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Separation of Unitary Representations of Exponential Lie Groups

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Abstract. We show that every irreducible unitary representation π of an exponential Lie group $G = \exp \mathfrak{g}$ is characterized by its generalized moment set.

1. Introduction

Let G be a real Lie group with Lie algebra $\mathfrak{g}$, $(\pi, \mathcal{H}_\pi)$ a unitary representation of G and $\mathcal{H}_\pi^\infty$ the space of C^∞ vectors of π. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. In [8], Wildberger has introduced the moment map Ψ_π of π. For all ξ in $\mathcal{H}_\pi^\infty \setminus \{0\}$, $\Psi_\pi(\xi)$ in $\mathfrak{g}^*$ is defined by:

$$\Psi_\pi(\xi)(X) = \frac{1}{i}\frac{\langle d\pi(X)\xi, \xi\rangle}{\langle \xi, \xi\rangle}, \quad X \in \mathfrak{g}. \tag{1.1}$$

The moment set I_π of the representation π is by definition the closure in $\mathfrak{g}^*$ of the image of the moment map:

$$\Psi_\pi : \mathcal{H}^\infty \setminus \{0\} \longrightarrow \mathfrak{g}^*.$$

Wildberger has given an explicit description of the moment set I_π when G is a connected simply connected nilpotent Lie group. More precisely, he shows (Theorem 4.2 in [8]) that I_π is the closure of the convex hull of the coadjoint orbit $\mathcal{O}_\pi$ associated to π via the Kirillov theory, i.e.

$$I_\pi = \overline{\mathrm{conv}(\mathcal{O}_\pi)}.$$

This result has been generalized by Arnal and Ludwig [1] for connected solvable Lie groups. Nevertheless, as shown in [8], the moment set does not characterize the representation for nilpotent Lie groups.

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We can extend the moment map to the dual of the complex universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ as follows:

$$\tilde{\Psi}_\pi(\xi)(A) = \mathfrak{Re}\left(\frac{1}{i}\frac{\langle d\pi(A)\xi,\xi\rangle}{\langle\xi,\xi\rangle}\right), \quad \xi \in \mathcal{H}^\infty \setminus \{0\}, \quad A \in \mathcal{U}(\mathfrak{g}) \tag{1.2}$$

and consider the convex hull $J(\pi)$ of the image of this generalized moment map $\tilde{\Psi}_\pi$. Let us call the set $J(\pi)$ the generalized moment set of π, i.e.

$$J(\pi) = \operatorname{conv}\left(\tilde{\Psi}_\pi(\mathcal{H}^\infty \setminus \{0\})\right).$$

Let $\mathcal{U}_n$ be the subspace of $\mathcal{U}(\mathfrak{g})$ consisting of elements A of degree less or equal to n. By restriction to $\mathcal{U}_n$, we define:

$$J(\pi)^n = \{f|_{\mathcal{U}_n}, \quad f \in J(\pi)\}.$$

In [2], we show:

Theorem 0. *Let G be a simply connected nilpotent Lie group. Then there exists an integer n such that, for any unitary irreducible representations π and ρ, one has $J(\pi)^n = J(\rho)^n$ if and only if π and ρ are equivalent.*

In this paper, we use the set $J(\pi)$ to characterize the irreducible unitary representations of exponential groups. More precisely, we prove the following:

Theorem 1. *Let $G = \exp \mathfrak{g}$ be an exponential Lie group. Let π and ρ be two unitary irreducible representations. Then $\pi \sim \rho$ if and only if $J(\pi) = J(\rho)$.*

2. Preliminaries

Let $\mathfrak{g}$ be a real Lie algebra, G its simply connected Lie group and let $\exp : \mathfrak{g} \to G$ be the exponential mapping. The group G is called an exponential group if the exponential mapping exp is a diffeomorphism. In particular this condition implies that G is solvable. Throughout this paper, we assume that G is an exponential group.

