Journal of Integer Sequences, Vol. 22 (2019),
Article 19.7.2

# On Second-Order Linear Sequences of Composite Numbers

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**Abstract**

We present a new proof of the following result of Somer:

*Let $(a, b) \in \mathbb{Z}^2$ and let $(x_n)_{n\geq 0}$ be the sequence defined by some initial values $x_0$ and $x_1$ and the second-order linear recurrence*

$$x_{n+1} = ax_n + bx_{n-1}$$

*for $n \geq 1$. Suppose that $b \neq 0$ and $(a, b) \neq (2, -1), (-2, -1)$. Then there exist two relatively prime positive integers $x_0$, $x_1$ such that $|x_n|$ is a composite integer for all $n \in \mathbb{N}$.*

The above theorem extends a result of Graham, who solved the problem when $(a, b) = (1, 1)$.

# 1 Introduction

We give a new proof of the following result of Somer:

**Theorem 1.** *[12, 3] Let $(a, b) \in \mathbb{Z}^2$ and let $(x_n)_{n\geq 0}$ be the sequence defined by some initial values $x_0$ and $x_1$ and the second-order linear recurrence*

$$x_{n+1} = ax_n + bx_{n-1} \tag{1}$$

*for $n \geq 1$. Suppose that $b \neq 0$ and $(a, b) \neq (2, -1), (-2, -1)$. Then there exist two relatively prime positive integers $x_0$, $x_1$ such that $|x_n|$ is a composite integer for all $n \in \mathbb{N}$.*

Throughout the paper we will use the following convention: a nonnegative integer $n$ is said to be *composite* if $n \neq 0, 1$, and $n$ is not a prime number.

Graham [5] considered the problem above in the particular case $(a, b) = (1, 1)$. He found two relatively prime positive integers $x_0$ and $x_1$ such that the sequence $x_{n+1} = x_n + x_{n-1}$, consists of composite numbers only. Graham's starting pair is

$$(x_0, x_1) = (331635635998274737472200656430763, 1510028911088401971189590305498785).$$

Graham's technique was successively refined by Knuth [6], Wilf [16], and Nicol [10], who all found smaller pairs $(x_0, x_1)$. The current record is due to Vsemirnov [15]:

$$(x_0, x_1) = (106276436867, 35256392432). \tag{2}$$

The above results are based on the fact that the Fibonacci sequence is a *divisibility sequence*, that is, $F_n \mid F_m$ whenever $n \mid m$, and on finding a finite *covering system of congruences* $r_i \pmod{m_i}$, $1 \leq i \leq t$, such that there exist distinct primes $p_1, p_2, \ldots p_t$, so that $p_i \mid F_{m_i}$ for all $i = 1, 2, \ldots, t$.

Somer's proof of Theorem 1 is relatively short since it relies on several results of Bilu, Hanrot and Voutier [1], Choi [2], and Parnami and Shorey [11].

A few years later, unaware of Somer's article, Dubickas, Novikas and Šiurys [3] published a new solution that, although somewhat lengthier, is essentially self-contained.

In this paper, we present another free-standing proof of this theorem that, while comparable to the one in [3], differs from it in several important ways.

We summarize our plan for proving Theorem 1 as follows.

In Section 2 we prove three easy lemmata, which will be useful later on. We believe that Lemma 2 is of independent interest. In Section 3 we study two simple cases: $(i)$ $a = 0$ and $(ii)$ $a^2 + 4b = 0$. In this section we also show that the condition $(a, b) \neq (\pm 2, -1)$ is necessary. Section 4 deals with the case $|b| \geq 2$. Following [3], we choose $x_1 \equiv 0 \pmod{|b|}$, since then (1) implies that $x_n \equiv 0 \pmod{|b|}$ for all $n \geq 2$. The main difficulty is to show that $x_0$ and $x_1$ can be chosen such that $x_n \neq -b, 0, b$ for every $n \geq 0$. In this case, Dubickas et al. present a mainly existential proof which relies on a series of six lemmata. In contrast, our proof is constructive, as we provide explicit expressions for $x_0$ and $x_1$ as polynomials in $a$ and $b$.

In Section 5 we consider the case $|b| = 1$. Dubickas, Novikas and Šiurys prove that except for finitely many values of $|a|$, one can take $x_0$ and $x_1$ so that each $x_n$ is divisible by one of five distinct appropriately chosen prime numbers. We prove that, with the exception of finitely many values of $|a|$, four primes suffice. The proof in the case $|b| = 1$ relies on the divisibility properties of the *Lucas sequence of the first kind* $(u_n)_{n \geq 0}$, defined as

$$u_0 = 0, u_1 = 1 \quad \text{and} \quad u_{n+1} = au_n + bu_{n-1}, \text{ for } n \geq 1. \tag{3}$$

Finally, in Section 6 we prove that if $|a| \geq 3$ and $b = -1$, then $u_n$ is composite for all $n \geq 3$. The interesting fact is that in this case it seems likely that there is no finite set of prime numbers $p_1, p_2, \ldots p_t$ such that each $u_n$ is divisible by some $p_i$, $i = 1, 2, \ldots t$.

## 2 Three useful lemmata

**Lemma 2.** *Consider the sequence* $(x_n)_{n \geq 0}$ *given by* (1). *Then*

$$x_{n+1}^2 - ax_nx_{n+1} - bx_n^2 = (-b)^n (x_1^2 - ax_0x_1 - bx_0^2). \tag{4}$$

*Proof.*

$$\begin{bmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{bmatrix} = \begin{bmatrix} ax_{n+1} + bx_n & ax_n + bx_{n-1} \\ x_{n+1} & x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_n & x_{n-1} \\ x_{n-1} & x_{n-2} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^2 \cdot \begin{bmatrix} x_n & x_{n-1} \\ x_{n-1} & x_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} x_2 & x_1 \\ x_1 & x_0 \end{bmatrix}.$$

Taking determinants on both sides we obtain

$$\det\begin{bmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{bmatrix} = \det\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^n \cdot \det\begin{bmatrix} x_2 & x_1 \\ x_1 & x_0 \end{bmatrix} \implies x_{n+2}\cdot x_n - x_{n+1}^2 = (-b)^n \cdot (x_2x_0 - x_1^2)$$

$\implies (ax_{n+1} + bx_n)x_n - x_{n+1}^2 = (-b)^n \cdot ((ax_1 + bx_0)x_0 - x_1^2)$, which after expanding becomes $x_{n+1}^2 - ax_nx_{n+1} - bx_n^2 = (-b)^n(x_1^2 - ax_0x_1 - bx_0^2)$, as claimed.

