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# Basic relative invariants associated to homogeneous cones and applications

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**Abstract.** In this paper, we determine the basic relative invariants on the ambient vector space of a homogeneous cone $\Omega$ under the action of the solvable linear Lie group acting on $\Omega$ simply transitively. The results are applied to a study of the Riesz distributions on $\Omega$ and to an algebraic description of the closure $\overline{\Omega}$ of $\Omega$.

## Introduction

In analysis on homogeneous cones, relatively invariant polynomials on the ambient vector spaces play significant roles. Among them, of particular importance are the basic relative invariants, that is, the polynomials by which every relatively invariant polynomial is expressed as a product of their powers. We determine in this paper the basic relative invariants under the action of the solvable Lie group acting simply transitively on a homogeneous cone. These are generalizations of principal minors of real symmetric matrices. As an application, we establish the condition that the Riesz distributions on the homogeneous cone are supported by the origin. We also give an algebraic description of the closure of the cone by introducing analogues of submatrices and minors.

Let $\Omega$ be a homogeneous cone in a real vector space $V$. Vinberg [14] shows that there exists a split solvable linear Lie group $H \subset \mathrm{GL}(V)$ which acts on $\Omega$ simply transitively. In order to describe $\Omega$ algebraically, he introduces in [14, Chapter 3] polynomials which we denote by $D_1, D_2, \ldots, D_r$ here, where $r$ is the rank of the cone $\Omega$ (see also [10, p. 73]). These $D_k$ are also treated in Gindikin [4] and called integral compound power functions there. As Gindikin observes, $D_k$ may have extra factors. In the present work, we give a simple method to factorize $D_k$'s into irreducibles in a general setting. As a result, exactly $r$ polynomials $\Delta_1, \Delta_2, \ldots, \Delta_r$ appear as irreducible factors, and we show that they are the basic relative invariants.

Our first application of the basic relative invariants is to a study of Riesz distributions. *The Riesz distribution* on $\Omega$ is characterized as the distribution

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whose Laplace transform is a relatively invariant function on the dual cone $\Omega^*$ (see (3.23)), where $\Omega^*$ is regarded as a homogeneous cone by the adjoint action of $H$. Let $\Delta_1^*, \dots, \Delta_r^*$ be the basic relative invariants associated to $\Omega^*$. We show that the Riesz distribution is supported by the one-point set $\{0\}$ if and only if its Laplace transform equals a product of powers of $\Delta_k^*$. This is a refinement of Gindikin's result [4, Proposition 3.4].

Our second application is to describe the closure $\overline{\Omega}$ of the cone $\Omega$. The motivation is the fact that a real symmetric matrix $x = (x_{ij})$ of size $r$ is positive semi-definite if and only if the determinants of the submatrices $(x_{ij})_{i,j \in I}$ are non-negative for all non-empty subsets $I \subset \{1, \dots, r\}$. Note that the non-negativity of the principal minors alone is not sufficient (see the beginning of Section 4). We give a criterion for the elements in $V$ to belong to $\overline{\Omega}$. Namely, after introducing polynomials $\Delta^I$ on $V$ which are analogues of the minors of matrices, we characterize $\overline{\Omega}$ by $2^r - 1$ inequalities $\Delta^I \geq 0$.

Let us explain the organization of this paper. According to [14], we introduce a certain algebra structure on $V$, called *the clan* of $\Omega$, and present Vinberg's polynomials $D_k$ in Section 1. In Proposition 1.4, we give recurrence formulas practical for the computation of $D_k$.

We consider $H$-relatively invariant polynomials in Section 2. The basic relative invariants are obtained in Theorem 2.2. We show in Proposition 2.3 that $x \in \Omega$ if and only if $\Delta_k(x) > 0$ $(k = 1, \dots, r)$. In Section 3, after describing the dual cone $\Omega^*$ as a homogeneous cone, we prove the result on the Riesz distributions mentioned above. Section 4 is devoted to getting the description of $\overline{\Omega}$ by the polynomials $\Delta^I$.

In Section 5, two examples are presented to write down $D_k$ and to get $\Delta_k$ from $D_k$ by factorization. The first example is the cone of real positive definite symmetric matrices, which is a typical homogeneous cone. It turns out that the polynomials $\Delta_k$ for this cone are equal to the principal minors. The second is so-called the Vinberg cone [14]. The cone and its dual cone are the lowest dimensional ones among the non-symmetric homogeneous cones. For these cones, our polynomials $\Delta_k$ coincide with Gindikin's in [6, p. 98 (e), (f)] as a matter of fact.