It is well known that each irreducible unitary representation π of G is associated to a coadjoint orbit $\mathcal{O}_\pi$ in $\mathfrak{g}^*$ by the following construction:

Let f be in $\mathfrak{g}^*$. We take polarizations in f, i.e. subalgebras $\mathfrak{h}$ such that:

$$\langle f, [\mathfrak{h},\mathfrak{h}]\rangle = 0$$

with maximal dimension. Among those polarizations, we can find one satisfying the Pukanszky's condition:

$$H.f = f + \mathfrak{h}^\perp \quad (\text{here } H = \exp(\mathfrak{h})).$$

For such a Pukanszky polarization, we define $\pi_{f,\mathfrak{h}}$ by:

$$\pi_{f,\mathfrak{h}} = \operatorname{Ind}_H^G \chi_f$$

where χ_f denotes the unitary character:

$$\chi_f(\exp X) = e^{if(X)} \qquad \forall X \in \mathfrak{h}$$

of H.

Then $\pi_{f,\mathfrak{h}}$ is irreducible and the class π of $\pi_{f,\mathfrak{h}}$ depends only of the coadjoint orbit of f.

Moreover, the map $f \longmapsto [\pi_{f,\mathfrak{h}}]$ defines a bijection K (the Kirillov correspondance) between the space $\mathfrak{g}^*/G$ of orbits and the the space $\hat{G}$ of classes of irreducible unitary representation of G [4], [5], [7].

Let now H be a closed subgroup of G and π_0 a unitary representation of H on the space $\mathcal{H}_{\pi_0}$. Let us describe explicitely the induced representation $\pi = \mathrm{Ind}_H^G \pi_0$. Let dg be a left Haar measure on G and Δ_G the modular function of G. Note $\Delta_{H,G}$ the positive character of H defined by:

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}.$$

Let $K(G,H)$ be the space of continuous functions F on G, with compact support modulo H and which verify:

$$F(gh) = \Delta_{H,G}(h)F(g) \qquad (g \in G,\ h \in H).$$

G acts on this space by left translation. It is well known ([4]) that, up to a multiplication by a scalar, there exists a unique G-invariant positive linear form:

$$F \mapsto \oint_{G/H} F(g)\, d\dot{g}$$

on $K(G,H)$.

Let $K(G,H,\pi_0)$ be the space of continuous mappings $F : G \longrightarrow \mathcal{H}_{\pi_0}$ with compact support modulo H and which verify:

$$F(gh) = \Delta_{H,G}^{1/2}(h)\pi_0(h)^{-1}F(g) \qquad (g \in G,\ h \in H).$$

If F belongs to $K(G,H,\pi_0)$, the function $g \mapsto \|F(g)\|_{\pi_0}^2$ belongs to $K(G,H)$. Set:

$$\|F\|_\pi^2 = \oint_{G/H} \|F(g)\|_{\pi_0}^2\, d\dot{g}.$$

The induced representation $\pi = \mathrm{Ind}_H^G\, \pi_0$ of G is realized by left translation on the Hilbert space $\mathcal{H}_\pi$ obtained by the completion of $K(G,H,\pi_0)$ with the norm defined above.

Definition 2.1 Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, a set $\mathcal{B} = \{X_1, \cdots, X_d\}$ of elements of $\mathfrak{g}$ is called a coexponential basis to $\mathfrak{h}$ in $\mathfrak{g}$ if the map:

$$(t_1, t_2, \cdots, t_d; Y) \to E_{\mathcal{B}}(t_1, t_2, \cdots, t_d; Y) = \exp t_1X_1 \exp t_2X_2, \ldots, \exp t_dX_d \exp Y$$

is a diffeomorphism from $\mathbb{R}^d \times \mathfrak{h}$ onto G.

For any subalgebra $\mathfrak{h}$, there exist coexponential basis (see [4], [5]). With such a basis, we can realize the space $\mathcal{H}_\pi$ of an induced representation $\pi = \mathrm{Ind}_H^G\, \pi_0$ as the space $L^2(\mathbb{R}^d, \mathcal{H}_{\pi_0})$.

3. Main lemma

The proof of theorem 1 will use an induction on the dimension of $\mathfrak{g}$. The following lemmata allow us to pass from $\mathfrak{g}$ to some subalgebra $\mathfrak{g}_0$ of lower dimension.

