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Jacobsthal and Jacobsthal-Lucas Numbers and Sums Introduced by Jacobsthal and Tverberg

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Abstract

We study the sums introduced by Jacobsthal and Tverberg and show that the extreme values of the sums are connected with Jacobsthal and Jacobsthal-Lucas numbers.

1 Introduction

Let $a, b \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. In 1957, Jacobsthal [4] introduced the sums of the form

$$S_{a,b;m}(K) = \sum_{k=0}^{K} f_{a,b;m}(k),$$

where

$$f_{a,b;m}(k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \tag{1}$$

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In the above equation and throughout this article, unless stated otherwise, k is an integer and K is a nonnegative integer. So we can consider $f_{a,b;m}$ and $S_{a,b;m}$ as functions of k and K defined on $\mathbb{Z}$ and on $\mathbb{N} \cup \{0\}$, respectively.

These sums are also studied by Carlitz [1, 2], Grimson [3] and recently by Tverberg [6]. In addition, Tverberg [6] extends the definition of $f_{a,b;m}(k)$ and $S_{a,b;m}(K)$ to the following form.

Definition 1. Let m and ℓ be positive integers and let C be a multiset of ℓ integers $a_1, a_2, \ldots, a_\ell$, i.e., $a_i = a_j$ is allowed for some $i \neq j$. Define $f_{C;m} : \mathbb{Z} \to \mathbb{Z}$ and $S_{C;m} : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ by

$$f_{C;m}(k) = \sum_{T \subseteq [1,\ell]} (-1)^{\ell - |T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor,$$

$$S_{C;m}(K) = \sum_{k=0}^{K} f_{C;m}(k).$$

We sometimes write $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ and $S_{a_1,a_2,\ldots,a_\ell;m}(K)$ instead of $f_{C;m}(k)$ and $S_{C;m}(K)$, respectively. The set $[1, \ell]$ appearing in the sum defining f is $\{1, 2, 3, \ldots, \ell\}$ and if $T = \emptyset$, then $\sum_{i \in T} a_i$ is defined to be zero.

For example, if $C = \{a, b\}$, then $f_{C;m}(k)$ given in Definition 1 is the same as $f_{a,b;m}(k)$ given in (1), and if $C = \{a_1, a_2, a_3\}$, then $f_{C;m}(k)$ is

$$\begin{aligned} f_{a_1,a_2,a_3;m}(k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

Jacobsthal [4] shows that for any $K \in \mathbb{N} \cup \{0\}$, we have

$$0 \leq S_{a,b;m}(K) \leq \left\lfloor \frac{m}{2} \right\rfloor, \tag{2}$$

which is a sharp inequality, that is, the lower bound 0 is actually the minimum value and the upper bound $\left\lfloor \frac{m}{2} \right\rfloor$ is the maximum value of $S_{a,b;m}(K)$. Tverberg [6] proves (2) in a different way and he also gives the extreme values of $S_{a_1,a_2,a_3;m}(K)$ without proof. Nevertheless, the extreme values of $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ (for $\ell \geq 2$) and $S_{a_1,a_2,\ldots,a_\ell;m}(K)$ (for $\ell \geq 4$) have not been calculated.

In this article, we calculate the extreme values of $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ for all $\ell \geq 2$ (see Theorem 8). We also introduce the function g in Definition 2, give its connection with $f_{a_1,a_2,\ldots,a_\ell;m}(k)$, and obtain its extreme values (see Proposition 3 and Theorem 4). Furthermore, we obtain the minimum value of $S_{a_1,a_2,\ldots,a_\ell;m}(K)$ when ℓ is odd and the maximum value of $S_{a_1,a_2,\ldots,a_\ell;m}(K)$ when ℓ is even (see Theorem 9).

The reader will see that the extreme values of the functions g and $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ are connected with Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n defined, respectively, by the recurrence relations

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2,$$

and

$$j_0 = 2, \quad j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \quad \text{for } n \geq 2.$$

The sequences $(J_n)_{n\geq 0}$ and $(j_n)_{n\geq 0}$ are, respectively, A001045 and A014551 in the OEIS [5]. The function g is defined as follows:

Definition 2. Let $g : \mathbb{R}^n \to \mathbb{Z}$ be given by

$$\begin{aligned} g(x_1, x_2, x_3, \ldots, x_n) = & \sum_{1\leq i\leq n} \lfloor x_i \rfloor - \sum_{1\leq i_1<i_2\leq n} \lfloor x_{i_1} + x_{i_2} \rfloor \\ & + \sum_{1\leq i_1<i_2<i_3\leq n} \lfloor x_{i_1} + x_{i_2} + x_{i_3} \rfloor - \cdots + (-1)^{n-1} \lfloor x_1 + x_2 + x_3 + \cdots + x_n \rfloor. \end{aligned}$$

In other words,

$$g(x_1, x_2, x_3, \ldots, x_n) = \sum_{\emptyset\neq T\subseteq[1,n]} (-1)^{|T|-1} \left\lfloor \sum_{i\in T} x_i \right\rfloor.$$

2 Main results

We begin this section by giving a relation between the functions f and g. Then we give the extreme values of g and f and their connection with Jacobsthal and Jacobsthal-Lucas numbers.

Proposition 3. *For each $\ell \geq 2$, we have*

(i) $f_{a_1,a_2,\ldots,a_\ell;m}(0) = (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right)$,

(ii) $f_{a_1,a_2,\ldots,a_\ell;m}(k) = (-1)^{\ell} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right)$.

Proof. This follows easily from the definitions of f and g but we give a proof for completeness. We have

$$\begin{aligned} f_{a_1,a_2,\ldots,a_\ell;m}(0) &= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= \sum_{\emptyset\neq T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= (-1)^{\ell-1} \sum_{\emptyset\neq T\subseteq[1,\ell]} (-1)^{1-|T|} \left\lfloor \sum_{i\in T} \left(\frac{a_i}{m}\right) \right\rfloor \\ &= (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_\ell}{m}\right). \end{aligned}$$

Next let $a_{\ell+1} = k$. Then we obtain

$$\begin{aligned}
&(-1)^{\ell} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}, \frac{k}{m}\right) + (-1)^{\ell-1} g\left(\frac{a_1}{m}, \frac{a_2}{m}, \cdots, \frac{a_\ell}{m}\right) \\
=&(-1)^{\ell}\left(\sum_{\emptyset \neq T \subseteq [1,\ell+1]} (-1)^{|T|-1} \left\lfloor \sum_{i \in T} \left(\frac{a_i}{m}\right) \right\rfloor - \sum_{\emptyset \neq T \subseteq [1,\ell]} (-1)^{|T|-1} \left\lfloor \sum_{i \in T} \left(\frac{a_i}{m}\right) \right\rfloor \right) \\
=&(-1)^{\ell} \sum_{\substack{T \subseteq [1,\ell+1] \\ \ell+1 \in T}} (-1)^{|T|-1} \left\lfloor \sum_{i \in T} \left(\frac{a_i}{m}\right) \right\rfloor \\
=&(-1)^{\ell} \sum_{T \subseteq [1,\ell]} (-1)^{|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor \\
=&f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

