t)}{t}dt + R(x)(\log x)^i \\
&= \sum_{j=0}^r \left(c_j(\log x)^{j+i} - \frac{c_j i}{j+i}(\log t)^{j+i}\Big|_1^x\right) + \\
&- i\int_1^\infty \frac{(\log t)^{i-1}R(t)}{t}dt + i\int_x^\infty \frac{(\log t)^{i-1}R(t)}{t}dt + R(x)(\log x)^i \\
&= \sum_{j=0}^r \frac{c_j j}{j+i}(\log x)^{j+i} + D_i + O(x^{-1/2+o(1)}),
\end{aligned}
$$

where

$$D_i := -i\int_1^\infty \frac{(\log t)^{i-1}R(t)}{t}dt$$

In the above, we used the fact that $|R(t)| \le t^{-1/2+o(1)}$ as $t \to \infty$ to deduce that the above integral converges and that its tail from x to infinity as well as the other errors are $O(x^{-1/2+o(1)})$ as $x \to \infty$. Using the binomial formula and the above arguments, we have

$$
\begin{aligned}
C_k(x) &:= \sum_{n\le x} f(n)(\log(x/n))^k \\
&= \sum_{i=0}^k (-1)^i \binom{k}{i} (\log x)^{k-i} \sum_{n\le x} f(n)(\log n)^i \\
&= \sum_{n\le x} f(n) + \sum_{i=1}^k (-1)^i \binom{k}{i} (\log x)^{k-i} B_i(x) \\
&= \sum_{\ell=0}^{k+r} C_\ell (\log x)^\ell + O(x^{-1/2+o(1)}),
\end{aligned}
$$

where C_ℓ are given by formula (4) for $\ell \ge k$. For $\ell = 1, \ldots, k-1$, the coefficient C_ℓ involves the expression D_ℓ. The deduction of (6) out of (5) is immediate by similar arguments. $\square$

3 The proof of Theorem 1

Let $f_0(n) := f(n)$. Recursively define $f_j(n)$ such that

$$f_{j-1}(n) = \sum_{d|n} f_j(d), \quad j = 1, 2, \ldots.$$

By Möbius inversion,

$$f_j(n) = \sum_{d|n} \mu(d) f_{j-1}(n/d).$$

On primes

$$f_j(p) = f_{j-1}(p) - 1, \quad j = 1, 2, \ldots.$$

Since $f_0(p) = k$, we get that $f_j(p) = k - j$. In particular, $f_k(p) = 0$. Further, for $\alpha \geq 2$, we have that

$$f_j(p^\alpha) = f_{j-1}(p^\alpha) - f_{j-1}(p^{\alpha-1}).$$

Since $f_0(p^\alpha) = \alpha^{O(1)}$ it follows that $f_j(p^\alpha) = \alpha^{O(1)}$ for all $j \geq 2$. The constant in $O(1)$ might depend on j. Further,

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right) \sum_{\alpha \geq 0} \frac{f_{j-1}(p^\alpha)}{p^\alpha}, \quad j = 1, 2, \ldots,$$

therefore

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha}, \quad j = 0, 1, \ldots$$

Put

$$E_j := \prod_{p \geq 2} \left(\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} \right) = \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha} \right).$$

Fix $j \geq 1$. Then

$$F_{j-1}(x) := \sum_{n \leq x} \frac{f_{j-1}(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d|n} f_j(d) = \sum_{d \leq x} f_j(d) \sum_{\substack{n \leq x \\ d|n}} \frac{1}{n}.$$

In the inner sum, we write an $n \leq x$ which is a multiple of d as $n = dm$ for some integer $m \leq x$. We get

$$\begin{aligned} F_{j-1}(x) &= \sum_{d \leq x} \frac{f_j(d)}{d} \sum_{m \leq x/d} \frac{1}{m} = \sum_{d \leq x} \frac{f_j(d)}{d} \left(\log(x/d) + \gamma + O(d/x)\right) \\ &= \sum_{d \leq x} \frac{f_j(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_j(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} |f_j(d)| \right) \end{aligned} \quad (7)$$

for $j = 1, 2, \ldots$. When $j = k$, since $f_k(p) = 0$, it follows that $f_k(d) = 0$ if d is not squarefull. Thus, when $j = k$ in the right–hand side of (7), we have

$$\sum_{d \le x} \frac{f_k(d)}{d} \log(x/d) + \gamma \sum_{d \le x} \frac{f_k(d)}{d} + O\left(\frac{1}{x} \sum_{d \le x} |f_k(d)|\right).$$

Note that

$$\begin{aligned}\sum_{d \le x} \frac{f_k(d)}{d} &= \sum_{d \ge 1} \frac{f_k(d)}{d} + O\left(\sum_{d > x} \frac{|f_k(d)|}{d}\right) = E_k + O\left(\sum_{\substack{d \ge x \\ d \text{ squarefull}}} \frac{1}{d^{1+o(1)}}\right) \\ &= E_k + O(x^{-1/2+o(1)}),\end{aligned} \tag{8}$$

where for the error term we used the fact that $|f_k(d)| = |\tau(d)|^{O(1)} = d^{o(1)}$ as $d \to \infty$ and the Abel summation formula to conclude that

$$\sum_{\substack{d > x \\ d \text{ squarefull}}} \frac{1}{d^{1+o(1)}} \le x^{-1/2+o(1)} \quad \text{as} \quad x \to \infty.$$