□

**Lemma 3.** *Consider the sequence $(x_n)_{n\geq 0}$ given by (1). Suppose that $1 \leq |x_0| < |x_1|$ and $|a| > |b| \geq 1$. Then the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing.*

*Proof.* We use induction on $n$. By hypothesis, the statement is true for $n = 0$.

Suppose that $|x_{n-1}| < |x_n|$ for some $n \geq 1$. We intend to prove that $|x_n| < |x_{n+1}|$. Indeed

$$|x_{n+1}| = |ax_n + bx_{n-1}| \geq |ax_n| - |bx_{n-1}| = |a||x_n| - |b||x_{n-1}| >$$
$$> |a||x_n| - |b||x_n| = (|a| - |b|)|x_n|, \text{ by the induction hypothesis.}$$

Using $|a| - |b| \geq 1$, we obtain $|x_{n+1}| > |x_n|$, which completes the induction. □

**Lemma 4.** *Let $n_1$, $n_2$ and $n_3$ be three positive integers such that no prime number $p$ divides all of them. Then there exists an integer $k \geq 2$ such that $n_1$ and $n_2 + kn_3$ are relatively prime.*

*Proof.* Let $d := \gcd(n_2, n_3)$. Note that $\gcd(n_1, d) = 1$, otherwise $d$ divides $n_1$, $n_2$ and $n_3$.

By Dirichlet's theorem on arithmetic progressions there exists a $k$ such that $n_2/d + kn_3/d$ is a prime number greater than $n_1$. Then $d\,(n_2/d + k \cdot n_3/d) = n_2 + k\,n_3$ is both greater and relatively prime to $n_1$. □

# 3 Two simple special cases: $(i)$ $a = 0$ and $(ii)$ $a^2 + 4b = 0$

*Case (i):* Since $a = 0$, it can be easily proved that $x_{2n} = b^nx_0$ and $x_{2n+1} = b^nx_1$. It suffices to take $x_0 = 4$ and $x_1 = 9$ to obtain that $x_n$ is composite for all $n \geq 0$.

*Case (ii):* If $a^2 + 4b = 0$, then $a$ must be even; $a = 2c$ and therefore $b = -a^2/4 = -c^2$. We divide the proof into two cases: $|b| \geq 2$ and $b = -1$.

If $|b| \geq 2$, we have $|c| \geq 2$. Let us now take $x_0 = 4c^2 - 1$ and $x_1 = 2c^3$. Then $x_0$ and $x_1$ are relative prime positive composite integers. One immediately obtains $x_2 = c^2$; thus $x_2$ is also composite. Also, it can be easily proved that $x_n = c^n\left((n-1) - (2n-4)c^2\right)$ for all $n \geq 3$.

Note that for any $n \geq 3$ one cannot have $x_n = 0$; otherwise, $4 \leq c^2 = (n-1)/(2n-4) \leq 1$, which is impossible. It follows that $|x_n|$ is composite for all $n \geq 3$.

If $b = -1$, then $|a| = 2$. If $a = 2$, then a simple induction shows that

$$x_{n+1} = (n+1)x_1 - nx_0 = x_1 + n(x_1 - x_0) \quad \text{for all } n \geq 0,$$

that is, $(x_n)_{n\geq 0}$ is an arithmetic sequence whose first term and common difference are relatively prime. By Dirichlet's theorem on primes in arithmetic progressions, it follows that $|x_n|$ is a prime number for infinitely many values of $n$. If $a = -2$, then one can show that $x_{n+1} = (-1)^n(x_1 + n(x_0 + x_1))$. In this case, $(x_n)_{n\geq 0}$ is the union of two arithmetic sequences, both of which have the first term and the common difference relatively prime. Again, for any choice of $x_0$ and $x_1$ relatively prime, $|x_n|$ is a prime for infinitely many $n$. This proves the necessity of the condition $(a, b) \neq (\pm 2, -1)$.

# 4 The case $|b| \geq 2$

Based on the results from the previous section, from now on we can assume that

$$|a| \geq 1 \quad \text{and} \quad a^2 + 4b \neq 0. \tag{5}$$

We divide the proof into three separate cases:

*Case I:* $|a| > |b|$,
*Case II:* $1 \leq |a| \leq |b|$ and $|b|$ is composite, and
*Case III:* $1 \leq |a| \leq |b|$ and $|b|$ is a prime.

*Case I:* $|a| > |b|$. To complete the proof of Theorem 1, in this case we take $x_0 = b^4 - 1, x_1 = b^4$. Clearly, $x_0$ and $x_1$ are both positive, relatively prime and composite. Moreover, $x_0 < x_1$. Then $x_n \equiv 0 \pmod{b}$ for all $n \geq 1$, and by using Lemma 3 it follows that $(|x_n|)_{n\geq 0}$ is strictly increasing. Hence, we have $|x_n| \geq x_0 > |b|$ for all $n \geq 0$, and therefore $|x_n|$ is composite for all $n \geq 0$.

*Case II:* $1 \leq |a| \leq |b|$ *and* $|b|$ *is composite.* Choose $x_0 = 4b^4 - 1$, $x_1 = 2b^2$. Clearly, $\gcd(x_0, x_1) = 1$, and $x_0, x_1$ are positive composite integers. It also follows that $x_n \equiv 0 \pmod{b}$ for all $n \geq 1$, and since $|b|$ is composite, it follows that $|x_n|$ is composite, unless $x_n = 0$ for some $n$. We will prove that this cannot happen.