Lemma 3.1. *Let G be an exponential Lie group. Let π and ρ be two irreducible representations of G such that $J(\pi) = J(\rho)$. Suppose there exists a connected subgroup $G_0 = \exp(\mathfrak{g}_0)$ of codimension 1 and two irreducible representations π_0 and ρ_0 of G_0 such that:*

$$\pi = Ind_{G_0}^{G}\pi_0 \quad \text{and} \quad \rho = Ind_{G_0}^{G}\rho_0.$$

Choose a coexponential basis $\{X\}$ to $\mathfrak{g}_0$ in $\mathfrak{g}$, realize π (resp. ρ) on the space $L^2(\mathbb{R}, \mathcal{H}_{\pi_0})$ (resp. $L^2(\mathbb{R}, \mathcal{H}_{\rho_0})$). Suppose furthermore that there exists u in $\mathcal{U}(\mathfrak{g}_0)$ such that:

$$(d\pi(u)(\xi))(t) = h(t)\xi(t) \quad \text{and} \quad (d\rho(u)(\eta))(t) = h(t)\eta(t),$$

for each ξ in $\mathcal{H}_\pi^\infty$ and η in $\mathcal{H}_\rho^\infty$, for every t, where h is a strictly increasing C^∞ function with $h(0) \geq 0$. Then:

$$J(\pi_0) = J(\rho_0).$$

Proof. We recall that all C^∞ vectors of π are C^∞ functions of the variable t in $\mathbb{R}$ with value in the space $\mathcal{H}_{\pi_0}^\infty$. Moreover, if ξ_0 is a vector in $\mathcal{H}_{\pi_0}^\infty$ and φ a C^∞ complex valued map with compact support, then $\xi(t) = \varphi(t)\xi_0$ is a C^∞ vector of π [6].

We fix such a vector ξ, so that:

$$\varphi(0) = 1, \ \|\xi_0\|_{\pi_0} = 1 \quad \text{and} \quad \|\xi\|_\pi = 1.$$

Since $J(\pi) = J(\rho)$, there exist normalized C^∞ vectors $\eta_1, \dots, \eta_n$ in $\mathcal{H}_\rho^\infty$ and $\lambda_1, \dots, \lambda_n$ in]0,1[with $\sum_{j=1}^n \lambda_j = 1$ such that for all A in $\mathcal{U}(\mathfrak{g})$:

$$\mathfrak{Re}\left(\frac{1}{i}\langle d\pi(A)\xi, \xi\rangle\right) = \mathfrak{Re}\left(\frac{1}{i}\sum_{j=1}^n \lambda_j \langle d\rho(A)\eta_j, \eta_j\rangle\right).$$

This implies:

$$\langle d\pi(A)\xi, \xi\rangle = \sum_{j=1}^n \lambda_j \langle d\rho(A)\eta_j, \eta_j\rangle \text{ for all } A \text{ in } \mathcal{U}(\mathfrak{g}). \tag{3.1}$$

In particular, for any integer k, we have:

$$\langle d\pi(u^k)\xi,\xi\rangle = \sum_{j=1}^{n} \lambda_j \langle d\rho(u^k)\eta_j,\eta_j\rangle.$$

Choose $M>0$ and $m<M$ such that the support of φ is contained in $[m,M]$. Then, for every odd k,

$$\begin{aligned}\langle d\pi(u^k)\xi,\xi\rangle &= \int_{\mathbb{R}} h^k(t)\|\xi(t)\|_{\pi_0}^2\,dt \\ &= \int_m^M h^k(t)\varphi^2(t)\|\xi_0\|_{\pi_0}^2\,dt \\ &\leq (h(M))^k\|\xi\|_\pi^2 = (h(M))^k.\end{aligned}$$

Hence:

$$\begin{aligned}\sum_{j=1}^{n}\lambda_j\int_{-\infty}^{+\infty} h^k(t)\|\eta_j(t)\|_\rho^2\,dt &= \sum_{j=1}^{n}\lambda_j\langle d\rho(u^k)\eta_j,\eta_j\rangle \\ &= \langle d\pi(u^k)\xi,\xi\rangle \leq (h(M))^k.\end{aligned}$$

Thus, for each $\varepsilon>0$, for all $M+\varepsilon\leq t$, we have $0<h(M+\varepsilon)\leq h(t)$ and:

$$h(M+\varepsilon)^k\sum_{j=1}^{n}\lambda_j\int_{M+\varepsilon}^{+\infty}\|\eta_j(t)\|_\rho^2\,dt \leq \sum_{j=1}^{n}\lambda_j\int_{M+\varepsilon}^{+\infty} h^k(t)\|\eta_j(t)\|_\rho^2\,dt \leq h(M)^k.$$