□

Theorem 4. *For each $n \geq 2$, the function g given in Definition 2 has maximum value $2^{n-2} - 1$ and minimum value -2^{n-2}. The minimum occurs at least when $x_k = \frac{1}{2}$ for every $1 \leq k \leq n$. The maximum occurs at least when $x_k = \frac{1}{2} - \frac{1}{n^2}$ for every $1 \leq k \leq n$.*

Proof. If $n = 2$, then the result is a well-known inequality

$$-1 \leq \lfloor x \rfloor + \lfloor y \rfloor - \lfloor x + y \rfloor \leq 0, \tag{3}$$

which holds for all $x, y \in \mathbb{R}$. The inequality (3) is sharp: if $x = y = \frac{1}{2}$ the left inequality in (3) becomes equality, and if $x = y = \frac{1}{4}$ the right inequality in (3) becomes equality. The result when $n \geq 3$ is obtained from the case $n = 2$ and a careful selection of pairs. For illustration purpose, we first give a proof for the case $n = 3$ and $n = 4$. Recall that

$$g(x_1, x_2, x_3) = \lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_1 + x_3 \rfloor - \lfloor x_2 + x_3 \rfloor + \lfloor x_1 + x_2 + x_3 \rfloor.$$

We obtain by (3) that

$$0 \leq \lfloor x_1 + x_2 + x_3 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_3 \rfloor \leq 1, \tag{4}$$

$$-1 \leq -\lfloor x_2 + x_3 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor \leq 0, \tag{5}$$

$$-1 \leq -\lfloor x_1 + x_3 \rfloor + \lfloor x_1 \rfloor + \lfloor x_3 \rfloor \leq 0. \tag{6}$$

Summing (4), (5), and (6), the middle terms give $g(x_1, x_2, x_3)$. Then $-2 \leq g(x_1, x_2, x_3) \leq 1$. Next we consider

$$\begin{aligned}
g(x_1, x_2, x_3, x_4) = &\lfloor x_1 \rfloor + \lfloor x_2 \rfloor + \lfloor x_3 \rfloor + \lfloor x_4 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_1 + x_3 \rfloor - \lfloor x_1 + x_4 \rfloor \\
&- \lfloor x_2 + x_3 \rfloor - \lfloor x_2 + x_4 \rfloor - \lfloor x_3 + x_4 \rfloor + \lfloor x_1 + x_2 + x_3 \rfloor + \lfloor x_1 + x_2 + x_4 \rfloor \\
&+ \lfloor x_1 + x_3 + x_4 \rfloor + \lfloor x_2 + x_3 + x_4 \rfloor - \lfloor x_1 + x_2 + x_3 + x_4 \rfloor.
\end{aligned}$$

Again, we obtain by (3) the following inequalities:

$$-1 \leq -\lfloor x_1 + x_2 + x_3 + x_4 \rfloor + \lfloor x_1 + x_2 + x_3 \rfloor + \lfloor x_4 \rfloor \leq 0, \tag{7}$$

$$0 \leq \lfloor x_1 + x_2 + x_4 \rfloor - \lfloor x_1 + x_2 \rfloor - \lfloor x_4 \rfloor \leq 1, \tag{8}$$

$$0 \leq \lfloor x_1 + x_3 + x_4 \rfloor - \lfloor x_1 + x_3 \rfloor - \lfloor x_4 \rfloor \leq 1, \tag{9}$$

$$0 \leq \lfloor x_2 + x_3 + x_4 \rfloor - \lfloor x_2 + x_3 \rfloor - \lfloor x_4 \rfloor \leq 1, \tag{10}$$

$$-1 \leq -\lfloor x_1 + x_4 \rfloor + \lfloor x_1 \rfloor + \lfloor x_4 \rfloor \leq 0, \tag{11}$$

$$-1 \leq -\lfloor x_2 + x_4 \rfloor + \lfloor x_2 \rfloor + \lfloor x_4 \rfloor \leq 0, \tag{12}$$

$$-1 \leq -\lfloor x_3 + x_4 \rfloor + \lfloor x_3 \rfloor + \lfloor x_4 \rfloor \leq 0. \tag{13}$$

Summing (7) to (13), we see that $-4 \leq g(x_1, x_2, x_3, x_4) \leq 3$.

Next we prove the general case $n \geq 5$. The expression of the form $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_k} \rfloor$ will be called a *k-bracket*. So for each $1 \leq k \leq n$, there are $\binom{n}{k}$ k-brackets appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$. We first pair up the n-bracket with an $(n-1)$-bracket and a 1-bracket as follows:

$$s_1 = (-1)^{n-1} \lfloor x_1 + x_2 + \cdots + x_n \rfloor + (-1)^{n-2} \lfloor x_1 + x_2 + \cdots + x_{n-1} \rfloor + (-1)^{n-2} \lfloor x_n \rfloor. \tag{14}$$

Notice that the sign of $\lfloor x_n \rfloor$ in (14) may or may not be the same as that appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$ but it is the same as the sign of $\lfloor x_1 + x_2 + \cdots + x_{n-1} \rfloor$ so that we can apply (3) to obtain the bound for s_1. Next we pair up the remaining $(n-1)$-brackets with $(n-2)$-brackets and 1-brackets as follows:

$$(-1)^{n-2} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}} \rfloor + (-1)^{n-3} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rfloor + (-1)^{n-3} \lfloor x_{i_{n-1}} \rfloor, \tag{15}$$

where $1 \leq i_1 < i_2 < \ldots < i_{n-1} \leq n$. We note again that the sign of $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}} \rfloor$ and $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rfloor$ in (15) are the same as those appearing in the sum defining $g(x_1, x_2, \ldots, x_n)$ while the sign of $\lfloor x_{i_{n-1}} \rfloor$ in (15) may or may not be the same, but we can apply (3) to obtain the bound of (15). Since $\lfloor x_1 + x_2 + \cdots + x_{n-1} \rfloor$ appears in (14), the term $x_{i_{n-1}}$ appearing in the $(n-1)$-brackets in (15) is always x_n. So in fact (15) is

$$(-1)^{n-2} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n \rfloor + (-1)^{n-3} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rfloor + (-1)^{n-3} \lfloor x_n \rfloor. \tag{16}$$

Then we sum (16) over all possibles $1 \leq i_1 < i_2 < \ldots < i_{n-2} < n$, and call it s_2. That is

$$s_2 = (-1)^{n-2} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-2} < n} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} + x_n \rfloor \\ + (-1)^{n-3} \sum_{1 \leq i_1 < i_2 < \ldots < i_{n-2} < n} \lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \rfloor + (-1)^{n-3} \binom{n-1}{n-2} \lfloor x_n \rfloor.$$