To get a contradiction, suppose that $x_{n+1} = 0$ for some $n \geq 2$. Then, by Lemma 2, we have $x_n^2 = (-b)^{n-1}(x_1^2 - ax_0x_1 - bx_0^2)$.

Since $x_0 = 4b^4 - 1$ and $x_1 = 2b^2$, we have

$$x_1^2 - ax_0x_1 - bx_0^2 = (-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1), \quad \text{which implies that}$$

$$x_n^2 = (-b)^n(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1). \tag{6}$$

If $n$ is even, this implies that $16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1$ is a perfect square. If $n$ is odd, this implies that $(-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1)$ is a perfect square. We will prove that none of these are possible if $a^2 + 4b \neq 0$.

Note first that since $|a| \leq |b|$, we have

$$(4b^4 + ab - 2)^2 < 16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1 < (4b^4 + ab)^2. \tag{7}$$

Indeed, these inequalities are equivalent to the following ones:

$$8b^4 - 2 - (4b^3 + (ab-1)^2) > 0, \quad \text{and} \quad 8b^4 + 4b^3 - 2 + (ab+1)^2 > 0. \tag{8}$$

We have

$$|a| \leq |b| \implies |ab| \leq b^2 \implies |ab - 1| \leq |ab| + 1 \leq b^2 + 1 \implies (ab-1)^2 \leq b^4 + 2b^2 + 1. \tag{9}$$

From this we obtain

$$|4b^3 + (ab-1)^2| \leq 4|b|^3 + (ab-1)^2 \leq b^4 + 4|b|^3 + 2b^2 + 1. \tag{10}$$

To prove the first inequality in (8), note that

$$\begin{aligned} &8b^4 - 2 - (4b^3 + (ab-1)^2) \geq 8b^4 - 2 - |4b^3 + (ab-1)^2| \geq \\ &\geq 8b^4 - 2 - (b^4 + 4|b|^3 + 2b^2 + 1) = 7b^4 - 4|b|^3 - 2b^2 - 3 = \\ &= 7|b|^4 - 4|b|^3 - 2|b|^2 - 3 = 4|b|^3(|b| - 1) + 2|b|^2(|b|^2 - 1) + (|b|^4 - 3) \geq \\ &\geq 4|b|^3 + 2|b|^2 + |b|^4 - 3 \geq 4 \cdot 2^3 + 2 \cdot 2^2 + 2^4 - 3 > 0. \end{aligned}$$

The second inequality in (8) is much easier to prove:

$$8b^4 + 4b^3 - 2 + (ab+1)^2 \geq 8b^4 + 4b^3 - 2 \geq 8|b|^4 - 4|b|^3 - 2 \geq 8 \cdot 2^4 - 4 \cdot 2^3 - 2 > 0.$$

Hence, the inequalities (7) hold.

If the middle term in (7) were to be a perfect square, the only option remaining is

$$16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1 = (4b^4 + ab - 1)^2, \text{ which implies } b^2(a^2 + 4b) = 0.$$

However, this is impossible if $a^2 + 4b \neq 0$. Hence, if one assumes that $|a| \leq |b|$ and $a^2 + 4b \neq 0$, then $16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1$ cannot be a perfect square.

Suppose next that $(-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1)$ is a perfect square. Since the two factors are mutually prime, it follows that both have to be perfect squares. In particular, the second one has to be a perfect square, and we have already proved that it cannot be. It follows that the sequence $(x_n)$ does not contain any terms equal to 0. This completes the proof of Theorem 1 in case II.

*Case III:* $1 \leq |a| \leq |b|$ *and* $|b|$ *is a prime.*

We divide the proof of this case into three subcases: $|a| = 1$, $|a| = |b|$ and $2 \leq |a| < |b|$.

*Subcase IIIa:* $|a| = 1$, $|b|$ *is a prime.*

If $a = 1$ and $b > 0$, then take $x_0 = (2b^2-1)^2, x_1 = b(b^2-1)$. Clearly, $x_0$ and $x_1$ are positive, relatively prime, composite integers. It follows immediately that $x_2 = b^3(4b^2 - 3)$ is also composite. Finally, an easy induction shows that $x_n \equiv 0 \pmod{b^2}$ and that $\frac{x_n}{b^2} \equiv -1 \pmod{b}$ for all $n \geq 3$. Hence, $|x_n|$ is necessarily composite for all $n \geq 3$.

The other situations can be dealt with similarly.

If $a = -1$ and $b < 0$, take $x_0 = (2b^2 - 1)^2, x_1 = -b(b^2 - 1)$. Then $x_2 = b^3(4b^2 - 3)$, and for $n \geq 3$ one can show that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv (-1)^{n+1} \pmod{b}$. Again, all $|x_n|$ are composite.

If $a = 1$ and $b < 0$, we take $x_0 = (2b^2 - 1)^2, x_1 = -b(b^2 + 1)$. Then $x_2 = b^3(4b^2 - 5)$, and for all $n \geq 3$ one can prove that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -1 \pmod{b}$. Hence, all $|x_n|$ are composite.

Finally, if $a = -1$ and $b > 0$, select $x_0 = (2b^2 - 1)^2, x_1 = b(b^2 + 1)$. Then $x_2 = b^3(4b^2 - 5)$, and for all $n \geq 3$ we have that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv (-1)^{n+1} \pmod{b}$. All $|x_n|$ are composite.

This completes the proof of Theorem 1 in subcase IIIa.

*Subcase IIIb:* $|a| = |b|$, $|b|$ *is a prime.*

If $a = b$, take $x_0 = 4b^4 - 1, x_1 = 2b^2$. Then $x_2 = b(4b^4 + 2b^2 - 1)$ is composite, and $x_n \equiv 0 \pmod{b^2}$ for all $n \geq 3$. It remains to show that $x_n \neq 0$ for all $n$.