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## 1. Preliminaries

Let $V$ be a real vector space and $\Omega$ an open convex cone in $V$ containing no line. We assume that $\Omega$ is *homogeneous*, that is, the linear automorphism group $G(\Omega) \subset \mathrm{GL}(V)$ of $\Omega$ acts transitively on the cone $\Omega$. Then it is known [14] that there exists a split solvable Lie subgroup $H \subset G(\Omega)$ which acts on $\Omega$ simply transitively. Let $\mathfrak{h} \subset \mathrm{End}(V)$ be the Lie algebra of $H$ and fix a point $E$ in $\Omega$. Then we have the linear isomorphism $\mathfrak{h} \ni T \overset{\sim}{\mapsto} T \cdot E \in V$ obtained by differentiating the orbit map $H \ni t \mapsto t \cdot E \in \Omega$. Let $j : V \to \mathfrak{h}$ be the inverse map of the isomorphism, and define a bilinear multiplication $\triangle$ on $V$ by $x \triangle y := (jx) \cdot y \in V$ $(x, y \in V)$. The algebra $(V, \triangle)$ is called *the clan* of the homogeneous cone $\Omega$ ([14, Chapter 2]). This algebra is left-symmetric:

$$x\triangle(y\triangle z)-(x\triangle y)\triangle z=y\triangle(x\triangle z)-(y\triangle x)\triangle z \quad (x,y,z\in V),$$

which is equivalent to

$$[jx,jy]=j(x\triangle y-y\triangle x) \quad (x,y\in V). \tag{1.1}$$

By [14, Chapter 2, Proposition 8] we have *a normal decomposition* of the space $V$:

$$V=\sum_{1\le k\le m\le r}^{\oplus} V_{mk}, \tag{1.2}$$

where $V_{kk}$ $(k=1,\dots,r)$ is the one-dimensional subspace spanned by an idempotent $E_k$, and $V_{mk}$ consists of the elements $x\in V$ such that

$$y\triangle x=(1/2)(y_m+y_k)x \quad \text{and} \quad x\triangle y=y_kx \tag{1.3}$$

for all $y=\sum_{i=1}^r y_iE_i\in V$ $(y_1,\dots,y_r\in\mathbb{R})$. It is known [14, p. 376] that the following relations hold:

$$\begin{gathered} V_{mk}\triangle V_{li}=\{0\} \quad (\text{if } k\ne l,i), \quad V_{mk}\triangle V_{ki}\subset V_{mi},\\ V_{mk}\triangle V_{lk}\subset V_{ml} \text{ or } V_{lm}. \end{gathered} \tag{1.4}$$

Put $\mathfrak{h}_{mk}:=jV_{mk}\subset\mathfrak{h}$ $(1\le k\le m\le r)$ and $A_k:=jE_k\in\mathfrak{h}_{kk}$ $(k=1,\dots,r)$. The space $\mathfrak{a}:=\sum_{1\le k\le r}^{\oplus}\mathfrak{h}_{kk}=\sum_{1\le k\le r}^{\oplus}\mathbb{R}A_k$ is a commutative subalgebra of $\mathfrak{h}$ by (1.1) and (1.4), and we get from (1.2)

$$\mathfrak{h}=\mathfrak{a}\oplus\sum_{1\le k<m\le r}^{\oplus}\mathfrak{h}_{mk}. \tag{1.5}$$

**Remark 1.1.** The semidirect product $V\rtimes\mathfrak{h}$ has a normal $j$-algebra structure and the decompositions (1.2) and (1.5) coincide with the root space decompositions of the normal $j$-algebra ([1], [7], [10]).

Now we introduce global coordinate systems on $H$ and $V$. For each $t\in H$, there exist unique $t_{kk}>0$ $(k=1,2,\dots,r)$ and $T_{mk}\in\mathfrak{h}_{mk}$ $(1\le k<m\le r)$ for which

$$t=\exp T_{11}\cdot\exp L_1\cdot\exp T_{22}\cdots\exp L_{r-1}\cdot\exp T_{rr} \tag{1.6}$$

with $T_{kk}:=(2\log t_{kk})A_k\in\mathfrak{a}$ and $L_k:=\sum_{m>k}T_{mk}$ (see [7, Proposition 2.1 (ii)] for the proof). On the other hand, for any $x\in V$, we take unique $x_{kk}\in\mathbb{R}$ $(k=1,2,\dots,r)$ and $X_{mk}\in V_{mk}$ $(1\le k<m\le r)$ such that

$$x=\sum_{k=1}^r x_{kk}E_k+\sum_{1\le k<m\le r}X_{mk} \tag{1.7}$$

according to (1.2). We call $t_{kk}$, $T_{mk}$ (resp. $x_{kk}$, $X_{mk}$) the coordinates of $t\in H$ (resp. $x\in V$). Let $E^*$ be the linear form on $V$ given by

$$\langle x,E^*\rangle:=\sum_{k=1}^r x_{kk} \qquad (x\in V). \tag{1.8}$$

It is deduced from [14, p. 376] that the bilinear form

$$(x|y) := \langle x \triangle y, E^* \rangle / 2 \qquad (x, y \in V) \tag{1.9}$$

defines an inner product on $V$, and that the normal decomposition (1.2) of $V$ is orthogonal with respect to this $(\cdot|\cdot)$ (cf. [7, Lemma 2.3]). Let $\|\cdot\|$ be the norm on $V$ defined by $\|x\|^2 := (x|x)$ $(x \in V)$. Transferring this norm by means of $j$, we also define a norm on $\mathfrak{h}$, that is, $\|jx\| := \|x\|$ $(x \in V)$. Then the coordinates of each element of $\Omega$ are expressed as follows:

