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The Hankel Transform and Some of its Properties

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Abstract

The Hankel transform of an integer sequence is defined and some of its properties discussed. It is shown that the Hankel transform of a sequence S is the same as the Hankel transform of the Binomial or Invert transform of S. If H is the Hankel matrix of a sequence and H=LU is the LU decomposition of H, the behavior of the first super-diagonal of U under the Binomial or Invert transform is also studied. This leads to a simple classification scheme for certain integer sequences.

1. Introduction.

The Hankel matrix H of the integer sequence $\{a_1, a_2, a_3, \cdots\}$ is the infinite matrix

$$H = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ a_4 & a_5 & a_6 & a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with elements $h_{i,j} = a_{i+j-1}$. The *Hankel matrix* H_n *of order n* of A is the upper-left n$\times$n submatrix of H, and h_n, the *Hankel determinant of order n* of A, is the determinant of the corresponding Hankel matrix of order n, $h_n = \det(H_n)$. For example, the Hankel matrix of order 4 of the Fibonacci sequence 1,1,2,3,5,… , is

$$H_4 = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 8 \\ 3 & 5 & 8 & 13 \end{bmatrix},$$

with 4^{th} order Hankel determinant $h_4 = 0$. Hankel matrices of integer sequences and their determinants have been studied in several recent papers by Ehrenborg [1] and Peart and Woan [2].

Given an integer sequence $A = \{a_1, a_2, a_3, \cdots\}$, the sequence $\{h_n\} = \{h_1, h_2, h_3, \cdots\}$ of Hankel determinants of A is called the *Hankel transform of A*, a term first introduced by the author in sequence A055878 of the On-Line Encyclopedia of Integer Sequences (EIS)[5]. For example, the Hankel matrix of order 4 of the sequence of Catalan numbers $\{1, 1, 2, 5, 14, 42, 132, \cdots\}$ (sequence A000108 in the EIS) is

$$H_4 = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix},$$

and the determinants of orders 1 through 4 give the Hankel transform $\{1,1,1,1,\ldots\}$.

The Hankel transform can easily be shown to be many-to-one, illustrated by the fact that a search of the EIS finds approximately twenty sequences besides A000108 that have the Hankel transform $\{1,1,1,1,\ldots\}$. The author and Michael Somos [6], working independently, found ten sequences in the EIS whose Hankel transform is $\{\prod_{i=0}^{n}(i!)^2\}$ (A055209), which was shown theoretically by Radoux [3] to be the Hankel transform of the *derangement*, or *rencontres*, numbers (A000166). Other examples of groups of sequences in the EIS all of which have the same Hankel transform may be found in the comments to sequences A000079 and A000178.

2. Invariance of the Hankel Transform.

Further computational investigation reveals numerous instances in which one member of a pair of sequences with the same Hankel transform is the Binomial or Invert transform of the other. Some examples are provided by A000166 and its Binomial transform A000142, both of which have the Hankel transform A055209, and by A005043 and its Invert transform A001006, both of which have {1, 1, 1, 1, …} for their Hankel transform. In the following it is shown that this invariance of the Hankel transform under applications of the Binomial or Invert transform holds in general. The definitions of the Binomial and Invert transforms may be found on the EIS web site [5].

Theorem 1. *Let A be an integer sequence and B its Binomial transform. Then A and B have the same Hankel transform.*

Proof. Let $A = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$, and define H* to be the matrix H* = RHC, where the elements of R, H, and C are given by

$$r_{i,j} = \begin{cases} 0, & if\ i < j, \\ \binom{i-1}{j-1}, & if\ i \geq j \end{cases}, \quad h_{k,m} = a_{k+m-1}, \quad and \quad c_{i,j} = \begin{cases} 0, & if\ i > j, \\ \binom{j-1}{i-1}, & if\ i \leq j \end{cases},$$

and $\binom{i}{j}$ denotes the usual binomial coefficient. Then the elements of H* are

$$h^*_{i,j} = \sum_{k=1}^{i} \sum_{m=1}^{j} \binom{i-1}{k-1} a_{k+m-1} \binom{j-1}{m-1},$$

which, by making slight changes of variables, gives

$$h^*_{i,j} = \sum_{k=0}^{i-1} \sum_{m=0}^{j-1} \binom{i-1}{k} \binom{j-1}{m} a_{k+m-1}.$$