This shows that:

$$\sum_{j=1}^{n}\lambda_j\int_{M+\varepsilon}^{+\infty}\|\eta_j(t)\|_\rho^2\,dt \leq \lim_{\substack{k\to\infty\\ k\ odd}}\left(\frac{h(M)}{h(M+\varepsilon)}\right)^k = 0.$$

Hence supp $\eta_j \subset]-\infty,M]$, for all j. Using the negative increasing function $t\longmapsto h(t)-h(M)$ on $]-\infty,M]$, one gets by (3.1):

$$\begin{aligned}\sum_{j=1}^{n}\lambda_j\int_{-\infty}^{M}(h(t)-h(M))^{2k}\|\eta_j(t)\|_\rho^2\,dt &= \sum_{j=1}^{n}\lambda_j\langle d\rho(u-h(M)1)^{2k}\eta_j,\eta_j\rangle \\ &= \langle d\pi(u-h(M)1)^{2k}\xi,\xi\rangle \\ &= \int_m^M (h(t)-h(M))^{2k}\varphi^2(t)dt.\end{aligned}$$

The function $t\mapsto(h(t)-h(M))^{2k}$ being decreasing,

$$\begin{aligned}\sum_{j=1}^{n}\lambda_j\int_{-\infty}^{M}(h(t)-h(M))^{2k}\|\eta_j(t)\|_\rho^2\,dt &= \int_m^M (h(t)-h(M))^{2k}\varphi^2(t)dt \\ &\leq (h(m)-h(M))^{2k}.\end{aligned}$$

Then for each $\varepsilon > 0$:

$$\sum_{j=1}^{n} \lambda_j \int_{-\infty}^{m-\varepsilon} (h(t)-h(M))^{2k} \|\eta_j(t)\|_\rho^2 \, dt \leq (h(m)-h(M))^{2k}.$$

If $t \leq m-\varepsilon$ then $(h(m-\varepsilon)-h(M))^{2k} \leq (h(t)-h(M))^{2k}$ thus:

$$\sum_{j=1}^{n} \lambda_j \int_{-\infty}^{m-\varepsilon} \|\eta_j(t)\|^2 dt \leq \lim_{k\to+\infty} \left(\frac{h(m)-h(M)}{h(m-\varepsilon)-h(M)}\right)^{2k} = 0.$$

Hence supp $\eta_j \subset [m, M]$ for each j.

Let now A be in $\mathcal{U}(\mathfrak{g}_0)$ and x a real number, for each integer k:

$$\begin{aligned}\int_m^M \frac{(ixh(t))^k}{k!} \langle (d\pi(A)\xi)(t), \xi(t)\rangle_{\pi_0}\, dt = \\ &= \langle \frac{1}{k!}(ixd\pi(u))^k d\pi(A)\xi, \xi\rangle \\ &= \sum_{j=1}^{n} \lambda_j \langle \frac{1}{k!}(ixd\pi(u))^k d\rho(A)\eta_j, \eta_j\rangle \\ &= \sum_{j=1}^{n} \lambda_j \int_m^M \frac{(ixh(t))^k}{k!} \langle (d\rho(A)\eta_j)(t), \eta_j(t)\rangle_{\rho_0}\, dt.\end{aligned}$$

Let $g(t) = \langle (d\pi(A)\xi)(t), \xi(t)\rangle_{\pi_0}$ and $g_j(t) = \langle (d\rho(A)\eta_j)(t), \eta_j(t)\rangle_{\rho_0}$, then g and g_j are C^∞ functions with compact support. Hence:

$$\int_{\mathbb{R}} e^{ixh(t)} g(t) dt = \sum_{j=1}^{n} \lambda_j \int_{\mathbb{R}} e^{ixh(t)} g_j(t) dt.$$

Let us make the change of variables $s = h(t)$. We set $\theta(s) = \frac{g(t)}{h'(t)}$ and $\theta_j(s) = \frac{g_j(t)}{h'(t)}$, so:

$$\hat{\theta}(x) = \int_{\mathbb{R}} e^{ixs}\theta(s) ds = \sum_{j=1}^{n} \lambda_j \int_{\mathbb{R}} e^{ixs}\theta_j(s) ds = \sum_{j=1}^{n} \lambda_j \hat{\theta}_j(x).$$