As in Case II, $x_{n+1} = 0$ implies that either $16b^8 + 8b^6 - 8b^4 - 4b^3 - 2b^2 + 1$ or $(-b)(16b^8 + 8b^6 - 8b^4 - 4b^3 - 2b^2 + 1)$ is a perfect square. One can use the same technique as in Case II to prove that this cannot happen if $a^2 + 4b = b^2 + 4b \neq 0$.

If $a = -b$, take $x_0 = 4b^4 - 1, x_1 = 2b^2$ (same choice). Then $x_2 = b(4b^4 - 2b^2 - 1)$ is composite, and $x_n \equiv 0 \pmod{b^2}$ for all $n \geq 3$. It remains to show that $x_n \neq 0$ for all $n$.

As in Case II, $x_{n+1} = 0$ implies that either $16b^8 - 8b^6 - 8b^4 - 4b^3 + 2b^2 + 1$ or $(-b)(16b^8 - 8b^6 - 8b^4 - 4b^3 + 2b^2 + 1)$ is a perfect square. The same approach as in Case II shows that this is impossible if $a^2 + 4b = b^2 + 4b \neq 0$.

*Subcase IIIc:* $2 \leq |a| < |b|$, $|b|$ *is a prime.*

It follows that $\gcd(a, b) = 1$.

If $a > 0, b > 0$, take $x_0 = a^3, x_1 = b(b^2 - a^2)$. Then $x_2 = ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -a^{n-1} \pmod{b}$. Since $\gcd(a, b) = 1$, $x_n \neq 0$.

If $a < 0, b < 0$, take $x_0 = -a^3, x_1 = -b(b^2 - a^2)$. Then $x_2 = -ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv a^{n-1} \pmod{b}$. Hence, $x_n \neq 0$ for all $n$.

If $a < 0, b > 0$, take $x_0 = -a^3, x_1 = b(b^2 + a^2)$. Then $x_2 = ab^3$, and for $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv a^{n-1} \pmod{b}$. So, $x_n \neq 0$ for all $n$.

Lastly, if $a > 0, b < 0$, take $x_0 = a^3, x_1 = -b(b^2 + a^2)$. Then $x_2 = -ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -a^{n-1} \pmod{b}$. Since $\gcd(a, b) = 1$, we have $x_n \neq 0$ for all $n$.

Hence, in each case one can choose $x_0$ and $x_1$ to be positive, composite and relatively prime so that $x_2$ is composite, and for all $n \geq 3$ we have that $x_n \equiv 0 \pmod{b^2}$ and $x_n \neq 0$. It follows that $|x_n|$ is composite for all $n \geq 0$, and this completes the proof of Theorem 1 in the case $|b| \geq 2$.

## 5 The case $|b| = 1$

The main idea behind the proof of Theorem 1 in this particular case can be summarized as follows: we want to find a finite set of primes $p_1, p_2, \ldots, p_t$ such that for every $n \geq 0$ the number $|x_n|$ is divisible by at least one of these primes.

We start the analysis with the simple case in which $|a|$ has at least two distinct prime factors.

**Lemma 5.** *Let $|b| = 1$, and suppose that $|a|$ has at least two distinct prime factors: $p_1$ and $p_2$, with $p_1 < p_2$. Then the sequence given by (1) and the initial terms $x_0 = p_1^2$, $x_1 = p_2^2$ satisfies the conditions in Theorem 1.*

*Proof.* Clearly, $x_0$ and $x_1$ are positive relatively prime composite integers. An easy induction shows that $x_n \equiv 0 \pmod{p_1}$ if $n$ is even and $x_n \equiv 0 \pmod{p_2}$ if $n$ is odd. Since $|x_0| < |x_1|$ and $1 = |b| < |a|$, the hypotheses of Lemma 3 are satisfied; hence $(|x_n|)_{n\geq 0}$ is strictly increasing. It follows that $x_n$ is composite for all $n \geq 0$, and therefore Theorem 1 is verified in this case. □

*Remark* 6. In all what follows we will assume that $|a| = p_1^s$ for some prime $p_1$ and some nonnegative integer $s \geq 0$. Note that this allows the possibility of $|a| = 1$.

Next, we introduce a special sequence $(u_n)_{n\geq 0}$ given by

$$u_0 = 0, u_1 = 1, \quad u_{n+1} = au_n + bu_{n-1} \quad \text{for all } n \geq 1. \tag{11}$$

This sequence is called the *Lucas sequence of the first kind*.

One can prove that $(u_n)_{n\geq 0}$ is a *divisibility sequence*; that is, $u_m$ divides $u_n$ whenever $m$ divides $n$. Indeed, suppose that $a^2 + 4b \neq 0$, which implies that the roots $\alpha$ and $\beta$ of the characteristic equation are distinct, $\alpha \neq \beta$. Assume that $n = mk$. Then

$$\begin{aligned} u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{mk} - \beta^{mk}}{\alpha - \beta} &= \frac{\alpha^m - \beta^m}{\alpha - \beta} \cdot \left(\alpha^{m(k-1)} + \alpha^{m(k-2)}\beta^m + \cdots + \beta^{m(k-1)}\right) \\ &= u_m \left(\alpha^{m(k-1)} + \alpha^{m(k-2)}\beta^m + \cdots + \beta^{m(k-1)}\right). \end{aligned}$$

The second factor of the last term in the equality above is a symmetric function in $\alpha$ and $\beta$, and therefore it can be written as a polynomial function of $\alpha + \beta = a$ and $\alpha\beta = -b$. It follows that $u_n/u_m$ is an integer, as claimed.

Of particular interest in the sequel are the values of $u_4$ and $u_6$:

$$u_4 = a(a^2 + 2b) \quad \text{and} \quad u_6 = a(a^2 + b)(a^2 + 3b). \tag{12}$$

The next two results give information regarding the prime factors of $u_4$ and $u_6$ when $|b| = 1$.