We continue doing this process as follows. For each $0 \leq \ell \leq n-1$, let c_ℓ be the sum of all $\lfloor x_{i_1} + x_{i_2} + \cdots + x_{i_{n-\ell}} \rfloor$ with $1 \leq i_1 < i_2 < \ldots < i_{n-\ell} \leq n$, a_ℓ the sum of all such terms with $i_{n-\ell} = n$, and b_ℓ the sum of all such terms with $i_{n-\ell} < n$. Therefore $c_\ell = a_\ell + b_\ell$. As usual, the empty sum is defined to be zero, so $b_0 = 0$. The number of $(n-\ell)$-brackets appearing in the sum defining c_ℓ is $\binom{n}{n-\ell}$, the number of $(n-\ell)$-brackets appearing in the sum defining a_ℓ is $\binom{n-1}{n-\ell-1}$, and the number of $(n-\ell)$-brackets appearing in the sum defining b_ℓ is $\binom{n-1}{n-\ell}$. In addition, we have

$$s_1 = (-1)^{n-1} a_0 + (-1)^{n-2} b_1 + (-1)^{n-2} \lfloor x_n \rfloor,$$
$$s_2 = (-1)^{n-2} a_1 + (-1)^{n-3} b_2 + (-1)^{n-3} \binom{n-1}{n-2} \lfloor x_n \rfloor.$$

In general, for each $1 \leq \ell \leq n-1$, we let

$$s_\ell = (-1)^{n-\ell} a_{\ell-1} + (-1)^{n-\ell-1} b_\ell + (-1)^{n-\ell-1} \binom{n-1}{n-\ell} \lfloor x_n \rfloor.$$

Then

$$\sum_{1 \leq \ell \leq n-1} s_\ell = (-1)^{n-1} a_0 + \sum_{2 \leq \ell \leq n-1} (-1)^{n-\ell} a_{\ell-1} + \sum_{1 \leq \ell \leq n-2} (-1)^{n-\ell-1} b_\ell + b_{n-1} \\ + \lfloor x_n \rfloor \sum_{1 \leq \ell \leq n-1} (-1)^{n-\ell-1} \binom{n-1}{n-\ell}. \tag{17}$$

Recall a well known identity $\sum_{0 \leq \ell \leq n} (-1)^\ell \binom{n}{\ell} = 0$ for all $n \geq 1$. Therefore the last sum on the right hand side of (17) is

$$-\sum_{1 \leq \ell \leq n-1} (-1)^{n-\ell} \binom{n-1}{n-\ell} = -\sum_{1 \leq \ell \leq n-1} (-1)^\ell \binom{n-1}{\ell} = -\sum_{0 \leq \ell \leq n-1} (-1)^\ell \binom{n-1}{\ell} + 1 = 1.$$

Therefore the last term in (17) is $\lfloor x_n \rfloor$. Replacing ℓ by $\ell+1$ in the first sum on the right hand side of (17), we see that

$$
\begin{aligned}
\sum_{1\le \ell\le n-1} s_\ell &= (-1)^{n-1}a_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}(a_\ell+b_\ell) + b_{n-1} + \lfloor x_n\rfloor \\
&= (-1)^{n-1}c_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}c_\ell + b_{n-1} + \lfloor x_n\rfloor \\
&= (-1)^{n-1}c_0 + \sum_{1\le \ell\le n-2}(-1)^{n-\ell-1}c_\ell + c_{n-1} \qquad (18)\\
&= \sum_{0\le \ell\le n-1}(-1)^{n-\ell-1}c_\ell \\
&= g(x_1,x_2,\ldots,x_n),
\end{aligned}
$$

where (18) can be obtained from the definition of c_{n-1}, b_{n-1}, and a_{n-1} that

$$
\begin{aligned}
c_{n-1} &= \lfloor x_1\rfloor + \lfloor x_2\rfloor + \cdots + \lfloor x_n\rfloor, \\
b_{n-1} &= \lfloor x_1\rfloor + \lfloor x_2\rfloor + \cdots + \lfloor x_{n-1}\rfloor, \\
a_{n-1} &= \lfloor x_n\rfloor, \quad \text{and} \\
c_{n-1} &= a_{n-1} + b_{n-1}.
\end{aligned}
$$

We apply (3) to (14) to obtain

$$0 \le s_1 \le 1 \text{ if } n \text{ is odd, and } -1 \le s_1 \le 0 \text{ if } n \text{ is even.}$$

Similarly, applying (3) to (16), we see that such sum lies in $[0,1]$ if n is even, and lies in $[-1,0]$ if n is odd. Therefore

$$0 \le s_2 \le \binom{n-1}{n-2} \text{ if } n \text{ is even, and } -\binom{n-1}{n-2} \le s_2 \le 0 \text{ if } n \text{ is odd.}$$

In general, for each $1 \le \ell \le n-1$, we have

$$
\begin{aligned}
0 \le s_\ell \le \binom{n-1}{n-\ell}&, \text{ if } n \text{ and } \ell \text{ have the same parity,} \\
-\binom{n-1}{n-\ell} \le s_\ell \le 0&, \text{ if } n \text{ and } \ell \text{ have a different parity.}
\end{aligned}
$$

Since $g(x_1,x_2,\ldots,x_n) = \sum_{1\le \ell\le n-1} s_\ell$, we obtain, for odd n,

$$-\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{n-\ell} \le g(x_1,x_2,\ldots,x_n) \le \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{n-\ell},$$

and for even n,

$$-\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}}\binom{n-1}{n-\ell} \le g(x_1,x_2,\ldots,x_n) \le \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}}\binom{n-1}{n-\ell}.$$

Recall a well known identity

$$\sum_{\substack{0\le k\le n\\ k \text{ is even}}} \binom{n}{k} = \sum_{\substack{0\le k\le n\\ k \text{ is odd}}} \binom{n}{k} = 2^{n-1}.$$

Therefore if n is odd, then

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}} \binom{n-1}{n-\ell} = \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}} \binom{n-1}{\ell} = 2^{n-2}-1, \text{ and}$$

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}} \binom{n-1}{n-\ell} = \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}} \binom{n-1}{\ell} = \sum_{\substack{0\le \ell\le n-1\\ \ell \text{ is odd}}} \binom{n-1}{\ell} = 2^{n-2}.$$

Similarly, if n is even, then

$$\sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is odd}}} \binom{n-1}{n-\ell} = 2^{n-2} \text{ and } \sum_{\substack{1\le \ell\le n-1\\ \ell \text{ is even}}} \binom{n-1}{n-\ell} = 2^{n-2}-1.$$