Then $\theta(x) = \sum_{j=1}^{n} \lambda_j \theta_j(x)$, by the injectivity of the Fourier transform. In particular $\theta(h(0)) = \sum_{j=1}^{n} \lambda_j \theta_j(h(0))$, i.e.:

$$\frac{1}{h'(0)} \langle (d\pi(A)\xi)(0), \xi(0)\rangle_{\pi_0} = \frac{1}{h'(0)} \sum_{j=1}^{n} \lambda_j \langle (d\rho(A)\eta_j)(0), \eta_j(0)\rangle_{\rho_0}.$$

Since

$$(d\pi(A)\xi)(0) = d\pi_0(A)(\xi(0)) = d\pi_0(A)\xi_0$$

and similarly for the representation ρ, we have:

$$\langle d\pi_0(A)(\xi_0), \xi_0\rangle_{\pi_0} = \sum_{j=1}^{n} \lambda_j \|\eta_j(0)\|^2_{\rho_0}. \frac{\langle d\rho_0(A)\eta_j(0), \eta_j(0)\rangle_{\rho_0}}{\|\eta_j(0)\|^2_{\rho_0}} \quad (3.2)$$

When $A = 1$, the formula (3.2) gives us

$$1 = \sum_{j=1}^{n} \lambda_j \|\eta_j(0)\|^2_{\rho_0}.$$

Hence $A \mapsto \langle d\pi_0(A)(\xi_0), \xi_0\rangle_{\pi_0}$ is a convex combination of some elements in $\tilde{\Psi}_{\rho_0}(\mathcal{H}^\infty_{\rho_0})$. So,

$$J(\pi_0) \subset J(\rho_0).$$

By exchanging the roles of π and ρ, we are done. ∎

By exactly the same arguments we can prove the following lemma:

Lemma 3.2. *Let G be an exponential Lie group. Let π and ρ be two irreducible representations of G such that $J(\pi) = J(\rho)$. Suppose that there exists a connected subgroup $G_0 = \exp(\mathfrak{g}_0)$ of codimension 2 and two irreducible representations π_0 and ρ_0 of G_0 such that:*

$$\pi = Ind_{G_0}^G \pi_0 \quad \text{and} \quad \rho = Ind_{G_0}^G \rho_0.$$

Choose a coexponential basis $\{X_1, X_2\}$ to $\mathfrak{g}_0$ in $\mathfrak{g}$, realize π (resp. ρ) on the space $L^2(\mathbb{R}^2, \mathcal{H}_{\pi_0})$ (resp. $L^2(\mathbb{R}^2, \mathcal{H}_{\rho_0})$). Suppose furthermore that there exist u_1, u_2 in $\mathcal{U}(\mathfrak{g}_0)$ and strictly increasing real C^∞ functions h_1, h_2 such that $h_j(0) \geq 0$ and:

$$(d\pi(u_1)(\xi))(t_1, t_2) = h_1(t_1)\xi(t_1, t_2) \quad \text{and} \quad (d\rho(u_1)(\eta))(t_1, t_2) = h_1(t_1)\eta(t_1, t_2)$$
$$(d\pi(u_2)(\xi))(t_1, t_2) = h_2(t_2)\xi(t_1, t_2) \quad \text{and} \quad (d\rho(u_2)(\eta))(t_1, t_2) = h_2(t_2)\eta(t_1, t_2).$$

for every ξ in $\mathcal{H}^\infty_\pi$, η in $\mathcal{H}^\infty_\rho$ and real numbers t_1, t_2. Then:

$$J(\pi_0) = J(\rho_0).$$

4. An example

Before we give the proof of theorem 1, we present an example where there is no polynomial invariant to separate the generic orbits. Let $\mathfrak{g}_\alpha$ be the Lie algebra spanned by X, Y_1, Y_2 such that:

$$[X, Y_1] = Y_1 - \alpha Y_2, \quad [X, Y_2] = \alpha Y_1 + Y_2 \quad \text{and} \quad [Y_1, Y_2] = 0.$$