**Lemma 7.** *Suppose that $b = -1$, $|a| \geq 4$ and $|a| = p_1^s$ for some prime $p_1$ and some $s \geq 1$. Then $u_6 = a(a^2 - 1)(a^2 - 3)$ has at least three additional distinct prime factors, $p_2, p_3$ and $p_4$, all three different from $p_1$.*

*Proof.* Note that $a^2-3$ is not divisible by 4 or 9. Since $a^2-3 \geq 13$, it follows that $a^2-3$ has an odd prime factor $p_4$ different from 3, and since $\gcd(a, a^2-3) \in \{1,3\}$ we have that $p_4 \neq p_1$. Clearly, neither $p_1$ nor $p_4$ are factors of $a^2-1$. Note that $a^2-1$ has at least two distinct prime factors. Indeed, if one assumes the opposite, then both $|a-1|$ and $|a+1|$ are powers of some prime. But this cannot happen, since $|a| \geq 4$. It follows that $a^2-1$ has at least two distinct prime factors, $p_2$ and $p_3$, both different from $p_1$ and $p_4$. □

To illustrate Lemma 7, let us take $(a,b) = (-9,-1)$. Then $a^2-3 = 78 = 2 \cdot 3 \cdot 13$ while $a^2-1 = 80 = 2^4 \cdot 5$. Hence, one has $p_1 = 3$, $p_4 = 13$, $p_2 = 2$, and $p_3 = 5$.

**Lemma 8.** *Suppose that $b = 1$, $|a| \geq 6$ and $|a| = p_1^s$ for some prime $p_1$ and some $s \geq 1$.*

*If $p_1 \neq 3$, then $u_4 = a(a^2+2)$ has at least two additional distinct prime factors, $p_2$ and $p_3$, both different from $p_1$.*

*If $|a| = 3^s$ for some $s \geq 2$, then $u_6 = a(a^2+1)(a^2+3)$ has at least three additional distinct prime factors, $p_2$, $p_3$ and $p_4$, all three different from $p_1$.*

*Proof.* Suppose first that $p_1 = 2$; that is, $|a| = 2^s$ for some $s \geq 3$. Note that $a^2+2 \equiv 0 \pmod 3$, hence one can choose $p_2 = 3$. Note that $a^2 = 2(2^{2s-1}+1)$ must have a prime factor different from 2 and 3. Indeed, if one assumes the opposite, then $2^{2s-1}+1 = 3^t$ for some $t \geq 2$. However, under the assumption that $2s-1 \geq 2$ and $t \geq 2$, the Catalan-Mihăilescu theorem implies that the only solution of this equation is $s = 2, t = 2$. But this implies that $|a| = 2^2 = 4$, a contradiction.

Consider next the case $|a| = p_1^s$ with $p_1 > 3$. In particular, $a^2+2$ is odd and therefore $\gcd(a, a^2+2) = 1$. Moreover, since $a^2+2 \equiv 0 \pmod 3$ one can safely take $p_2 = 3$. We claim that $a^2+2$ has at least one other prime factor, $p_3$, different from $p_1$ and $p_2$. Indeed, if one assumes otherwise, then $a^2+2 = p_1^{2s}+2 = 3^t$ for some $t \geq 2$. However, it was shown by Ljunggren [7] that the more general equation $x^2+2 = y^n$, $n \geq 2$ has the unique solution $x = 5, y = 3, n = 3$. This would give $|a| = 5$, which is excluded from our analysis.

Finally, suppose that $p_1 = 3$ and therefore $|a| = 3^s$ for some $s \geq 2$. Then both $a^2+1$ and $a^2+3$ are even, so one can take $p_2 = 2$. Note that $a^2+1 = 9^s+1$; hence $a^2+1 \equiv 10 \pmod{72}$ and $a^2+3 \equiv 12 \pmod{72}$. Hence, there exists a positive integer $c$ such that $a^2+1 = 2(36c+5)$ and $a^2+3 = 12(6c+1)$. Let $p_3$ be a prime factor of $36c+5$ and let $p_4$ be a prime factor of $6c+1$. It follows immediately that $p_1 = 3, p_2 = 2, p_3$, and $p_4$ are all distinct. □

We present a couple of particular instances covered by the above lemma.

Suppose first that $a = 8$ and $b = 1$. Then $u_2 = a = 2^3$ while $u_4 = 2^4 \cdot 3 \cdot 11$. Thus, in this case one can take $p_1 = 2, p_2 = 3, p_3 = 11$.

Suppose next that $a = -49$ and $b = 1$. Then $u_2 = a = -7^2$, while $u_4 = -7^2 \cdot 3^3 \cdot 89$. It follows that we can choose $p_1 = 7, p_2 = 3, p_3 = 89$.

Finally, let us assume that $a = 9$ and $b = 1$. Then $u_2 = a = 3^2$, while $a^2+1 = 2 \cdot 41$, $a^2+3 = 2^2 \cdot 3 \cdot 7$. Hence, in this case we have $p_1 = 3, p_2 = 2, p_3 = 41, p_4 = 7$.

The next lemma uses the concept of covering system introduced by Erdős in [4].

**Definition 9.** A collection of residue classes $r_i \pmod{m_i}$, $0 \le r_i < m_i$, where $1 \le i \le t$ is said to be a *covering system* if every integer $n$ satisfies at least one equality $n \equiv r_i \pmod{m_i}$.

In the proof of the theorem when $|b| = 1$, except for finitely many values of $a$ we will use one of the following two covering systems

$$\begin{aligned}&\{0 \pmod 2, \quad 1 \pmod 6, \quad 3 \pmod 6, \quad 5 \pmod 6\} \quad \text{or} \\ &\{0 \pmod 2, \quad 1 \pmod 4, \quad 3 \pmod 4\}.\end{aligned}$$

**Lemma 10.** *Let $a, b$ be two integers such that $|a| \ge 2$ and $|b| = 1$. Let $(u_n)_{n\ge 0}$ be the Lucas sequence defined in (11), and suppose that there exists a finite collection of triples $(p_i, m_i, r_i)$, $1 \le i \le t$ with the following properties:*

*(i)* *All primes $p_i$ are distinct.*

*(ii)* *The residue classes $r_i \pmod{m_i}$ form a covering system.*

*(iii)* $p_i \mid u_{m_i}$ *for all* $1 \le i \le t$.