**Proposition 1.2.** [7, Proposition 2.5] *For $t \in H$ let $x$ be the element $t \cdot E$ of $\Omega$. Then one has*

$$x_{kk} = (t_{kk})^2 + \sum_{i<k} \|T_{ki}\|^2 \quad (k = 1, \ldots, r),$$
$$X_{mk} = t_{kk} T_{mk} \cdot E_k + \sum_{i<k} T_{mi} T_{ki} \cdot E_i \quad (1 \le k < m \le r).$$

Taking $\tau_{ki} \in V_{ki}$ for which $T_{ki} = j\tau_{ki}$, we have $T_{ki} \cdot E_i = \tau_{ki} \triangle E_i = \tau_{ki}$ by (1.3). Thus the equalities in Proposition 1.2 are rewritten as

$$\begin{aligned} x_{kk} &= (t_{kk})^2 + \sum_{i<k} \|\tau_{ki}\|^2 \quad (k = 1, \ldots, r), \\ X_{mk} &= t_{kk} \tau_{mk} + \sum_{i<k} \tau_{mi} \triangle \tau_{ki} \quad (1 \le k < m \le r). \end{aligned} \tag{1.10}$$

Vinberg [14] solved $t_{ii}$ and $\tau_{li}$ $(l > i)$ in the equations (1.10) as functions of $x_{kk}$ and $X_{mk}$ (see also [4, p. 16]) by utilizing certain polynomials on $V$ determined as follows: For $x \in V$ and $i = 1, \ldots, r$, define $x^{(i)} = \sum_{k=1}^{r} x_{kk}^{(i)} E_k + \sum_{m>k} X_{mk}^{(i)} \in V$ by

$$\begin{aligned} x^{(1)} &:= x, \\ x_{kk}^{(i+1)} &:= x_{ii}^{(i)} x_{kk}^{(i)} - \|X_{ki}^{(i)}\|^2 \quad (i < k \le r), \\ X_{mk}^{(i+1)} &:= x_{ii}^{(i)} X_{mk}^{(i)} - X_{mi}^{(i)} \triangle X_{ki}^{(i)} \quad (i < k < m \le r), \end{aligned} \tag{1.11}$$

and put

$$D_k(x) := x_{kk}^{(k)} \in \mathbb{R} \ \ (k = 1, \ldots, r), \quad Y_{mk}(x) := X_{mk}^{(k)} \in V_{mk} \ \ (1 \le k < m \le r).$$

In particular, $D_1(x) = x_{11}$ and $Y_{m1}(x) = X_{m1}$ $(m > 1)$. We call these $D_k$, $Y_{mk}$ *the determinant type polynomials* associated to the cone $\Omega$. We remark that $D_k$ appears also in [4, Lemma 3.1] as an integral compound power function. It is easily seen from (1.11) that the degrees of $D_k$ and $Y_{mk}$ are $2^{k-1}$. We write simply $D$ for the polynomial $D_r$ and call it *the composite determinant.* Then Vinberg's results are stated as follows:

**Proposition 1.3.** [14, Chapter 3, Section 3] (i) *If* $x = t \cdot E \in \Omega$ *for* $t \in H$, *then*

$$(t_{kk})^2 = \frac{D_k(x)}{\prod_{i<k} D_i(x)} \quad (k = 1, \dots, r), \tag{1.12}$$

$$t_{kk}\tau_{mk} = \frac{Y_{mk}(x)}{\prod_{i<k} D_i(x)} \quad (1 \le k < m \le r). \tag{1.13}$$

(ii) *One has*

$$\Omega = \{\, x \in V\,;\ D_k(x) > 0\ (k = 1, \dots, r)\,\}.$$

Although the recurrence formula (1.11) is quite simple, actual calculation of the composite determinant $D$ is hard to carry out because we must compute all the quantities $x^{(i)}_{kk}$ and $X^{(i)}_{mk}$ before getting to $D(x) = x^{(r)}_{rr}$ for $x \in V$. The following proposition is useful for the computation of $D_k$ and $Y_{mk}$.

**Proposition 1.4.** *The determinant type polynomials* $D_k$ *and* $Y_{mk}$ *satisfy the following recurrence relations:*

$$D_k = D_1 D_2 \cdots D_{k-1} x_{kk} - \sum_{i<k-1} D_{i+1} \cdots D_{k-1} \|Y_{ki}\|^2 - \|Y_{k,k-1}\|^2$$

$$(k = 2, \dots, r),$$

$$Y_{mk} = D_1 D_2 \cdots D_{k-1} X_{mk} - \sum_{i<k-1} D_{i+1} \cdots D_{k-1} Y_{mi} \triangle Y_{ki} - Y_{m,k-1} \triangle Y_{k,k-1}$$

$$(2 \le k < m \le r).$$

**Proof.** It is sufficient to show the relations in the case of $x = t \cdot E \in \Omega$ $(t \in H)$. Substituting (1.12) and (1.13) to (1.10), we have

$$x_{kk} = \frac{D_k}{D_1 \cdots D_{k-1}} + \sum_{i<k} \frac{\|Y_{ki}\|^2}{(t_{ii})^2 (D_1 \cdots D_{i-1})^2},$$

$$X_{mk} = \frac{Y_{mk}}{D_1 \cdots D_{k-1}} + \sum_{i<k} \frac{Y_{mi} \triangle Y_{ki}}{(t_{ii})^2 (D_1 \cdots D_{i-1})^2}.$$