The group $G_\alpha = \exp \mathfrak{g}_\alpha$ is called the Grélaud group. Let $\{X^*,\ Y_1^*,\ Y_2^*\}$ be the dual basis of $\{X,\ Y_1,\ Y_2\}$ in $\mathfrak{g}_\alpha^*$. The coadjoint orbits of G_α which are not reduced to a point are the orbits $\mathcal{O}_\theta$ of $f_\theta = \cos\theta Y_1^* + \sin\theta Y_2^*$ $(0 < \theta \leq 2\pi)$. It is easy to see that:

$$\mathcal{O}_\theta = \left\{ sX^* + e^{-t}\cos(\theta - \alpha t)Y_1^* + e^{-t}\sin(\theta - \alpha t)Y_2^*,\ s,\ t \in \mathbb{R} \right\}.$$

We consider the Pukanszky polarisation $\mathfrak{h} = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ in f_θ, we choose $\{-X\}$ as coexponential basis to $\mathfrak{h}$ in $\mathfrak{g}$. The unitary irreducible representation π, associated to the orbit $\mathcal{O}_\theta$ acts on the space $L^2(\mathbb{R})$ as:

$$\begin{aligned}
\pi(\exp xX)\xi(t) &= \xi(t+x) \\
\pi(\exp y_1 Y_1)\xi(t) &= e^{-iy_1 e^t \cos(\theta+\alpha t)}\xi(t) \\
\pi(\exp y_2 Y_2)\xi(t) &= e^{-iy_2 e^t \sin(\theta+\alpha t)}\xi(t).
\end{aligned}$$

The self-adjoint operators :

$$id\pi(X) = i\frac{d}{dt}, \quad id\pi(Y_1) = e^t\cos(\theta+\alpha t) \quad \text{and} \quad id\pi(Y_2) = e^t\sin(\theta+\alpha t)$$

on $L^2(\mathbb{R})$ characterize the representation π. Since $id\pi(Y_1)$ and $id\pi(Y_2)$ are multiplication operators, we could obtain the value of θ which characterizes the orbit $\mathcal{O}_\theta$, directly from these operators by the following transcendental expression:

$$e^{i\theta}Id = \frac{id\pi(Y_1) - d\pi(Y_2)}{\sqrt{-d\pi(Y_1)^2 - d\pi(Y_2)^2}} e^{-\frac{i}{2}\alpha \log(-d\pi(Y_1)^2 - d\pi(Y_2)^2)}. \tag{4.1}$$

But we can also use lemma 3.1. Indeed let $f_\psi = \cos\psi Y_1^* + \sin\psi Y_2^*$ and ρ the representation associated to $\mathcal{O}_\psi$. Let us put:

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \pi_0 = \chi_{f_\theta} \quad \rho_0 = \chi_{f_\psi}, \quad u = (iX)^2 + (iY)^2, \quad h(t) = e^{2t}.$$

If $J(\pi) = J(\rho)$, then with lemma 3.1:

$$J(\pi_0) = J(\chi_{f_\theta}) = J(\rho_0) = J(\chi_{f_\psi})$$

and $f_\theta = f_\psi$, $\pi \sim \rho$.

5. Proof of theorem 1

Now we are going to prove theorem 1. It is clear that if $\pi \sim \rho$ then $J(\pi) = J(\rho)$. We shall show the converse by induction on the dimension of G. It is obviously true for one dimensional G.

Let f (resp. g) be an element of the Kirillov orbit $\mathcal{O}_\pi$ of π (resp. $\mathcal{O}_\rho$ of ρ). If there exists a non trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$ on which f vanishes, then $d\pi(X) = 0$ for all X in $\mathfrak{a}$ and since $J(\pi) = J(\rho)$:

$$\langle d\rho(X)\eta, \eta\rangle = \sum_{j=1}^{n} \lambda_j \langle d\pi(X)\xi_i, \xi_j\rangle = 0, \qquad (X \in \mathfrak{a}, \quad \eta \in \mathcal{H}_\rho).$$

Hence $d\rho(X) = 0$ for each X in $\mathfrak{a}$ and $g|_\mathfrak{a}$ vanishes. Let $\bar{G} = G/A$ where $A = \exp \mathfrak{a}$ and $p : G \to \bar{G}$ be the canonical projection. This p defines projections (still denoted p) for the corresponding Lie algebras and enveloping algebras. We can write $f = \bar{f}{\circ}p = p^t(\bar{f})$ and $\pi = \bar{\pi}{\circ}p$ (resp. $g = \bar{g}{\circ}p$, $\rho = \bar{\rho}{\circ}p$). By construction:

$$J(\bar{\pi}) = p^t\left(J(\pi)\right) = p^t\left(J(\rho)\right) = J(\bar{\rho})$$

and $\bar{\pi}$ and $\bar{\rho}$ are equivalent by the induction hypothesis. Finally we get $\pi \sim \rho$.