*Then there exist two relatively prime positive integers $x_0$ and $x_1$ such that each term of the sequence $(x_n)_{n\ge 0}$ defined in (1) is composite.*

*Proof.* Let $P = p_1 p_2 \dots p_t$.

By the Chinese remainder theorem, there exist $y, z \in \{0, 1, \dots P-1\}$ satisfying

$$\begin{aligned} y &\equiv u_{m_i - r_i} \pmod{p_i}, \\ z &\equiv u_{m_i - r_i + 1} \pmod{p_i}, \end{aligned} \tag{13}$$

for $i = 1, 2, \dots, t$. Note that there is no prime which divides $y, z$, and $P$ simultaneously. Indeed, if such a prime $p_j$ were to exist, then there would be two consecutive terms $u_n$ and $u_{n+1}$ both divisible by $p_j$. Since $u_{n+1} = au_n + bu_{n-1}$ and $|b| = 1$, then $p_j \mid u_{n-1}$. By induction, it follows that $p_j \mid u_1$ which is impossible since $u_1 = 1$.

Let $x_0 \equiv y \pmod P$ and $x_1 \equiv z \pmod P$.

Then we have $x_0 \equiv u_{m_i - r_i} \pmod{p_i}$ and $x_1 \equiv u_{m_i - r_i + 1} \pmod{p_i}$ for all $i = 1, 2, \dots, t$. By induction on $n$, we obtain $x_{n+1} \equiv u_{m_i - r_i + n} \pmod{p_i}$ for every $n \ge 0$ and every $1 \le i \le t$.

Since the residue classes $r_i \pmod{m_i}$ form a covering system, each nonnegative integer $n$ belongs to one of these classes, say $n = r_i + k\, m_i$ for some $k \ge 0$ and some $i \in \{1, 2, \dots, t\}$. This implies that

$$x_{n+1} \equiv u_{m_i - r_i + n} \pmod{p_i} \equiv u_{m_i(k+1)} \pmod{p_i} \equiv 0 \pmod{p_i}, \tag{14}$$

since $p_i \mid u_{m_i}$ and $u_{m_i} \mid u_{m_i(k+1)}$. Hence, every term of the sequence $(x_n)_n$ is divisible by some prime $p_i$. It remains to choose $x_0$ and $x_1$, relatively prime positive integers $x_0 \equiv y \pmod P$ and $x_1 \equiv z \pmod P$ such that $|x_n| \ge P$ for every $n \in \mathbb{N}$.

In order to achieve this, take $x_0 = y+P$ and $x_1 = z+kP$ where $k \geq 2$ and $\gcd(x_0, x_1) = 1$. Using Lemma 4 with $n_1 = y + P$, $n_2 = z$ and $n_3 = P$ shows that such a choice is always possible. Recall that we proved earlier that $\gcd(y, z, P) = 1$. Such a choice implies that $0 < P \leq x_0 < x_1$, and since $|a| > |b| = 1$, Lemma 3 implies that $(|x_n|)_n$ is a strictly increasing sequence. It follows that $|x_n| \geq P$ for all $n \geq 0$, and therefore each such $x_n$ is composite. □

We can now prove the theorem if $|b| = 1$.

Suppose first that $b = -1$ and $|a| = p_1^s \geq 4$. Then, by Lemma 7, there are four distinct primes $p_1, p_2, p_3, p_4$ dividing $u_2, u_6, u_6, u_6$, respectively. The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 6, 1), (p_3, 6, 3), (p_4, 6, 5)$.

As a numerical illustration, suppose that $a = -9, b = -1$. Then, as described in the paragraph following the proof of Lemma 7, we have $p_1 = 3, p_2 = 2, p_3 = 5$ and $p_4 = 13$.

The system (13) becomes

$$\begin{aligned} y &\equiv u_2 \pmod{3} \qquad & z &\equiv u_3 \pmod{3} \\ y &\equiv u_5 \pmod{2} \qquad & z &\equiv u_6 \pmod{2} \\ y &\equiv u_3 \pmod{5} \qquad & z &\equiv u_4 \pmod{5} \\ y &\equiv u_1 \pmod{13} \qquad & z &\equiv u_2 \pmod{13} \end{aligned}$$

and its solution is $P = p_1p_2p_3p_4 = 390, y = 105, z = 134$. Since $y < z$ and $\gcd(y, z) = 1$, one can safely take $x_0 = y = 105$ and $x_1 = z = 134$. Then the sequence $(|x_n|)_{n \geq 0}$ is strictly increasing, and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod{3}, x_{6n+1} \equiv 0 \pmod{2}, x_{6n+3} \equiv 0 \pmod{5}$ and $x_{6n+5} \equiv 0 \pmod{13}$. It follows that all terms of the sequence are composite.

Next suppose that $b = 1$ and $|a| = p_1^s \geq 6$ for some prime $p_1$. If $p_1 \neq 3$, then by Lemma 8 there exist three distinct primes $p_1, p_2, p_3$, dividing $u_2, u_4, u_4$, respectively.

The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 4, 1), (p_3, 4, 3)$. As in the case $b = -1$, we present the details in a couple of particular cases.