Since $D_i = (t_{ii})^2 D_1 \cdots D_{i-1}$ by (1.12), the above equalities are rewritten as

$$\frac{D_k}{D_1 \cdots D_{k-1}} = x_{kk} - \sum_{i<k} \frac{\|Y_{mi}\|^2}{D_1 \cdots D_{i-1} D_i},$$

$$\frac{Y_{mk}}{D_1 \cdots D_{k-1}} = X_{mk} - \sum_{i<k} \frac{Y_{mi} \triangle Y_{ki}}{D_1 \cdots D_{i-1} D_i}.$$

Multiplying the both sides by $D_1 \cdots D_{k-1}$, we obtain the desired formulas. ∎

## 2. Basic relative invariants

In this section, we determine all the polynomials on $V$ relatively invariant under the action of $H$. Actually, we factorize the polynomials $D_1, D_2, \dots, D_r$ into irreducibles to obtain the basic relative invariants, that is, the generators of the set of $H$-relatively invariant polynomials. Then the cone $\Omega$ is distinguished as the set on which all the basic relative invariants are positive.

For $s = (s_1, \dots, s_r) \in \mathbb{C}^r$, we define the one-dimensional representation $\chi_s$ of $H$ by

$$\chi_s(\exp(\textstyle\sum_{k=1}^r c_k A_k)) = e^{s_1 c_1 + \cdots + s_r c_r} \quad (c_1, \dots, c_r \in \mathbb{R}). \tag{2.14}$$

Let $F$ be an $H$-relatively invariant rational function on $V$ with $\chi_s$ the multiplier. Namely, $F(t \cdot x) = \chi_s(t) F(x)$ for all $t \in H$ and $x \in V$. Since $H$ acts transitively on the open cone $\Omega$ in $V$, the rational function $F$ is determined uniquely (up to constant) by $\chi_s$. Throughout this section, we assume that rational functions $f$ are normalized as $f(E) = 1$ unless otherwise stated.

**Proposition 2.1.** (i) *Putting*

$$\mu(k) = \begin{cases} (1, 0, \dots, 0) & (k = 1), \\ (2^{k-2}, 2^{k-3}, \dots, 1, 1, 0, \dots, 0) & (k = 2, \dots, r), \end{cases}$$

*one has*

$$D_k(t \cdot x) = \chi_{\mu(k)}(t) D_k(x) \quad (t \in H,\ x \in V).$$

(ii) *There exists an $H$-relatively invariant rational function corresponding to the multiplier $\chi_s$ if and only if $s \in \mathbb{Z}^r$.*

**Proof.** (i) Although the statement can be found in [4, Lemma 3.1], we give a proof here for completeness. The equality (1.12) implies that for $t \in H$

$$\begin{aligned} D_1(t \cdot E) &= (t_{11})^2, \\ D_k(t \cdot E) &= (t_{11})^{2^{k-1}} (t_{22})^{2^{k-2}} \cdots (t_{k-1,k-1})^2 \cdot (t_{kk})^2 \quad (k = 2, \dots, r). \end{aligned} \tag{2.15}$$

In other words, we have $D_k(t \cdot E) = \chi_{\mu(k)}(t)$, whence the assertion (i) follows.

(ii) Take an arbitrary $s \in \mathbb{Z}^r$. Since $\mu(k)_k = 1$ and $\mu(k)_m = 0$ $(m > k)$ for $k = 1, \dots, r$, there exist integers $a_1, \dots, a_r$ for which $s = a_1\mu(1) + a_2\mu(2) + \cdots + a_r\mu(r)$. Then the assertion (i) tells us that the product $(D_1)^{a_1}(D_2)^{a_2} \dots (D_r)^{a_r}$ is an $H$-relatively invariant rational function corresponding to $\chi_s$, which proves the "if" part of the statement. Next, we show the "only if" part. Let $F$ be an $H$-relatively invariant rational function and $\chi_s$ its multiplier. For $x_{11}, \dots, x_{rr} > 0$, let $t$ be the element of $H$ given by $t_{kk} := \sqrt{x_{kk}}$ $(k = 1, \dots, r)$ and $T_{mk} := 0$ $(1 \le k < m \le r)$. Then we have $t \cdot E = x_{11}E_1 + \cdots + x_{rr}E_r$ by Proposition 1.2, so that

$$\begin{aligned} F(x_{11}E_1 + x_{22}E_2 + \cdots + x_{rr}E_r) &= (t_{11})^{2s_1}(t_{22})^{2s_2} \cdots (t_{rr})^{2s_r} F(E) \\ &= (x_{11})^{s_1}(x_{22})^{s_2} \cdots (x_{rr})^{s_r}. \end{aligned} \tag{2.16}$$

Therefore, since $F$ is rational, we conclude that $s_1, s_2, \dots, s_r$ are integers. ∎

The relation (2.16) also implies that, if $F$ is a polynomial, then $s_1, \dots, s_r$ are non-negative integers. Now we give an algorithm to find the generators of $H$-relatively invariant polynomials. The algorithm is valid for any homogeneous cone.