Hence $-2^{n-2} \le g(x_1, x_2, \ldots, x_n) \le 2^{n-2}-1$, as required. Next we show that the lower bound -2^{n-2} and the upper bound $2^{n-2}-1$ are actually the minimum and the maximum of $g(x_1, x_2, \ldots, x_n)$, respectively. Recall that the fractional part of a real number x, denoted by $\{x\}$, is defined by $\{x\} = x - \lfloor x \rfloor$. Let $x_k = \frac{1}{2}$ for every $k = 1, 2, \ldots, n$. Then

$$\begin{aligned} g(x_1, x_2, \ldots, x_n) &= \sum_{1\le k\le n} (-1)^{k-1} \left\lfloor \frac{k}{2} \right\rfloor \binom{n}{k} \\ &= \sum_{1\le k\le n} (-1)^{k-1} \left(\frac{k}{2}\right) \binom{n}{k} - \sum_{1\le k\le n} (-1)^{k-1} \left\{\frac{k}{2}\right\} \binom{n}{k} \\ &= \frac{1}{2} \sum_{1\le k\le n} (-1)^{k-1} k \binom{n}{k} - \frac{1}{2} \sum_{\substack{1\le k\le n\\ k \text{ is odd}}} \binom{n}{k}, \end{aligned} \tag{19}$$

where the last equality is obtained from the fact that $\left\{\frac{k}{2}\right\} = 0$ if k is even and $\left\{\frac{k}{2}\right\} = \frac{1}{2}$ if k is odd. By differentiating both sides of

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{20}$$

and substituting $x = -1$, we obtain a well-known identity

$$\sum_{k=1}^{n} (-1)^{k-1} k \binom{n}{k} = 0, \text{ which holds for all } n \ge 2. \tag{21}$$

In addition, we know that

$$\sum_{\substack{1\le k\le n\\ k \text{ is odd}}} \binom{n}{k} = 2^{n-1}.$$

Therefore (19) becomes

$$g(x_1, x_2, \ldots, x_n) = 0 - \frac{1}{2}\left(2^{n-1}\right) = -2^{n-2}.$$

This shows that -2^{n-2} is the minimun value of g. Next let $x_k = \frac{1}{2} - \frac{1}{n^2}$ for every $k = 1, 2, \ldots, n$. Then

$$g(x_1, x_2, \ldots, x_n) = \sum_{1\le k\le n} (-1)^{k-1} \left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor \binom{n}{k}. \tag{22}$$

If $1 \le k \le n$ and k is even, then $\left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor = \frac{k}{2} - 1 = \left\lfloor \frac{k-1}{2} \right\rfloor$. If $1 \le k \le n$ and k is odd, then $\left\lfloor \frac{k}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k-1}{2} + \frac{1}{2} - \frac{k}{n^2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor$. Therefore (22) becomes

$$g(x_1, x_2, \ldots, x_n) = \sum_{1\le k\le n} (-1)^{k-1} \left\lfloor \frac{k-1}{2} \right\rfloor \binom{n}{k}. \tag{23}$$

Now we can evaluate the sum (23) by using the same method as in (19). We write $\left\lfloor \frac{k-1}{2} \right\rfloor = \frac{k-1}{2} - \left\{\frac{k-1}{2}\right\}$ and we know that $\left\{\frac{k-1}{2}\right\} = 0$ if k is odd and $\left\{\frac{k-1}{2}\right\} = \frac{1}{2}$ if k is even. Then (23) can be written as

$$g(x_1, x_2, \ldots, x_n) = \frac{1}{2} \sum_{1\le k\le n} (-1)^{k-1} k \binom{n}{k} - \frac{1}{2} \sum_{1\le k\le n} (-1)^{k-1} \binom{n}{k} + \frac{1}{2} \sum_{\substack{1\le k\le n\\ k \text{ is even}}} \binom{n}{k}.$$

The first sum is zero by (21). The second sum is 1 by substituting $x = -1$ in (20). Therefore

$$g(x_1, x_2, \ldots, x_n) = 0 - \frac{1}{2} + \frac{1}{2}\left(2^{n-1} - 1\right) = 2^{n-2} - 1.$$

□

Recall that the Binet forms of Jacobsthal numbers J_n and Jacobsthal-Lucas numbers j_n are

$$J_n = \frac{2^n - (-1)^n}{3} \quad \text{and} \quad j_n = 2^n + (-1)^n \tag{24}$$

for every $n \ge 0$. Therefore we obtain the connection between Jacobsthal and Jacobsthal-Lucas numbers and sums introduced by Jacobsthal [4] and Tverberg [6] as follows.

Corollary 5. *If n is odd, then the maximum and the minimum value of $g(x_1, x_2, x_3, \ldots, x_n)$ are j_{n-2} and $-1 - j_{n-2}$, respectively. If n is even, then the maximum and the minimum value of $g(x_1, x_2, x_3, \ldots, x_n)$ are $3J_{n-2}$ and $1 - j_{n-2}$, respectively.*

Proof. This follows immediately from (24) and Theorem 4. □

Remark 6. From this point on, we will apply the well-known identities which are already recalled without reference.

Next we give the extreme values of $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. Although we can write $f_{a_1,a_2,\ldots,a_\ell;m}(k)$ in terms of $g(x_1,x_2,\ldots,x_n)$ as given in Proposition 3, we do not know the proof which applies Theorem 4 to obtain Theorem 8. Nevertheless, we can use the same idea in the proof of Theorem 4 together with the following lemma to prove Theorem 8.

Lemma 7. *The following statements hold.*

(i) *For each $i \in \{1,2,\ldots,n\}$ and $q \in \mathbb{Z}$, we have*

$$g(x_1,x_2,\ldots,x_i+q,\ldots,x_n) = g(x_1,x_2,\ldots,x_n).$$

In particular, g has period 1 in each variable.

(ii) *For each $i \in \{1,2,\ldots,\ell\}$ and $q \in \mathbb{Z}$, we have*

$$f_{a_1,a_2,\ldots,a_i+qm,\ldots,a_\ell;m}(k) = f_{a_1,a_2,\ldots,a_\ell;m}(k) = f_{a_1,a_2,\ldots,a_\ell;m}(k+qm).$$

In particular, f has period m in each variable $a_1,a_2,\ldots,a_\ell$ and k.