Suppose first that $a = 8, b = 1$. Then, as described in the paragraph following the proof of Lemma 8, we have $p_1 = 2, p_2 = 3, p_3 = 11$. The system (13) becomes

$$\begin{aligned} y &\equiv u_0 \pmod{2} \qquad & z &\equiv u_1 \pmod{2} \\ y &\equiv u_3 \pmod{3} \qquad & z &\equiv u_4 \pmod{3} \\ y &\equiv u_1 \pmod{11} \qquad & z &\equiv u_2 \pmod{11} \end{aligned}$$

and its solution is $P = p_1p_2p_3 = 66, y = 56, z = 63$. Note that in this case $\gcd(y, z) = 7 > 1$. Still, one can safely take $x_0 = y = 56$ and $x_1 = z + P = 129$, and now $\gcd(x_0, x_1) = 1$ as desired. Then, since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n \geq 0}$ is strictly increasing and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod{2}, x_{4n+1} \equiv 0 \pmod{3}$, and $x_{4n+3} \equiv 0 \pmod{11}$. It follows that all terms of the sequence are composite.

Next assume that $a = -49, b = 1$. Then $p_1 = 7, p_2 = 3, p_3 = 89$, and the system (13) becomes

$$\begin{array}{llll} y \equiv u_0 & \pmod 7 & z \equiv u_1 & \pmod 7 \\ y \equiv u_3 & \pmod 3 & z \equiv u_4 & \pmod 3 \\ y \equiv u_1 & \pmod{89} & z \equiv u_2 & \pmod{89}. \end{array}$$

The solution is $P = p_1p_2p_3 = 1869, y = 980, z = 1464$. Notice that in this case $\gcd(y, z) = 4 > 1$. Still, one can safely take $x_0 = y = 980$ and $x_1 = z+P = 3333$ and now $\gcd(x_0, x_1) = 1$ as desired. Moreover, since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing, and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod 7, x_{4n+1} \equiv 0 \pmod 3$, and $x_{4n+3} \equiv 0 \pmod{89}$. It follows that all terms of the sequence are composite.

For the case $b = 1$ and $|a| = 3^s$ with $s \geq 2$, we use the second part of Lemma 8 to conclude that there are four distinct primes $p_1, p_2, p_3, p_4$, dividing $u_2, u_6, u_6, u_6$, respectively. The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 6, 1), (p_3, 6, 3), (p_4, 6, 5)$.

We show the full details if $a = 9, b = 1$. Then, as mentioned in the paragraph following the proof of Lemma 8, we have $p_1 = 3, p_2 = 2, p_3 = 41, p_4 = 7$. The system (13) becomes

$$\begin{array}{llll} y \equiv u_2 & \pmod 3 & z \equiv u_3 & \pmod 3 \\ y \equiv u_5 & \pmod 2 & z \equiv u_6 & \pmod 2 \\ y \equiv u_3 & \pmod{41} & z \equiv u_4 & \pmod{41} \\ y \equiv u_1 & \pmod 7 & z \equiv u_2 & \pmod 7. \end{array}$$

Solving, we obtain $P = p_1p_2p_3p_4 = 1722, y = 1107, z = 1444$. In this case we can simply choose $x_0 = y = 1107$ and $x_1 = z = 1144$.

Then $\gcd(x_0, x_1) = 1$, and since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing. Moreover, for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod 3, x_{6n+1} \equiv 0 \pmod 2, x_{6n+3} \equiv 0 \pmod{41}$, and $x_{6n+5} \equiv 0 \pmod 7$. It follows that all terms of the sequence are composite as desired.

At this point we have proved the main theorem when $|b| = 1$ for all but finitely many values of $a$. We still have to study what happens when $b = -1$ and $|a| \leq 3$ as well as the cases when $b = 1$ and $|a| \leq 5$. Recall that the cases $a = 0$ and $(a, b) = (\pm 2, -1)$ were already handled in section 2.

For most of these cases, we will still use Lemma 10; the only difference is that the set of triples $\{p_i, m_i, r_i\}_{i=1}^{i=t}$ is occasionally going to be slightly more numerous.

We summarize our findings in the table below. We invite the reader to verify that the collections $\{p_i, m_i, r_i\}_{i=1}^{i=t}$ do indeed satisfy the three conditions in Lemma 10. Note that in each case we have $0 < x_0 < x_1$, and since $|a| > |b| = 1$, Lemma 3 implies that $(|x_n|)_{n\geq 0}$ is strictly increasing.

It remains to see what happens when $|a| = |b| = 1$, as in these cases Lemma 10 does not apply.

| $a$ | $b$ | $\{(p_i, m_i, r_i)\}$ | $x_0$ | $x_1$ |
|---|---|---|---|---|
| 5 | 1 | $(5,2,0),(2,6,1),(7,6,3),(13,6,5)$ | 495 | 1136 |
| $-5$ | 1 | $(5,2,0),(2,6,1),(7,6,3),(13,6,5)$ | 495 | 866 |
| 4 | 1 | $(2,2,0),(3,4,1),(7,8,3),(23,8,7)$ | 116 | 165 |
| $-4$ | 1 | $(2,2,0),(3,4,1),(7,8,3),(23,8,7)$ | 116 | 801 |
| 3 | 1 | $(3,2,0),(11,4,1),(7,8,3),(17,8,7)$ | 1803 | 3454 |
| $-3$ | 1 | $(3,2,0),(11,4,1),(7,8,3),(17,8,7)$ | 1803 | 3091 |
| 2 | 1 | $(2,2,0),(5,3,0),(3,4,1),(7,6,5),(11,12,7)$ | 260 | 807 |
| $-2$ | 1 | $(2,2,0),(5,3,0),(3,4,1),(7,6,5),(11,12,7)$ | 260 | 1503 |
| 3 | $-1$ | $(3,2,0),(2,3,0),(7,4,3),(47,8,5),(23,12,5),(1103,24,1)$ | 7373556 | 2006357 |
| $-3$ | $-1$ | $(3,2,0),(2,3,0),(7,4,3),(47,8,5),(23,12,5),(1103,24,1)$ | 7373556 | 14686445 |

Table 1: Covering triples for the cases $b = 1, a = \pm 2, \pm 3, \pm 4, \pm 5$ and $b = -1, a = \pm 3$

If $a = -1, b = -1$, then it can be easily verified that the sequence given by the recurrence $x_{n+1} = -x_n - x_{n-1}$ has period 3. Hence, if one chooses $x_0 = 8$ and $x_1 = 27$, then $x_2 = -35$, and due to the periodic behavior all terms of the sequence are composite.