By the well-known Vandermonde convolution formula [4] and another slight change of variable, this reduces to

$$h^*_{i,j} = \sum_{s=1}^{i+j-1} \binom{i+j-2}{s-1} a_s,$$

which, by the definition of the Binomial transform (see [5]), is b_{i+j-1}, thus showing that H* is the Hankel matrix of sequence B. Thus the terms of the Hankel transforms of the sequences A and B are det(H_n) and det($R_n H_n C_n$), respectively, where R_n, H_n, and C_n are the upper-left submatrices of order n of H, R, and C, respectively. But R_n and C_n are both triangular with all 1's on the main diagonal, thus det(R_n) and det(C_n) are both 1, and therefore det(H_n) = det($R_n H_n C_n$), completing the proof. ∎

Theorem 2. *Let A be an integer sequence and B its Invert transform. Then A and B have the same Hankel transform.*

Proof. Let $A = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$, and define H* to be the matrix H*=RHC, where the elements of R, H, and C are given by

$$r_{i,k} = \begin{cases} 0, & if\ k > i, \\ b_{i-k}, & if\ k \le i \end{cases}, \quad h_{k,m} = a_{k+m-1}, \quad \text{and } c_{m,j} = \begin{cases} 0, & if\ j < m, \\ b_{j-m}, & if\ j \ge m \end{cases},$$

where b_0 is defined to be 1. Then the (i, j-1)-element of H* given by

$$\begin{aligned} h^*_{i,j-1} &= \sum_{k=1}^{i} \sum_{m=1}^{j-1} b_{i-k}\ a_{k+m-1}\ b_{j-m-1} \\ &= \sum_{k=2}^{i} \sum_{m=1}^{j-1} b_{i-k}\ a_{k+m-1}\ b_{j-m-1} + b_{i-1} \sum_{m=1}^{j-1} a_m\ b_{j-m-1} \\ &= \sum_{k=1}^{i-1} \sum_{m=1}^{j-1} b_{i-1-k}\ a_{k+m}\ b_{j-m-1} + b_{i-1} \left[\sum_{m=1}^{j-2} a_m\ b_{j-m-1} + a_{j-1} \right] \\ &= \sum_{k=1}^{i-1} \sum_{m=2}^{j} b_{i-1-k}\ a_{k+m-1}\ b_{j-m} + b_{i-1} b_{j-1} \\ &= \sum_{k=1}^{i-1} \sum_{m=1}^{j} b_{i-1-k}\ a_{k+m-1}\ b_{j-m} + b_{j-1} \sum_{k=1}^{i-1} b_{i-1-k}\ a_k + b_{i-1} b_{j-1} \\ &= h^*_{i-1,j}\ , \end{aligned}$$

showing that elements of H* are constant along anti-diagonals. But, clearly,

$$\begin{aligned} h^*_{1,j} &= \sum_{k=1}^{1} \sum_{m=1}^{j} b_{1-k}\ a_{k+m-1}\ b_{j-m} \\ &= b_0 \sum_{m=1}^{j} a_m b_{j-m} \\ &= b_j\ , \end{aligned}$$

the last step following from the definition of the Invert transform (see [5]), which shows that $h^*_{i,j} = b_{i+j-1}$ or, in other words, that H* is the Hankel matrix of B. Since L and R are triangular with diagonals consisting of all 1's, this shows that the Hankel determinants of B are the same as those for A, and thus A and B have the same Hankel transform. ∎