Proof. Since $\lfloor q+x\rfloor = q+\lfloor x\rfloor$ for every $q\in\mathbb{Z}$ and $x\in\mathbb{R}$, we see that

$$\begin{aligned}
g(x_1,x_2,\ldots,x_i+q,\ldots,x_n) &= \left(q+\sum_{i=1}^{n}\lfloor x_i\rfloor\right) - \left(\binom{n-1}{1}q + \sum_{1\le i_1<i_2\le n}\lfloor x_{i_1}+x_{i_2}\rfloor\right)\\
&\quad + \left(\binom{n-1}{2}q + \sum_{1\le i_1<i_2<i_3\le n}\lfloor x_{i_1}+x_{i_2}+x_{i_3}\rfloor\right)\\
&\quad - \cdots + (-1)^{n-1}\left(\binom{n-1}{n-1}q + \lfloor x_1+x_2+\cdots+x_n\rfloor\right)\\
&= g(x_1,x_2,\ldots,x_n) + q\sum_{0\le k\le n-1}(-1)^k\binom{n-1}{k}\\
&= g(x_1,x_2,\ldots,x_n).
\end{aligned}$$

This proves (i). Next we prove (ii). By Proposition 3 and by (i), we obtain

$$\begin{aligned}
f_{a_1,a_2,\ldots,a_i+qm,\ldots,a_\ell;m}(k) &= (-1)^\ell g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_i}{m}+q,\ldots,\frac{a_\ell}{m},\frac{k}{m}\right)\\
&\quad + (-1)^{\ell-1} g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_i}{m}+q,\ldots,\frac{a_\ell}{m}\right)\\
&= (-1)^\ell g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_\ell}{m},\frac{k}{m}\right) + (-1)^{\ell-1} g\left(\frac{a_1}{m},\frac{a_2}{m},\ldots,\frac{a_\ell}{m}\right)\\
&= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

Similarly, $f_{a_1,a_2,\ldots,a_\ell;m}(k+qm) = f_{a_1,a_2,\ldots,a_\ell;m}(k)$. This completes the proof. □

Theorem 8. *For each $\ell \geq 2$, $a_1, a_2, \ldots, a_\ell, k \in \mathbb{Z}$ and $m \geq 1$, we have*

$$-2^{\ell-2} \leq f_{a_1,a_2,\ldots,a_\ell;m}(k) \leq 2^{\ell-2}.$$

Moreover, $-2^{\ell-2}$ and $2^{\ell-2}$ are best possible in the sense that there are $a_1, a_2, \ldots, a_\ell, m, k$ which make the inequality becomes equality. More precisely the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = -2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = 2^{\ell-2}$.*

(ii) *If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2}$.*

Proof. By Lemma 7, we can assume that $a_i \in [0, m-1]$ for every $1 \leq i \leq \ell$. Therefore

$$\left\lfloor \frac{a_i}{m} \right\rfloor = 0 \text{ for every } i \in \{1, 2, \ldots, \ell\}. \tag{25}$$

If $\ell = 2$, then the result follows from (25) and (3), and we have

$$0 \leq \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 1, \tag{26}$$

and

$$-1 \leq -\left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \leq 0. \tag{27}$$

Summing (26) and (27), we obtain $-1 \leq f_{a_1,a_2;m}(k) \leq 1$. The result when $\ell \geq 3$ is based on a careful selection of pairs and the case $\ell = 2$. For illustration purpose, we first give a proof for the case $\ell = 3$ and $\ell = 4$. Recall that

$$\begin{aligned} f_{a_1,a_2,a_3;m}(k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_1 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

We obtain by (3) and (25) that

$$0 \leq \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \leq 1, \tag{28}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 0, \tag{29}$$

$$-1 \leq -\left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + k}{m} \right\rfloor \leq 0, \tag{30}$$

$$0 \leq \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \leq 1. \tag{31}$$

Summing (28), (29), (30), and (31), we see that the middle term is $f_{a_1,a_2,a_3,m}(k)$. Therefore $-2 \leq f_{a_1,a_2,a_3;m}(k) \leq 2$. Next we consider

$$\begin{aligned} f_{a_1,a_2,a_3,a_4;m}(k) = &\left\lfloor \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + a_4 + k}{m} \right\rfloor \\ &- \left\lfloor \frac{a_1 + a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_4 + k}{m} \right\rfloor \\ &+ \left\lfloor \frac{a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_3 + k}{m} \right\rfloor - \left\lfloor \frac{a_4 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \end{aligned}$$

Again, we obtain by (3) and (25) the following inequalities:

$$0 \leq \left\lfloor \frac{a_1 + a_2 + a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + a_3 + k}{m} \right\rfloor \leq 1, \tag{32}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_2 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor \leq 0, \tag{33}$$

$$-1 \leq -\left\lfloor \frac{a_1 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_1 + a_3 + k}{m} \right\rfloor \leq 0, \tag{34}$$

$$-1 \leq -\left\lfloor \frac{a_2 + a_3 + a_4 + k}{m} \right\rfloor + \left\lfloor \frac{a_2 + a_3 + k}{m} \right\rfloor \leq 0, \tag{35}$$

$$0 \leq \left\lfloor \frac{a_1 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor \leq 1, \tag{36}$$

$$0 \leq \left\lfloor \frac{a_2 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor \leq 1, \tag{37}$$

$$0 \leq \left\lfloor \frac{a_3 + a_4 + k}{m} \right\rfloor - \left\lfloor \frac{a_3 + k}{m} \right\rfloor \leq 1, \tag{38}$$

$$-1 \leq -\left\lfloor \frac{a_4 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \leq 0. \tag{39}$$

Summing (32) to (39), we see that $-4 \leq f_{a_1,a_2,a_3,a_4,m}(k) \leq 4$.

Next we prove the general case $\ell \geq 5$. The expression of the form $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ will be called an r-*bracket*. So for each $1 \leq r \leq \ell$, there are $\binom{\ell}{r}$ r-brackets appearing in the sum defining $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. We follow closely the method used in the proof of Theorem 4. So we first pair up the ℓ-bracket with an $(\ell-1)$-bracket as follows:

$$s_1 = \left\lfloor \frac{a_1 + a_2 + \cdots + a_\ell + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + a_2 + \cdots + a_{\ell-1} + k}{m} \right\rfloor, \tag{40}$$

and we can apply (3) and (25) to obtain the bound for s_1. Next we pair up the remaining $(\ell-1)$-brackets with $(\ell-2)$-brackets as follows:

$$-\left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-1}} + k}{m} \right\rfloor + \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right\rfloor, \tag{41}$$

and we sum (41) over all $1 \le i_1 < i_2 < \ldots < i_{\ell-1} \le \ell$ and call it s_2. Since a_ℓ does not appear in the second term on the right hand side of (40), the term $a_{i_{\ell-1}}$ appearing in (41) is always a_ℓ. So in fact

$$\begin{aligned} s_2 = &-\sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + a_\ell + k}{m} \right\rfloor \\ &+ \sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} \left\lfloor \frac{a_{i_1} + a_{i_2} + \cdots + a_{i_{\ell-2}} + k}{m} \right\rfloor. \end{aligned}$$