We can thus assume than f does not vanish on any non trivial ideal. In particular the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ satisfies dim $\mathfrak{z}(\mathfrak{g}) \leq 1$ and if $\mathfrak{z}(\mathfrak{g})$ is one dimensional, f is non trivial on $\mathfrak{z}(\mathfrak{g})$.

Let $\mathfrak{a}$ be a non central minimal ideal of $\mathfrak{g}$. Then $\mathfrak{a}$ is abelian and of dimension equal to one, two or three (see [3]). Let us call $\mathfrak{a}^f$ the subspace:

$$\mathfrak{a}^f = \{T \in \mathfrak{g},\ \langle f, [T, \mathfrak{a}]\rangle = 0\}.$$

We shall prove in the sequel that $\mathfrak{a}^f = \mathfrak{a}^g$. Then according to [3], π and ρ are induced from the subgroup $G_0 = \exp \mathfrak{a}^f = \exp \mathfrak{a}^g$:

$$\pi = \mathrm{Ind}_{G_0}^G \pi_0, \quad \rho = \mathrm{Ind}_{G_0}^G \rho_0.$$

Case 1 dim $\mathfrak{z}(\mathfrak{g}) = 0$: Then the dimension of $\mathfrak{a}$ is one or two.

Subcase 1.1 $\dim(\mathfrak{a}) = 1$: Take Y in $\mathfrak{a}$ such that $f(Y) = 1$, then $g(Y) = \alpha \neq 0$ and so:

$$\mathfrak{a}^f = \{U \in \mathfrak{g},\ \ [U, Y] = 0\} = \mathfrak{a}^g.$$

Choose X in $\mathfrak{g}$ such that $[X, Y] = -Y$ as coexponential basis to $\mathfrak{a}^f$, then we have (see the notations of lemma 3.1):

$$\left(d\pi(Y)\xi\right)(t) = ie^t\xi(t), \quad \left(d\rho(Y)\eta\right)(t) = ie^t\eta(t) \qquad (\xi \in \mathcal{H}_\pi^\infty,\ \eta \in \mathcal{H}_\rho^\infty).$$

Then:

$$\alpha \int e^t \|\eta(t)\|_{\rho_0}^2\, dt = \frac{1}{i}\langle d\rho(Y)\eta, \eta\rangle = \sum_{j=1}^{n} \lambda_j \frac{1}{i}\langle d\pi(Y)\xi_j, \xi_j\rangle$$
$$= \sum_{j=1}^{n} \lambda_j \int e^t \|\xi_j(t)\|_{\pi_0}^2\, dt.$$

Hence α is non negative and if we replace g by $Ad^*(\exp \log \alpha X)g$, one can suppose that $g(Y) = 1$.

Now with $h(t) = e^t$, we can apply lemma 3.1 and the induction hypothesis, thus $\pi \sim \rho$.

Subcase 1.2 dim $\mathfrak{a} = 2$: Then there exist X, Y_1, Y_2 such that $\mathfrak{a} = \mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ with the brackets:

$$[X, Y_1] = Y_1 - \alpha Y_2, \quad [X, Y_2] = Y_2 + \alpha Y_1, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

This is a generalization of the Grélaud group case. As in subcase 1.1, we have:

$$\mathfrak{a}^f = \{U \in \mathfrak{g}, \ [U, Y_1] = [U, Y_2] = 0\} = \mathfrak{a}^g$$

and

$$\pi = \mathrm{Ind}_{G_0}^G \pi_0, \quad \rho = \mathrm{Ind}_{G_0}^G \rho_0 \qquad (\mathfrak{g}_0 = \mathfrak{a}^f = \mathfrak{a}^g).$$

Let us put $f_1 = f(Y_1)$, $f_2 = f(Y_2)$. Then $f_1^2 + f_2^2 \neq 0$ and since

$$f'^2_1 + f'^2_2 = e^{-2x}(f_1^2 + f_2^2) \quad \text{if} \quad f' = Ad^*(\exp xX)f,$$

we can suppose that $f_1^2 + f_2^2 = 1$ and similarly $g_1^2 + g_2^2 = 1$ if $g_1 = g(Y_1)$, $g_2 = g(Y_2)$. Then:

$$-d\pi(Y_1^2 + Y_2^2)\xi(t) = e^{2t}\xi(t), \quad -d\rho(Y_1^2 + Y_2^2)\eta(t) = e^{2t}\eta(t) \quad (\xi \in \mathcal{H}_\pi^\infty, \ \eta \in \mathcal{H}_\rho^\infty).$$

As above, we get $\pi \sim \rho$.