Put $\Delta_1 = D_1$, and for $k \geq 2$ let $\Delta_k$ be *the* polynomials on $V$ determined by requiring that $D_k$ are written as

$$D_k = \Delta_k \cdot (\Delta_1)^{a_{k1}} (\Delta_2)^{a_{k2}} \cdots (\Delta_{k-1})^{a_{k,k-1}} \tag{2.17}$$

with the following two conditions:

(i) $a_{k1}, a_{k2}, \dots, a_{k,k-1}$ are non-negative integers,

(ii) $\Delta_k$ is not divisible by any of $\Delta_1, \dots, \Delta_{k-1}$.

**Theorem 2.2.** (i) *The polynomial $\Delta_k$ is irreducible and relatively invariant under the action of $H$. Let $\sigma(k)$ be the element of $\mathbb{Z}^r$ such that $\chi_{\sigma(k)}$ is the multiplier of $\Delta_k$. Then $\sigma(k)_k = 1$ and $\sigma(k)_m = 0$ $(m = k+1, \dots, r)$.*

(ii) *Let $\sigma$ be the $r \times r$ matrix whose $(k,m)$-component is $\sigma(k)_m$. For $s \in \mathbb{Z}^r$, let $(a_1, \dots, a_r)$ be the row vector $(s_1, \dots, s_r)\sigma^{-1}$. Then, there exists an $H$-relatively invariant polynomial corresponding to the multiplier $\chi_s$ if and only if all $a_1, \dots, a_r$ are non-negative integers. In this case, the polynomial equals $(\Delta_1)^{a_1} (\Delta_2)^{a_2} \dots (\Delta_r)^{a_r}$ up to constant.*

In the proof of Theorem 2.2, we quote a well-known fact about prehomogeneous vector spaces, noting the fact that the action of $H$ on $V$ is algebraic ([14]).

**Proof.** (i) We shall show the assertion by induction on $k$. The case $k = 1$ is obvious. For $k \geq 2$, assume that the assertion holds for $\Delta_1, \dots, \Delta_{k-1}$. Let

$$D_k = \varphi_1 \varphi_2 \cdots \varphi_N \tag{2.18}$$

be the factorization of $D_k$ into irreducible polynomials. Then each $\varphi_n$ $(n = 1, \dots, N)$ is $H$-relatively invariant by [12, Proposition 2 (2)]. Let $s^n = (s^n_1, \dots, s^n_r)$ be the $r$-tuple of non-negative integers such that $\chi_{s^n}$ is the multiplier of $\varphi_n$. By (2.18) and Proposition 2.1 (i) we have $\mu(k) = s^1 + \dots + s^N$. In particular, we have for $m = k+1, \dots, r$

$$0 = \mu(k)_m = s^1_m + s^2_m + \dots + s^N_m,$$

which implies $s^1_m = s^2_m = \dots = s^N_m = 0$. Similarly, we have

$$1 = \mu(k)_k = s^1_k + s^2_k + \dots + s^N_k,$$

whence it follows that the only one of $s^1_k, \dots, s^N_k$ equals 1 and the others are 0. Assume that $s^1_k = 1$. Put $\tilde{\sigma} := s^2 + \dots + s^N$. Then we have $\tilde{\sigma}_m = 0$ for $m \geq k$, so that we can take integers $a_{k1}, a_{k2}, \dots, a_{k,k-1}$ for which $\tilde{\sigma} = a_{k1}\sigma(1) + a_{k2}\sigma(2) + \dots + a_{k,k-1}\sigma(k-1)$ thanks to the induction hypothesis on $\sigma(i)$ $(i < k)$. Then

$$\varphi_2 \dots \varphi_N = (\Delta_1)^{a_{k1}} (\Delta_2)^{a_{k2}} \cdots (\Delta_{k-1})^{a_{k,k-1}}$$

because the both sides are $H$-relatively invariant polynomials corresponding to the same multiplier $\chi_{\tilde{\sigma}} = (\chi_{\sigma(1)})^{a_{k1}} (\chi_{\sigma(2)})^{a_{k2}} \dots (\chi_{\sigma(k-1)})^{a_{k,k-1}}$. We see from

the irreducibility of $\Delta_1, \dots, \Delta_{k-1}$ that integers $a_{k1}, \dots, a_{k,k-1}$ are non-negative. Therefore we have $\Delta_k = \varphi_1$ and $\sigma(k) = s^1$, whence the assertion follows.
(ii) It is immediate from the definitions that $s = a_1\sigma(1) + \cdots + a_r\sigma(r)$, so that $(\Delta_1)^{a_1} \dots (\Delta_r)^{a_r}$ is an $H$-relatively invariant function corresponding to $\chi_s$. This function is polynomial if and only if $a_k$'s are non-negative integers since $\Delta_k$'s are irreducible by (i). Hence the theorem follows from the one-to-one correspondence between relative invariants and multipliers. ∎