3. The LU Decomposition and the First Super-Diagonal.

If the LU-decomposition of the Hankel matrix of an integer sequence A is H = LU, then the main diagonal of U clearly determines the Hankel transform of the sequence, and vice versa. By Theorem 1, if H* = L*U* is the LU-decomposition of the Hankel matrix H* of the Binomial or Invert transform of A, then the main diagonals of U and U* are identical. Thus the main diagonal of U is not sufficient to determine the sequence A, a point already noted. It is easy to see, however, that the main diagonal of U and the first superdiagonal, taken together, do determine A. It suffices to note, in proof, that H_n, the Hankel matrix of order n, consists of the first 2n-1 terms $a_1, a_2, a_3, \cdots$ of A and that the main diagonal and first superdiagonal of U_n contain 2n-1 elements whose values are linear combinations of the a's. Thus, $U_{1,1}$ determines a_1, $U_{1,2}$ and $U_{2,2}$ determine a_2 and a_3, and, by recursion, $U_{n-1,n}$ and $U_{n,n}$ determine a_{2n-2} and a_{2n-1}.

Because of the result just stated, it is of some interest to know how the first superdiagonal of U* is related to the first superdiagonal of U, where H=LU and H*=L*U*. The following two theorems give this relationship when A* is the Binomial transform or Invert transform of A.

Theorem 3. *Let H and H* be the Hankel matrices of the integer sequence A and its Binomial transform A*, respectively, and let H=LU and H*=L*U* be their LU-decompositions. Then the first super-diagonals of U and U* are related by* $U^*_{i,i+1} = U_{i,i+1} + iU_{i,i}$.

Proof. We have H = LU and, by the proof of the previous theorem, H* = RHC = RLUC, where the matrices R and C are as defined in that theorem. Thus U* = UC or, in terms of elements,

$$U^*_{i,j} = \sum_{k=1}^{j} U_{i,k} \binom{j-1}{k-1},$$

which, since $U_{i,k}$ is upper triangular, can be written

$$U^*_{i,j} = \sum_{k=i}^{j} U_{i,k} \binom{j-1}{k-1}.$$

The elements on the first super-diagonal are therefore given by

$$U^*_{i,i+1} = U_{i,i}\binom{i}{i-1} + U_{i,i+1}\binom{i}{i},$$

which reduces immediately to

$$U^*_{i,i+1} = U_{i,i+1} + iU_{i,i},$$

as was to be proved. ∎

A special case, which is of some interest because of a fairly large number of examples found in the EIS, follows immediately and is stated in the following corollary.

Corollary 1. *Let A be an integer sequence with Hankel transform {1, 1, 1, 1, 1, …} and let H and H* be the Hankel matrices of A and its Binomial transform A*, respectively. Then, if H=LU and H*=L*U* are the LU-decompositions of H and H*, the first superdiagonals of U and U* are related by* $U^*_{i,i+1} = U_{i,i} + i$.

The analogous results for the Hankel matrix of the Invert transform of a sequence follow.

Theorem 4. *Let A be an integer sequence, with Hankel matrix H, and let B be the Invert transform of A, with Hankel transform H*. Let H=LU be the LU-decomposition of H and H*=L*U* the LU-decomposition of H*. Then the elements of the first superdiagonals of U and U* are related by* $u^*_{i,i+1} = u_{i,i+1} + a_1 u_{i,i}$.

Proof. Let the matrices R and C be as in the proof of Theorem 3. Then H* = L*U* = RHC =RLUC, from which it follows that U* = UC. Thus we have, in general,

$$u^*_{i,j} = \sum_{m=i}^{j} u_{i,m}\, b_{j-m},$$

and, in particular,

$$\begin{aligned} u^*_{i,i+1} &= \sum_{m=i}^{i+1} u_{i,m}\, b_{i+1-m} \\ &= u_{i,i}\, b_1 + u_{i,i+1}\, b_0 \\ &= u_{i,i+1} + a_1\, u_{i,i}, \end{aligned}$$

completing the proof. ∎

Again, because of the large number of sequences in the EIS with Hankel transform {1, 1, 1, 1, 1, …}, we state the following corollary.

Corollary 2. *Let A be an integer sequence with Hankel transform {1, 1, 1, 1, 1, …} and let H and H* be the Hankel matrices of A and its Invert transform A*, respectively. Then, if H=LU and H*=L*U* are the LU-decompositions of H and H*, the first superdiagonals of U and U* are related by* $U^*_{i,i+1} = U_{i,i} + 1$.