Case 2 dim $\mathfrak{z}(\mathfrak{g}) = 1$: Let $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}Z$. Then both $f(Z)$ and $g(Z)$ are not zero and since $J(\pi) = J(\rho)$, we can choose Z such that $f(Z) = g(Z) = 1$.

Subcase 2.1 dim $\mathfrak{a} = 2$: We choose a basis $\{Y, Z\}$ of $\mathfrak{a}$ such that $f(Y) = 0$ (Z is central and $f(Z) = 1$). Thus:

$$[U, Y] = \lambda(U)Y + \mu(U)Z \qquad (U \in \mathfrak{g})$$

with either λ and μ linearly independant in $\mathfrak{g}^*$ or $\lambda = 0$ and $\mu \neq 0$ (see [4]). Hence:

$$\mathfrak{a}^f = \{U \in \mathfrak{g}, \ \mu(U) = 0\} = \mathfrak{a}^g.$$

We choose X in $\mathfrak{g}$ such that $\lambda(X) = 0$ and $\mu(X) = 1$. Then acting on g with $Ad^*(\exp g(Y)X)$, we can suppose $g(Y) = 0$. Then:

$$\pi = \mathrm{Ind}_{G_0}^G \pi_0, \quad \rho = \mathrm{Ind}_{G_0}^G \rho_0 \qquad (\mathfrak{g}_0 = \mathfrak{a}^f = \mathfrak{a}^g).$$

Moreover $\{X\}$ is a coexponential basis to $\mathfrak{g}_0$ and:

$$d\pi(-iY)\xi(t) = t\xi(t), \qquad d\rho(-iY)\eta(t) = t\eta(t) \qquad (\xi \in \mathcal{H}_\pi^\infty, \ \eta \in \mathcal{H}_\rho^\infty).$$

and by lemma 3.1 and induction hypothesis, $\pi \sim \rho$.

Subcase 2.2 dim $\mathfrak{a} = 3$: We choose a basis $\{Y_1, Y_2, Z\}$ of $\mathfrak{a}$ such that $f(Y_1) = f(Y_2) = 0$ (Z is central and $f(Z) = 1$). Thus for each U in $\mathfrak{g}$:

$$[U, Y_1] = \lambda(U)(Y_1 + \alpha Y_2) + \mu_1(U)Z, \quad [U, Y_2] = \lambda(U)(Y_2 - \alpha Y_1) + \mu_2(U)Z$$

with λ, μ_1 and μ_2 linearly independant in $\mathfrak{g}^*$ (see [4]). Then:

$$\mathfrak{a}^f = \{U \in \mathfrak{g}, \ \ \mu_1(U) = \mu_2(U) = 0\} = \mathfrak{a}^g.$$

Choose X_1, X_2 in $\mathfrak{g}$ such that:

$$\lambda(X_1) = \lambda(X_2) = 0 \quad \text{and} \quad \mu_i(X_j) = \delta_{ij}.$$

Then $\{X_1, \ X_2\}$ is a coexponential basis to $\mathfrak{g}_0$, acting with $Ad^*(\exp g(Y_1)X_1)$ and $Ad^*(\exp g(Y_2)X_2)$ on g, we can suppose $g(Y_1) = g(Y_2) = 0$. Thus for any ξ in $\mathcal{H}_\pi^\infty$ and η in $\mathcal{H}_\rho^\infty$:

$$d\pi(-iY_1)\xi(t_1, t_2) = t_1\xi(t_1, t_2), \quad d\rho(-iY_1)\eta(t_1, t_2) = t_1\eta(t_1, t_2),$$
$$d\pi(-iY_2)\xi(t_1, t_2) = t_2\xi(t_1, t_2), \quad d\rho(-iY_2)\eta(t_1, t_2) = t_2\eta(t_1, t_2).$$

This ends the proof of theorem 1 by using lemma 3.2 and the induction hypothesis. ∎

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