We write $\Delta$ for $\Delta_r$ and call it *the reduced determinant polynomial* associated to the cone $\Omega$. When $\Omega$ is an irreducible symmetric cone, the polynomials $\Delta_1, \dots, \Delta_r$ coincide with the principal minors for the Euclidean Jordan algebra [3, p. 114]. ¿From (2.17) and Proposition 1.3 (ii), we obtain the following result.

**Proposition 2.3.** *The cone $\Omega$ is described as*

$$\Omega = \{\, x \in V \,;\, \Delta_k(x) > 0 \ (k = 1, \dots, r) \,\}.$$

## 3. Application to Riesz distributions

For $A \in \mathrm{End}(V)$, we denote by $A^*$ the adjoint operator of $A$: $A^* \cdot \xi := \xi \circ A$ ($\xi \in V^*$). Let $H^*$ be the subgroup $\{\, t^* \,;\, t \in H \,\}$ of $\mathrm{GL}(V^*)$. Then $H^*$ acts simply transitively on the dual cone $\Omega^*$ of $\Omega$ ([14, Chapter 1, Proposition 9]). We shall consider the determinant type polynomials on $V^*$ associated to $\Omega^*$.

Taking $E^*$ as the base point (see (1.8)), we define the clan $(V^*, \triangle')$ of the homogeneous cone $\Omega^*$. Let $\mathfrak{E}_k$ ($k = 1, \dots, r$) be the element of $V^*$ given by $\langle x, \mathfrak{E}_k \rangle := x_{r+1-k,r+1-k}$ ($x \in V$). Then $V^*$ allows a normal decomposition with respect to the idempotents $\mathfrak{E}_k$, where the $(m,k)$-th component equals the dual space of $V_{r+1-k,r+1-m}$ (see [6, p. 86], [14, Chapter 3, Section 6] and also [8, Section 2]). Using this algebra structure, we define the determinant type polynomials $D^*_k$, $Y^*_{mk}$, and $\Delta^*_k$ on $V^*$. Let us observe the relative invariance of $\Delta^*_k$. Since (1.3) leads us to

$$\begin{aligned}\langle x, A^*_k \cdot E^* \rangle &= \langle A_k \cdot x, E^* \rangle = \langle x_{kk}E_k + \textstyle\sum_{m>k} X_{mk}/2, E^* \rangle \\ &= x_{kk} = \langle x, \mathfrak{E}_{r+1-k} \rangle,\end{aligned}$$

we have $\mathfrak{E}_k \triangle' \xi = A^*_{r+1-k} \cdot \xi$ ($\xi \in V^*$). Thus, defining the one-dimensional representation $\chi^*_s$ ($s = (s_1, \dots, s_r) \in \mathbb{C}^r$) of $H^*$ by

$$\chi^*_s(\exp(\textstyle\sum_{k=1}^r c_k A^*_{r+1-k})) = e^{s_1c_1 + \cdots + s_rc_r} \quad (c_1, \dots, c_r \in \mathbb{R}), \tag{3.19}$$

we see from Theorem 2.2 (i) that there exists $\rho(k) \in \mathbb{Z}^r$ for which

$$\Delta^*_k(\tau \cdot \xi) = \chi^*_{\rho(k)}(\tau)\Delta^*_k(\xi) \quad (\xi \in V^*,\ \tau \in H^*). \tag{3.20}$$

We see also from Theorem 2.2 (i) that $\rho(k)_k = 1$ and $\rho(k)_m = 0$ ($m > k$). Comparing (2.14) and (3.19), we have

$$\chi^*_s(t^*) = \chi_{s^*}(t) \qquad (t \in H), \tag{3.21}$$

where $s^* := (s_r, \dots, s_1) \in \mathbb{C}^r$.