4. Sequences with Hankel Transform {1, 1, 1, 1, 1, …}.

A search of the EIS database found almost twenty sequences with Hankel transform {1, 1, 1, 1, 1, …}, of which seventeen are related through the Binomial and Invert transforms. In a few cases an initial term or two must be added or deleted. It is rather surprising that all of these sequences exhibit a linear polynomial behavior of the first super-diagonal when reduced to upper triangular form. Table 1 below illustrates the relationships among these 17 sequences. Each sequence in the table is the Binomial transform of the sequence in the adjacent column to its left and the Invert transform of the sequence in the adjacent row just above the given entry. The linear polynomial written just below the EIS sequence number gives the elements of the first super-diagonal of U in the LU decomposition H=LU, where H is the Hankel matrix of the sequence. The parameter i is the row index of U. The operator (E) denotes the shift operator and is used here to denote the deletion of the first term of the sequence. Added initial terms are shown in braces.

Note that, because of Corollaries 1 and 2 governing the behavior of the elements of the first superdiagonal under the action of the Binomial or Invert transforms, the constant terms increase by 1 for each row change from top to bottom and the first degree coefficient increases by 1 for each column change from left to right. In one case, in the bottom row, in which the first superdiagonal is described by the polynomial i+2, the sequence, which is the Binomial transform of A054341 and the Invert transform of (E)A005773 and whose initial terms are {1,3,10,34,117, …}, was not found to be listed in the EIS. It has since been listed, and now appears in the encyclopedia as sequence A059738.

Table 1.

		$\{1,0\} \cup$ A000957 2i - 2	A033321 3i - 2	A033543 4i - 2
	A005043 i - 1	A000108 2i - 1	A007317 3i - 1	
	A001006 I	(E)A000108 2i	A002212 3i	A005572 4i
A001405 1	(E)A005773 i + 1	A001700 2i + 1	A026378 3i + 1	A005573 4i + 1
A054341 2	{1,3,10,34,117, …} i + 2	A049027 2i + 2		

In order to illustrate the significance of this table, we look at A000108 (the Catalan numbers, with many combinatorial interpretations, one of which is the number of ways to insert n pairs of parentheses in a word of n letters) in row 2 and column 3 of the table. The sequence is {1, 1, 2, 5, 14, 42, 132, 429, 1430,…}, with Hankel matrix

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 14 & \cdots \\ 1 & 2 & 5 & 14 & 42 & \cdots \\ 2 & 5 & 14 & 42 & 132 & \cdots \\ 5 & 14 & 42 & 132 & 429 & \cdots \\ 14 & 42 & 132 & 429 & 1430 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which row-reduces to the upper-triangular form

$$U = \begin{bmatrix} 1 & 1 & 2 & 5 & 14 \\ 0 & 1 & 3 & 9 & 28 \\ 0 & 0 & 1 & 5 & 24 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which clearly exhibits the Hankel transform of {1, 1, 1, 1, 1, …} and the 2i-1 polynomial behavior of the first super-diagonal {1, 3, 5, 7, …}, as indicated in the table. If we now take the Binomial transform of A000108, we get {1, 2, 5, 15, 51, 188, 731, …} = A007317, with Hankel matrix

$$\begin{bmatrix} 1 & 2 & 5 & 15 & 51 \\ 2 & 5 & 15 & 51 & 188 \\ 5 & 15 & 51 & 188 & 731 \\ 15 & 51 & 188 & 731 & 2950 \\ 51 & 188 & 731 & 2950 & 12235 \end{bmatrix},$$

which, in turn reduces to the upper-triangular form

$$U = \begin{bmatrix} 1 & 2 & 5 & 15 & 51 \\ 0 & 1 & 5 & 21 & 86 \\ 0 & 0 & 1 & 8 & 46 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

showing the Hankel transform {1, 1, 1, 1, 1, …} and the first super-diagonal {2, 5, 8, 11, …} = {3i-1}, again in agreement with the table.