We continue doing this process as follows. For each $1 \le r \le \ell$, let c_r be the sum of all $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ with $1 \le i_1 < i_2 < \cdots < i_r \le \ell$, a_r the sum of all such terms with $i_r = \ell$, and b_r the sum of all such terms with $i_r < \ell$. Therefore $c_r = a_r + b_r$, the number of summands of c_r is $\binom{\ell}{r}$, the number of summands of a_r is $\binom{\ell-1}{r-1}$, and the number of summands of b_r is $\binom{\ell-1}{r}$. As usual, the empty sum is defined to be zero, so $b_\ell = 0$. We have $s_1 = a_\ell - b_{\ell-1}$ and $s_2 = -a_{\ell-1} + b_{\ell-2}$. In general, for each $1 \le r \le \ell - 1$, we let

$$s_r = (-1)^{r+1} a_{\ell-r+1} + (-1)^r b_{\ell-r} \text{ and } s_\ell = (-1)^{\ell+1} a_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor.$$

Then

$$0 \le s_r \le \binom{\ell-1}{\ell-r} \text{ if } r \text{ is odd, and } -\binom{\ell-1}{\ell-r} \le s_r \le 0 \text{ if } r \text{ is even,}$$

$$\begin{aligned}
\sum_{1\le r\le \ell} s_r &= a_\ell + \sum_{2\le r\le \ell-1} (-1)^{r+1} a_{\ell-r+1} + \sum_{1\le r\le \ell-2} (-1)^r b_{\ell-r} + (-1)^{\ell-1} b_1 + s_\ell \\
&= a_\ell + \sum_{1\le r\le \ell-2} (-1)^r (a_{\ell-r} + b_{\ell-r}) + (-1)^{\ell-1} b_1 + (-1)^{\ell+1} a_1 + \left\lfloor \frac{k}{m} \right\rfloor \\
&= c_\ell + \sum_{1\le r\le \ell-2} (-1)^r c_{\ell-r} + (-1)^{\ell-1} c_1 + \left\lfloor \frac{k}{m} \right\rfloor \\
&= \sum_{0\le r\le \ell-1} (-1)^r c_{\ell-r} + \left\lfloor \frac{k}{m} \right\rfloor \\
&= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

Therefore

$$-\sum_{\substack{1\le r\le \ell \\ r \text{ is even}}} \binom{\ell-1}{\ell-r} \le f_{a_1,a_2,\ldots,a_\ell;m}(k) \le \sum_{\substack{1\le r\le \ell \\ r \text{ is odd}}} \binom{\ell-1}{\ell-r}.$$

Replacing r by $r+1$, we see that

$$\sum_{\substack{1\le r\le \ell \\ r \text{ is odd}}} \binom{\ell-1}{\ell-r} = \sum_{\substack{0\le r\le \ell-1 \\ r \text{ is even}}} \binom{\ell-1}{\ell-1-r} = 2^{\ell-2}.$$

Similarly,

$$-\sum_{\substack{1\le r\le \ell \\ r \text{ is even}}} \binom{\ell-1}{\ell-r} = -2^{\ell-2}.$$

Therefore

$$-2^{\ell-2} \le f_{a_1,a_2,\ldots,a_\ell;m}(k) \le 2^{\ell-2}, \tag{42}$$

as required. If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$, we obtain by Proposition 3 and Theorem 4 that $f_{a_1,a_2,\ldots,a_\ell;m}(0) = g\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) = -2^{\ell-2}$and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = (-1)^\ell g\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) + (-1)^{\ell-1} g\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) = 2^{\ell-2}$. If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$, we obtain similarly that $f_{a_1,a_2,\ldots,a_\ell;m}(0) = 2^{\ell-2}$ and $f_{a_1,a_2,\ldots,a_\ell;m}(\frac{m}{2}) = -2^{\ell-2}$. So $2^{\ell-2}$ and $-2^{\ell-2}$ in (42) cannot be improved. This completes The proof. □

We obtain the extreme values of $S_{a_1,a_2,\ldots,a_\ell;m}$ for some cases $\ell \ge 4$ as well. More precisely, we have the following result.

Theorem 9. *For each $\ell \ge 2$, $a_1, a_2, \ldots, , a_\ell \in \mathbb{Z}$, $m \in \mathbb{N}$, and $K \in \mathbb{N} \cup \{0\}$, we have*

$$-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor \le S_{a_1,a_2,\ldots,a_\ell;m}(K) \le 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor. \tag{43}$$

Moreover, If ℓ is odd, then the lower bound $-2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ is sharp and if ℓ is even, then the upper bound $2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$ is sharp in the sense that there are $a_1, a_2, \ldots, a_\ell, m, k$ which make the inequality becomes equality. More precisely, the following statements hold.

(i) *If ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $S_{a_1,a_2,\ldots,a_\ell;m}(K) = -2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$.*

(ii) *If ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $i = 1, 2, \ldots, \ell$, then $S_{a_1,a_2,\ldots,a_\ell;m}(K) = 2^{\ell-2}\left\lfloor\frac{m}{2}\right\rfloor$.*

Proof. If $\ell = 2$, then the result is already proved by Jacobsthal [4]. See also another proof by Tverberg [6]. We recall the result when $\ell = 2$ for easy reference as follows:

$$0 \leq S_{a,b;m}(K) \leq \left\lfloor\frac{m}{2}\right\rfloor. \tag{44}$$

As before the result when $\ell \geq 3$ is based on the case $\ell = 2$ and a careful selection of pairs, and we first illustrate the idea by giving the proof for the case $\ell = 3$ and $\ell = 4$. Recall that

$$\begin{aligned} f_{a_1,a_2,a_3;m}(k) = &\left\lfloor\frac{a_1+a_2+a_3+k}{m}\right\rfloor - \left\lfloor\frac{a_1+a_2+k}{m}\right\rfloor - \left\lfloor\frac{a_1+a_3+k}{m}\right\rfloor - \left\lfloor\frac{a_2+a_3+k}{m}\right\rfloor \\ &+ \left\lfloor\frac{a_1+k}{m}\right\rfloor + \left\lfloor\frac{a_2+k}{m}\right\rfloor + \left\lfloor\frac{a_3+k}{m}\right\rfloor - \left\lfloor\frac{k}{m}\right\rfloor. \end{aligned}$$

We have

$$f_{a_1+a_2,a_3;m}(k) = \left\lfloor\frac{a_1+a_2+a_3+k}{m}\right\rfloor - \left\lfloor\frac{a_1+a_2+k}{m}\right\rfloor - \left\lfloor\frac{a_3+k}{m}\right\rfloor + \left\lfloor\frac{k}{m}\right\rfloor, \tag{45}$$

$$-f_{a_1,a_3;m}(k) = -\left\lfloor\frac{a_1+a_3+k}{m}\right\rfloor + \left\lfloor\frac{a_1+k}{m}\right\rfloor + \left\lfloor\frac{a_3+k}{m}\right\rfloor - \left\lfloor\frac{k}{m}\right\rfloor, \tag{46}$$

$$-f_{a_2,a_3;m}(k) = -\left\lfloor\frac{a_2+a_3+k}{m}\right\rfloor + \left\lfloor\frac{a_2+k}{m}\right\rfloor + \left\lfloor\frac{a_3+k}{m}\right\rfloor - \left\lfloor\frac{k}{m}\right\rfloor. \tag{47}$$