Now we recall the definition of *the Riesz distributions* on the homogeneous cone $\Omega$ ([4], [7]). Let $\mu$ be an $H$-invariant measure on $\Omega$, and $\Upsilon_s$ ($s \in \mathbb{C}^r$) the function on $\Omega$ given by $\Upsilon_s(t \cdot E) := \chi_s(t)$ $(t \in H)$. Set $p_k := \sum_{i<k} \dim \mathfrak{h}_{ki}$ $(1 \le k \le r)$. When $\Re s_k > p_k/2$ for all $k = 1, \dots, r$, we define a tempered distribution $\mathcal{R}_s$ on $V$ by

$$\langle \mathcal{R}_s, f \rangle := \frac{c}{\prod_{k=1}^r \Gamma(s - p_k/2)} \int_\Omega f(x) \Upsilon_s(x)\, d\mu(x),$$

where $f$ is a rapidly decreasing Schwartz function on $V$ and $c$ is the normalizing constant determined in such a way that $\langle \mathcal{R}_s, e^{-\langle \cdot, E^* \rangle} \rangle = 1$. For general $s \in \mathbb{C}^r$, we define $\langle \mathcal{R}_s, f \rangle$ by analytic continuation. The distribution $\mathcal{R}_s$'s are called *the Riesz distributions* on $\Omega$. The support of $\mathcal{R}_s$ is contained in the closure $\overline{\Omega}$ of $\Omega$, and the distribution $\mathcal{R}_s$ is $H$-relatively invariant in the following sense:

$$\langle \mathcal{R}_s, f \circ t \rangle = \chi_{-s}(t) \langle \mathcal{R}_s, f \rangle \quad (t \in H). \tag{3.22}$$

Using this relative invariance, we can easily compute the Laplace transforms of $\mathcal{R}_s$. In fact, if $\xi = t^* \cdot E^* \in \Omega^*$ $(t \in H)$, we see from (3.22) that

$$\langle \mathcal{R}_s, e^{-\langle \cdot, \xi \rangle} \rangle = \chi_{-s}(t) \, \langle \mathcal{R}_s, e^{-\langle \cdot, E^* \rangle} \rangle = \chi_{-s}(t).$$

This together with (3.21) gives us the $H^*$-relative invariance of the Laplace transform of $\mathcal{R}_s$:

$$\langle \mathcal{R}_s, e^{-\langle \cdot, \tau \cdot \xi \rangle} \rangle = \chi^*_{-s^*}(\tau) \, \langle \mathcal{R}_s, e^{-\langle \cdot, \xi \rangle} \rangle \quad (\xi \in \Omega^*,\ \tau \in H^*). \tag{3.23}$$

For each polynomial $\phi$ on $V^*$ let $\phi(\partial_x)$ be the differential operator determined by

$$\phi(\partial_x) e^{\langle x, \xi \rangle} = \phi(\xi) e^{\langle x, \xi \rangle} \quad (\xi \in V^*). \tag{3.24}$$

Then we obtain the following theorem, which is a refinement of Gindikin's result [4, Proposition 3.4]. Recall the element $\rho(k)$ of $\mathbb{Z}^r$ in (3.20).

**Theorem 3.1.** *Let $\rho$ be the $r \times r$ matrix whose $(k, m)$-component is $\rho(k)_m$. For $s \in \mathbb{C}^r$, put $(a_1, a_2, \dots, a_r) := (-s_r, -s_{r-1}, \dots, -s_1)\rho^{-1}$. Then the Riesz distribution $\mathcal{R}_s$ is supported by the origin if and only if all $a_1, \dots, a_r$ are non-negative integers, and in this case, one has*

$$\mathcal{R}_s = \Delta_1^*(\partial_x)^{a_1} \Delta_2^*(\partial_x)^{a_2} \cdots \Delta_r^*(\partial_x)^{a_r} \delta,$$

*where $\delta$ is the Dirac distribution at $\{0\}$.*

**Proof.** It is known that a distribution supported by $\{0\}$ is of the form $\phi(\partial_x)\delta$ with some polynomial $\phi \in \mathcal{P}(V^*)$, and by (3.24), its Laplace transform $\langle \phi(\partial_x)\delta, e^{-\langle \cdot, \xi \rangle} \rangle$ equals $\phi(\xi)$. On the other hand, (3.23) says that the Laplace transform of $\mathcal{R}_s$ is $H^*$-relatively invariant with multiplier $\chi^*_{-s^*}$. Therefore, applying Theorem 2.2 (ii) to this situation, we conclude that $a_k$'s are non-negative integers and the Laplace transform $\langle \mathcal{R}_s, e^{-\langle \cdot, \xi \rangle} \rangle$ equals the polynomial function $\Delta_1^*(\xi)^{a_1} \Delta_2^*(\xi)^{a_2} \cdots \Delta_r^*(\xi)^{a_r}$ of $\xi \in \Omega^*$. Hence the injectivity of the Laplace transform completes the proof. ∎

## 4. Description of $\overline{\Omega}$

Proposition 2.3 implies that, if $x \in \overline{\Omega}$, then $\Delta_k(x) \geq 0$ $(k = 1, \dots, r)$. However the converse is false. For instance, although the principal minors of $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ are all non-negative, the matrix is not positive semi-definite. Thus we need more polynomials to describe the closure $\overline{\Omega}$ of $\Omega$. In this section, we define a polynomial $\Delta^I$ for each non-empty subset $I \subset \{1, \dots, r\}$, so that $\overline{\Omega}$ is characterized by the inequalities $\Delta^I \geq 0$.