If we now return to A000108 and take its Invert transform, we get (E)A000108 = {1, 2, 5, 14, 42, 132, 429, 1430, 4862, …}, that is, A000108 with the first term deleted. The Hankel matrix of this sequence row-reduces to

$$U = \begin{bmatrix} 1 & 2 & 5 & 14 & 42 \\ 0 & 1 & 4 & 14 & 48 \\ 0 & 0 & 1 & 6 & 27 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

again revealing the Hankel transform {1, 1, 1, 1, 1, …} and the polynomial behavior of 2i for the first super-diagonal, {2, 4, 6, 8, …}, in agreement with row 3, column 3 of the table.

Three other sequences have been found in the EIS which have the Hankel transform {1, 1, 1, 1, 1, …} but do not have a linear polynomial behavior of the first super-diagonal when reduced to upper-triangular form. These are A054391, A054393, and A055879, with first super-diagonals {1, 3, 4, 5, 6, …}, {1, 3, 5, 6, 7, …}, and {1, 2, 2, 3, 3, 4, 4, …}, respectively.

5. Other Families of Sequences.

Several other families of sequences, each member of which has the same Hankel transform sequence, have been found in the EIS, but the relationships among the members of the family via the Binomial and Invert transforms is much less complete than that indicated in Table 1 for the case of Hankel transform {1, 1, 1, 1, 1, …}.

Seven sequences have been found with Hankel transform {1, 2, 4, 8, 16, …}: A000984, A002426, A026375, A026569, A026585, and A026671. Four of these are related by the Binomial and Invert transforms, as shown in the following Table 2 in which each sequence listed is the Binomial transform of the sequence just to the left and the Invert transform of the sequence just above.

Table 2.

A002426	A000984	A026375
	A026671	

Seven sequences were found with Hankel transform {1, 1, 2, 12, 288, …}: A000085, A000110, A000296, A005425, A005493, A005494, and A045379. These are all related to at least one other by the Binomial transform, as shown in Table 3, in which each sequence is the Binomial transform of the sequence just to its left. No Invert transform relations hold among adjacent rows.

Table 3.

A000296	A000110	(E)A000110	A005493	A005494	A045379

A000085	A005425

Several of the sequences listed above, with Hankel transform {1, 1, 2, 12, 288, …}, as well as some of those below, with Hankel transform {1, 1, 4, 144, 82944, …}, were discussed by Ehrenborg in [1].

Nine sequences were found with Hankel transform {1, 1, 4, 144, 82944, …}: A000142, A000166, A000522, A003701, A010483, A010842, A052186, A053486, and A053487. Seven of these are related to at least one other by the Binomial or Invert transform. Table 4 shows these relationships, following the same format as used for Table 1.

Table 4.

	A052186				
A000166	A000142	A000522	A010842	A053486	A053487

6. Concluding Remarks.

Among questions raised by this investigation into properties of the Hankel transform we mention two which seem to be deserving of further study.

First, is there a combinatorial significance to the fact that essentially all studied sequences listed in the EIS [5] that have the Hankel transform {1, 1, 1, 1,…} and are related by the Binomial or Invert transform, have a first super-diagonal which, when reduced to upper-diagonal form, is linear in the row index with small coefficients, with constant terms ranging from -2 to 2 and first degree terms ranging from 0 to 4, as shown in Table 1?

Second, are there other interesting transforms, T, of an integer sequence S, in addition to the Binomial and Invert transforms studied in this paper, with the property that the Hankel transform of S is the same as the Hankel transform of the T transform of S?

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(Concerned with sequences A000079, A000085, A000108, A000110, A000142, A000166, A000178, A000296, A000522, A000957, A000984, A001006, A001405, A001700, A002212, A002426, A003701, A005043, A005425, A005493, A005494, A005572, A005773, A007317, A010483, A010842, A026375, A026378, A026569, A026585, A026671, A033321, A033543, A045379, A049027, A052186, A053486, A053487, A054341, A054391, A054393, A055209, A055878, A055879, A059738.)

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