Summing (45), (46), and (47), we see that

$$f_{a_1,a_2,a_3;m}(k) = f_{a_1+a_2,a_3;m}(k) - f_{a_1,a_3;m}(k) - f_{a_2,a_3;m}(k). \tag{48}$$

By the definition of $S_{a_1,a_2,a_3;m}(K)$, (48), and (44), we obtain

$$\begin{aligned} S_{a_1,a_2,a_3;m}(K) &= \sum_{k=0}^{K} f_{a_1,a_2,a_3;m}(k) \\ &= \sum_{k=0}^{K} f_{a_1+a_2,a_3;m}(k) - \sum_{k=0}^{K} f_{a_1,a_3;m}(k) - \sum_{k=0}^{K} f_{a_2,a_3;m}(k) \\ &= S_{a_1+a_2,a_3;m}(K) - S_{a_1,a_3;m}(K) - S_{a_2,a_3;m}(K) \\ &\geq 0 - \left\lfloor\frac{m}{2}\right\rfloor - \left\lfloor\frac{m}{2}\right\rfloor = -2\left\lfloor\frac{m}{2}\right\rfloor. \end{aligned}$$

Similarly,

$$S_{a_1,a_2,a_3;m}(K) \le \left\lfloor \frac{m}{2} \right\rfloor - 0 - 0 = \left\lfloor \frac{m}{2} \right\rfloor \le 2 \left\lfloor \frac{m}{2} \right\rfloor.$$

Similarly, we have the following equalities:

$$f_{a_1+a_2+a_3,a_4;m}(k) = \left\lfloor \frac{a_1+a_2+a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \quad (49)$$

$$-f_{a_1+a_2,a_4;m}(k) = -\left\lfloor \frac{a_1+a_2+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \quad (50)$$

$$-f_{a_1+a_3,a_4;m}(k) = -\left\lfloor \frac{a_1+a_3+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_1+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \quad (51)$$

$$-f_{a_2+a_3,a_4;m}(k) = -\left\lfloor \frac{a_2+a_3+a_4+k}{m} \right\rfloor + \left\lfloor \frac{a_2+a_3+k}{m} \right\rfloor + \left\lfloor \frac{a_4+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \quad (52)$$

$$f_{a_1,a_4;m}(k) = \left\lfloor \frac{a_1+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \quad (53)$$

$$f_{a_2,a_4;m}(k) = \left\lfloor \frac{a_2+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor, \quad (54)$$

$$f_{a_3,a_4;m}(k) = \left\lfloor \frac{a_3+a_4+k}{m} \right\rfloor - \left\lfloor \frac{a_3+k}{m} \right\rfloor - \left\lfloor \frac{a_4+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \quad (55)$$

Summing (49) to (55) and recalling the definition of $f_{a_1,a_2,a_3,a_4;m}(k)$, we see that

$$\begin{aligned} f_{a_1,a_2,a_3,a_4;m}(k) &= f_{a_1+a_2+a_3,a_4;m}(k) - f_{a_1+a_2,a_4;m}(k) - f_{a_1+a_3,a_4;m}(k) - f_{a_2+a_3,a_4;m}(k) \\ &\quad + f_{a_1,a_4;m}(k) + f_{a_2,a_4;m}(k) + f_{a_3,a_4;m}(k). \end{aligned} \quad (56)$$

Then we obtain from (56) and (44) that

$$\begin{aligned} S_{a_1,a_2,a_3,a_4;m}(K) &= S_{a_1+a_2+a_3,a_4;m}(K) - S_{a_1+a_2,a_4;m}(K) - S_{a_1+a_3,a_4;m}(K) - S_{a_2+a_3,a_4;m}(K) \\ &\quad + S_{a_1,a_4;m}(K) + S_{a_2,a_4;m}(K) + S_{a_3,a_4;m}(K) \\ &\le \left\lfloor \frac{m}{2} \right\rfloor - 0 - 0 - 0 + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor = 4 \left\lfloor \frac{m}{2} \right\rfloor. \end{aligned}$$

Similarly, $S_{a_1,a_2,a_3,a_4;m}(K) \ge -4 \left\lfloor \frac{m}{2} \right\rfloor$. Next we prove the general case $\ell \ge 5$. The expression of the form $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ will be called an *r-bracket*. So for each $0 \le r \le \ell$, there are $\binom{\ell}{r}$ r-brackets appearing in the sum defining $f_{a_1,a_2,\ldots,a_\ell;m}(k)$. We first pair up the ℓ-bracket with an $(\ell-1)$-bracket, a 1-bracket and a 0-bracket as follows:

$$s_1(k) = \left\lfloor \frac{a_1+a_2+\cdots+a_\ell+k}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2+\cdots+a_{\ell-1}+k}{m} \right\rfloor - \left\lfloor \frac{a_\ell+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor. \quad (57)$$

So $s_1(k)$ is in fact $f_{a_1+a_2+\cdots+a_{\ell-1},a_\ell;m}(k)$ and we can apply (44) to obtain the inequality

$$0 \le S_{a_1+a_2+\cdots+a_{\ell-1},a_\ell;m}(K) = \sum_{k=0}^{K} s_1(k) \le \left\lfloor \frac{m}{2} \right\rfloor.$$

Next we pair up the remaining $(\ell-1)$-brackets with $(\ell-2)$-brackets, 1-brackets and 0-brackets as follows:

$$-\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-1}}+k}{m} \right\rfloor + \left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}}+k}{m} \right\rfloor + \left\lfloor \frac{a_{i_{\ell-1}}+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor, \quad (58)$$

and we sum (58) over all $1 \le i_1 < i_2 < \cdots < i_{\ell-1} \le \ell$ and call it $s_2(k)$. Since a_ℓ does not appear in the second term on the right hand side of (57), the term $a_{i_{\ell-1}}$ appearing in (58) is always a_ℓ. So in fact (58) is $-f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(k)$ and

$$s_2(k) = -\sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(k)$$

Furthermore,

$$\sum_{k=0}^{K} s_2(k) = -\sum_{1\le i_1<i_2<\ldots<i_{\ell-2}<\ell} S_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-2}},a_\ell;m}(K) \le 0,$$