Similarly, if $a = 1, b = -1$, then the sequence given by the recurrence $x_{n+1} = x_n - x_{n-1}$ has period 6. Again, if one chooses $x_0 = 8$ and $x_1 = 35$, then the first few terms of the sequence are $8, 35, 27, -8, -35, -37, 8, 35, 27, \ldots$; that is, $x_n$ is always composite.

If $a = b = 1$, then Vsemirnov's pair $v_0 = 106276436867$, $v_1 = 35256392432$ shows that all the numbers

$$v_n = v_{n-1} + v_{n-2} = v_1 F_n + v_0 F_{n-1} \tag{15}$$

are composite. Here, $F_n$ is the $n$th Fibonacci number, where $F_{-1} = 1, F_0 = 0, F_1 = 1$.

For the case $a = -1, b = 1$, we follow the solution in [3].

It can be easily checked that the general term of the sequence $x_{n+1} = -x_n + x_{n-1}$ can be written as

$$x_n = (-1)^{n+1} x_1 F_n + (-1)^n x_0 F_{n-1}, \quad n \geq 0. \tag{16}$$

We choose $x_0 = v_0 - v_1 = 71020044435$ and $x_1 = v_0 = 106276436867$. It is easy to check that $x_0$ and $x_1$ are relatively prime composite integers. Moreover, from (15) and (16) we obtain that

$$\begin{aligned} x_n = (-1)^{n+1} v_0 F_n + (-1)^n (v_0 - v_1) F_{n-1} + = (-1)^{n+1} v_1 F_{n-1} + (-1)^{n+1} v_0 (F_n - F_{n-1}) = \\ = (-1)^{n+1} v_1 F_{n-1} + (-1)^{n+1} v_0 F_{n-2} = (-1)^{n+1} \left( v_1 F_{n-1} + v_0 F_{n-2} \right) = (-1)^{n-1} v_{n-1}. \end{aligned}$$

Hence, $|x_n| = v_{n-1}$ is composite for all $n \geq 0$. The proof of the theorem is now complete.

# 6 A surprising result

In this section we prove the following:

**Theorem 11.** *Consider the integers $a, b$ such that $|a| \geq 3$ and $b = -1$. Let $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = au_n + bu_{n-1} = au_n - u_{n-1}$ be the Lucas sequence of the first kind associated with $a$ and $-1$. Then $|u_n|$ is composite for all $n \geq 3$.*

*Proof.* One has $u_3 = a^2 - 1$, and $u_4 = a^3 - 2a$, which are obviously composite. Since $|a| > |b| = 1$, Lemma 3 implies that the sequence $(|u_n|)_{n\geq 0}$ is strictly increasing. Suppose for the sake of contradiction that there exists an $n \geq 2$ such that $|u_{n+1}| = p$, where $p$ is some prime number. Since $|u_{n+1}| \geq |u_3| = a^2 - 1$ it follows that necessarily $p > |a|$.

Now using Lemma 2 for the sequence $(u_n)_{n\geq 0}$, equality (4) becomes

$$u_{n+1}^2 - au_n u_{n+1} + u_n^2 = u_1^2 - au_1 u_0 + u_0^2,$$

and since $u_0 = 0$, $u_1 = 1$, and $|u_{n+1}| = p$, we obtain that

$$u_n^2 \pm apu_n + p^2 - 1 = 0. \tag{17}$$

Regard the above equation as a quadratic in $u_n$. Since $u_n \in \mathbb{Z}$, it is necessary that the discriminant is a perfect square, that is, there exist a nonnegative integer $c$ such that

$$a^2p^2 - 4(p^2 - 1) = c^2 \quad \text{from which} \quad (a^2 - 4)p^2 = c^2 - 4 = (c-2)(c+2). \tag{18}$$

Since $|a| \geq 3$, one can assume that $c \geq 3$. Since $p$ is a prime and $p^2$ divides $(c-2)(c+2)$, we have two possibilities. If $p$ divides both $c - 2$ and $c + 2$ then $p$ divides 4, which means that $p = 2$. However, this is impossible since $p > |a| \geq 3$. Otherwise, $p^2$ divides either $c - 2$ or $c + 2$. In either case we obtain that $c + 2 \geq p^2$. Using this inequality in (18), it follows that

$$(a^2 - 4)p^2 = (c-2)(c+2) \geq p^2(p^2 - 4) \implies |a| \geq p, \quad \text{a contradiction.}$$

This completes the proof. □

In particular, the above theorem holds if $a = 4$ and $b = -1$, thus answering a question of Vos Post (see A001353 in [14]).

What is remarkable about this situation is that while $u_n$ is composite for every $n \geq 3$, it seems likely that there is no finite set of primes $p_1, p_2, \ldots p_t$ such that every $u_n$ is divisible by some $p_i$, $1 \leq i \leq t$. In fact, we suspect the following is true.

**Conjecture 12.** Let $(u_n)_{n\geq 0}$ be the Lucas sequence of the first kind associated with some $|a| \geq 3$ and $b = -1$. Then for any two different primes $p$ and $q$, $u_p$ and $u_q$ are relatively prime.

If true, this conjecture would immediately imply that there is no finite set of primes $p_1, p_2, \ldots p_t$ such that every $u_n$ is divisible by some $p_i$, $1 \leq i \leq t$.

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2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37, 11A07, 11Y55.
*Keywords:* second-order linear recurrence, Lucas sequence of the first kind, composite number, covering system of congruences.

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(Concerned with sequences A001906, A001353, A004254, and A001109.)

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Received January 7 2019; revised versions received February 4 2019; September 8 2019; September 24 2019. Published in *Journal of Integer Sequences*, September 24 2019.

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