Further, we have

$$\begin{aligned}\sum_{d \le x} \frac{f_k(d)}{d}(-\log d + \gamma) &= \sum_{d \ge 1} \frac{f_k(d)(-\log d + \gamma)}{d} + O\left(\sum_{\substack{d > x \\ d \text{ squarefull}}} \frac{|f_k(d)| \log d}{d}\right) \\ &:= F_k + O(x^{-1/2+o(1)})\end{aligned} \tag{9}$$

as $x \to \infty$, by a similar argument since $|f_k(d)| \log d \le d^{o(1)}$ as $d \to \infty$. Finally

$$\sum_{d \le x} |f_k(d)| \le x^{1/2+o(1)}, \tag{10}$$

again since $f_k(d) = 0$ if d is not squarefull. Collecting (8), (9) and (10) and putting them into (7) with $j = k$, we get

$$F_{k-1}(x) = \sum_{n \le x} \frac{f_{k-1}(n)}{n} = E_k \log x + F_k + O(x^{-1/2+o(1)}).$$

In a similar way,

$$G_{k-1}(x) := \sum_{n \le x} \frac{|f_{k-1}(n)|}{n} = E_k' \log x + F_k' + O(x^{-1/2+o(1)}).$$

for some (maybe different) constants E'_k and F'_k. We now apply Lemma 2 in order to find recursively $F_{k-2}(x), F_{k-3}(x), \ \ldots, \ F_0(x)$. We claim, by induction on j, that

$$F_{k-j}(x) = A_j(\log x)^j + B_j(\log x)^{j-1} + O((\log x)^{j-2}) \tag{11}$$

for $j = 2, \ldots, k$. At $j = 1$, this is so with $A_1 = E_k$, $B_1 = F_k$ and the error term is better, namely $O(x^{-1/2+o(1)})$. In order to realize the induction step from $j = 1$ to $j = 2$, we use the first part of Lemma 1 with $r = 1$, whereas for the induction step from $j \geq 2$ to $j+1$ we use the second part of Lemma 2 with $r = j$ and $t = j-1$. Assuming that (11) holds for $j \geq 1$, we have, by (7),

$$\begin{aligned} F_{k-j-1}(x) &= \sum_{d\leq x} \frac{f_{k-j-1}(d)}{d} = \sum_{d\leq x} \frac{f_{k-j}(d)}{d} \log(x/d) + \gamma \sum_{d\leq x} \frac{f_{k-j}(d)}{d} \\ &+ O\left(\frac{1}{x}\sum_{d\leq x} |f_{k-j}(d)|\right). \end{aligned}$$

By Lemma 2, we get that the right hand side is

$$\begin{aligned} & \frac{A_j}{j+1}(\log x)^{j+1} + \left(\frac{B_j}{j} + \gamma A_j\right)(\log x)^j \\ + \ & O\left((\log x)^{j-1} + \frac{1}{x}\sum_{d\leq x} |f_{k-j}(d)|\right) \\ & := A_{j+1}(\log x)^{j+1} + B_{j+1}(\log x)^j + O\left((\log x)^{j-1} + \frac{1}{x}\sum_{d\leq x} |f_{k-j}(d)|\right), \end{aligned}$$

where

$$A_{j+1} = \frac{A_j}{j+1}, \quad \text{and} \quad B_{j+1} = \gamma A_j + \frac{B_j}{j}.$$

Thus, we note that $A_j = E_k/j!$. It remains to deal with the sum in the error term. But the exact same approach applies to $|f_{k-j}(n)|$. That is $g_0(n) = |f_{k-j}(n)|$ satisfies the same conditions as our initial $f_0(n)$ with k replaced by $k-j$. Thus,

$$\sum_{d\leq x} \frac{|f_{k-j}(d)|}{d} = C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2}),$$

where for $j = 1$, the error term is $O(x^{-1/2+o(1)})$ as $x \to \infty$. By Abel summation, we get that

$$\begin{aligned} \sum_{d\leq x} |f_{k-j}(d)| &= x(C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2})) \\ &- \int_1^x (C_j(\log t)^j + D_j(\log t)^{j-1} + O((\log t)^{j-2}))dt \\ &= O(x(\log x)^{j-1}), \end{aligned}$$

which is sufficient for us. This completes the induction procedure and shows that at $j = k$ we have

$$\sum_{n \le x} \frac{f(n)}{n} = \frac{1}{k!} E_k (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}).$$

Abel summation formula once again gives

$$\begin{aligned}\sum_{n \le x} f(n) &= \left(\frac{E_k}{k!} (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}) \right) x \\ &- \int_1^x \left(\frac{E_k}{k!} (\log t)^k + B_k (\log t)^{k-1} + O((\log t)^{k-2}) \right) dt \\ &= \frac{E_k}{(k-1)!} x (\log x)^{k-1} + O(x (\log x)^{k-2}),\end{aligned}$$

which is what we wanted.

4 Acknowledgments

We thank the referee for a careful reading of the manuscript. This work was done when both authors visited the Max Planck Institute of Mathematics in Bonn, Germany in February 2017. They thank the Institution for the invitation and support.

References

[1] M. B. Nathanson, *Elementary Methods in Number Theory*, Graduate Texts in Mathematics, Vol. 195, Springer-Verlag, 2000.

[2] S. Ramanujan, Some formulæ in the analytic theory of numbers, *Messenger of Math.* **45** (1915), 81–84.

[3] B. M. Wilson, Proofs of some formulae enunciated by Ramanujan, *Proc. London Math. Soc.* **21** (1922), 235–255.

2010 *Mathematics Subject Classification*: Primary 11A35; Secondary 11N37.
Keywords: number of divisors, Möbius inversion.

(Concerned with sequence A000005.)

Received March 20 2017; revised versions received June 27 2017; June 28 2017. Published in *Journal of Integer Sequences*, July 2 2017.