where the last inequality is obtained from (44). We continue doing this process and follow closely the method used in the proof of Theorems 4 and 8. The well-known identities previously recalled will be applied without reference. For each $1 \le r \le \ell$, let $c_r(k)$ be the sum of all $\left\lfloor \frac{a_{i_1}+a_{i_2}+\cdots+a_{i_r}+k}{m} \right\rfloor$ with $1 \le i_1 < i_2 < \cdots < i_r \le \ell$, $a_r(k)$ the sum of all such terms with $i_r = \ell$, and $b_r(k)$ the sum of all such terms with $i_r < \ell$. Therefore $c_r(k) = a_r(k) + b_r(k)$, the number of r-brackets appearing in the sum defining $c_r(k)$ is $\binom{\ell}{r}$, the number of r-brackets appearing in the sum defining $a_r(k)$ is $\binom{\ell-1}{r-1}$, and the number of r-brackets appearing in the sum defining $b_r(k)$ is $\binom{\ell-1}{r}$. As usual, the empty sum is defined to be zero, so $b_\ell(k) = 0$. We have $s_1(k) = a_\ell(k) - b_{\ell-1}(k) - a_1(k) + \left\lfloor \frac{k}{m} \right\rfloor$ and $s_2(k) = -a_{\ell-1}(k) + b_{\ell-2}(k) + \binom{\ell-1}{\ell-2} a_1(k) - \binom{\ell-1}{\ell-2} \left\lfloor \frac{k}{m} \right\rfloor$. In general, for each $1 \le r \le \ell-1$, we let

$$\begin{aligned} s_r(k) &= (-1)^{r+1} a_{\ell-r+1}(k) + (-1)^r b_{\ell-r}(k) + (-1)^r \binom{\ell-1}{\ell-r} a_1(k) + (-1)^{r+1} \binom{\ell-1}{\ell-r} \left\lfloor \frac{k}{m} \right\rfloor \\ &= (-1)^{r+1} \sum_{1\le i_1<i_2<\ldots<i_{\ell-r}<\ell} f_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-r}},a_\ell;m}(k). \end{aligned}$$

Then

$$\sum_{k=0}^{K} s_r(k) = (-1)^{r+1} \sum_{1\le i_1<i_2<\ldots<i_{\ell-r}<\ell} S_{a_{i_1}+a_{i_2}+\cdots+a_{i_{\ell-r}},a_\ell;m}(K).$$

So by (44), we see that

$$0 \le \sum_{k=0}^{K} s_r(k) \le \binom{\ell-1}{\ell-r}\left\lfloor \frac{m}{2} \right\rfloor \text{ if } r \text{ is odd, and } -\binom{\ell-1}{\ell-r}\left\lfloor \frac{m}{2} \right\rfloor \le \sum_{k=0}^{K} s_r(k) \le 0 \text{ if } r \text{ is even.}$$

Similar to the proof of Theorems 4 and 8, we obtain

$$\begin{aligned}
\sum_{1\le r\le \ell-1} s_r(k) &= a_\ell + \sum_{2\le r\le \ell-1} (-1)^{r+1} a_{\ell-r+1} + \sum_{1\le r\le \ell-2} (-1)^r b_{\ell-r} + (-1)^{\ell-1} b_1 \\
&\quad + (-1)^{\ell+1} a_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor \\
&= a_\ell + \sum_{1\le r\le \ell-2} (-1)^r (a_{\ell-r} + b_{\ell-r}) + (-1)^{\ell-1} b_1 + (-1)^{\ell+1} a_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor \\
&= c_\ell + \sum_{1\le r\le \ell-2} (-1)^r c_{\ell-r} + (-1)^{\ell-1} c_1 + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor \\
&= \sum_{0\le r\le \ell-1} (-1)^r c_{\ell-r} + (-1)^\ell \left\lfloor \frac{k}{m} \right\rfloor \\
&= f_{a_1,a_2,\ldots,a_\ell;m}(k).
\end{aligned}$$

Therefore

$$-\sum_{\substack{1\le r\le \ell-1 \\ r \text{ is even}}} \binom{\ell-1}{\ell-r}\left\lfloor \frac{m}{2} \right\rfloor \le \sum_{k=0}^{K} f_{a_1,a_2,\ldots,a_\ell;m}(k) \le \sum_{\substack{1\le r\le \ell-1 \\ r \text{ is odd}}} \binom{\ell-1}{\ell-r}\left\lfloor \frac{m}{2} \right\rfloor. \tag{59}$$

The middle term in (59) is $S_{a_1,a_2,\ldots,a_\ell;m}(K)$. The left and right most terms in (59) are, respectively, equal to $-2^{\ell-2}\left\lfloor \frac{m}{2} \right\rfloor$ and $2^{\ell-2}\left\lfloor \frac{m}{2} \right\rfloor$ which can be evaluated by the well-known identity previously recalled. This proves the first part of the theorem. Next we show that one of the upper bound or lower bound is sharp. Let $C = \{a_1, a_2, \ldots, a_\ell\}$. Suppose ℓ is odd, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$. Then we obtain by Proposition 3 and Theorem 4 that $f_{C;m}(0) = g\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) = -2^{\ell-2}$. Let $0 < k < \frac{m}{2}$. By the definition of $f_{C;m}(k)$, we see that

$$\begin{aligned}
f_{C;m}(k) &= \sum_{T\subseteq[1,\ell]} (-1)^{\ell-|T|} \left\lfloor \frac{k}{m} + \frac{|T|}{2} \right\rfloor \\
&= \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{\ell}{r} \left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor
\end{aligned} \tag{60}$$

Since $0 < k < \frac{m}{2}$, we have $\frac{r}{2} < \frac{k}{m} + \frac{r}{2} < \frac{r+1}{2}$. So if r is even, then $\left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor = \frac{r}{2} = \left\lfloor \frac{r}{2} \right\rfloor$ and if r is odd, then $\left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor = \frac{r-1}{2} = \left\lfloor \frac{r}{2} \right\rfloor$. In any case, $\left\lfloor \frac{k}{m} + \frac{r}{2} \right\rfloor = \frac{r}{2} = \left\lfloor \frac{0}{m} + \frac{r}{2} \right\rfloor$. This implies that $f_{C;m}(k) = f_{C;m}(0)$ for every $k = 0, 1, 2, \ldots, \frac{m}{2} - 1$. Then

$$S_{C;m}\left(\frac{m}{2} - 1\right) = \sum_{k=0}^{\frac{m}{2}-1} f_{C;m}(k) = \frac{m}{2} f_{C;m}(0) = -2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$$

So $-2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ in (43) cannot be improved when ℓ is odd. Next suppose ℓ is even, m is even, and $a_i = \frac{m}{2}$ for every $1 \le i \le \ell$. Then we obtain similarly that $f_{C;m}(k) = f_{C;m}(0) = 2^{\ell-2}$ for every $k = 0, 1, 2, \ldots, \frac{m}{2} - 1$. Then $S_{C;m}(\frac{m}{2} - 1) = 2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$. So $2^{\ell-2} \left\lfloor \frac{m}{2} \right\rfloor$ in (43) cannot be improved when ℓ is even. This completes the proof.

□

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