For integers $a \leq b$, we denote by $\lfloor a, b \rfloor$ the set $\{a, a+1, \dots, b\}$. Let $I = \{i_1, i_2, \dots, i_l\}$ $(i_1 < i_2 < \cdots < i_r)$ be a subset of $\lfloor 1, r \rfloor$. We put

$$V^I := \sum_{1 \leq \alpha \leq \beta \leq l}^{\oplus} V_{i_\beta i_\alpha}. \tag{4.25}$$

Then we see from (1.3) and (1.4) that $(V^I, \triangle)$ is a subalgebra of $(V, \triangle)$ with $E^I := \sum_{\alpha=1}^{l} E_{i_\alpha} \in V^I$ the unit element (cf. [14, p. 377]). Put

$$\mathfrak{h}^I := jV^I = \sum_{\alpha=1}^{l}{}^{\oplus} \mathbb{R} A_{i_\alpha} \oplus \sum_{1 \leq \alpha < \beta \leq l}^{\oplus} \mathfrak{h}_{i_\beta i_\alpha}. \tag{4.26}$$

Clearly $V^I$ is stable under the action of $\mathfrak{h}^I$, and (1.1) implies that $\mathfrak{h}^I$ is a Lie subalgebra of $\mathfrak{h}$. Let $H^I$ be the Lie subgroup $\exp \mathfrak{h}^I$ of $H$. In view of [14, Chapter 2, Section 2], the orbit $\Omega^I := H^I \cdot E^I \subset V^I$ is the homogeneous cone whose clan is $(V^I, \triangle)$. Let $P_I : V \to V^I$ be the orthogonal projection and denote by $x_I$ the image $P_I(x)$ of $x \in V$.

**Lemma 4.1.** (i) *The projection $P_I$ is $H^I$-equivariant:*

$$P_I(\tilde{t} \cdot x) = \tilde{t} \cdot P_I(x) \quad (x \in V,\ \tilde{t} \in H^I).$$

(ii) *The image of $\overline{\Omega}$ under $P_I$ is equal to $\overline{\Omega^I}$.*

**Proof.** (i) Let $A^I$ be the element $jE^I = \sum_{\alpha=1}^{l} A_{i_\alpha}$ of $\mathfrak{h}^I$. For $\lambda \in \mathbb{R}$, let $V(\lambda; A^I)$ be the eigenspace $\{ x \in V\,;\, A^I \cdot x = \lambda x \}$ of the operator $A^I$. Then $V = V(1; A^I) \oplus V(1/2; A^I) \oplus V(0; A^I)$. In fact, by (1.3) and (4.26) we have $V(1; A^I) = V^I$, $V(0; A^I) = V^{\lfloor 1, r \rfloor \setminus I}$ and

$$V(1/2; A^I) = \sum_{\alpha=1}^{l}{}^{\oplus} \Bigg( \sum_{m > i_\alpha, m \notin I}^{\oplus} V_{m i_\alpha} \oplus \sum_{k < i_\alpha, k \notin I}^{\oplus} V_{i_\alpha k} \Bigg).$$

The decomposition $V = V^I \oplus V(1/2; A^I) \oplus V^{\lfloor 1, r \rfloor \setminus I}$ is orthogonal, and since $[\mathfrak{h}^I, A^I] = \{0\}$ by (1.1), (1.3) and (4.26), the action of $H^I = \exp \mathfrak{h}^I$ preserves each of the subspaces. Hence the assertion (i) holds.
(ii) Clearly we have $\Omega^I = H^I \cdot E^I \subset \overline{\Omega}$, so that $\overline{\Omega^I} = P_I(\overline{\Omega^I}) \subset P_I(\overline{\Omega})$. Let us show the converse inclusion $\overline{\Omega^I} \supset P_I(\overline{\Omega})$. Let $E_I^*$ be the linear form on $V^I$ given by

$$\langle \tilde{x}, E_I^* \rangle := \sum_{k \in I} \tilde{x}_{kk} \quad (\tilde{x} \in V^I). \tag{4.27}$$

Then $E_I^*$ belongs to the dual cone $(\Omega^I)^*$ ([11, Theorem 4.15]). Thus [7, Lemma 1.2] implies that an element $\tilde{x}$ of $V^I$ belongs to $\overline{\Omega^I}$ if and only if $\langle \tilde{t}\cdot\tilde{x}, E_I^*\rangle \geq 0$ for all $\tilde{t} \in H^I$. Take $x \in \Omega$ and $\tilde{t} \in H^I$. Then we have $\langle \tilde{t}\cdot P_I(x), E_I^*\rangle = \langle P_I(\tilde{t}\cdot x), E_I^*\rangle$ by (i). Since $\tilde{t}\cdot x \in \Omega$, we can take $t \in H$ for which $\tilde{t}\cdot x = t\cdot E$. We then obtain by Proposition 1.2 and (4.27)

$$\langle P_I(\tilde{t}\cdot x), E_I^*\rangle = \sum_{k\in I}\Big((t_{kk})^2 + \sum_{i<k}\|T_{ki}\|^2\Big) \geq 0.$$

Thus we get $P_I(x) \in \overline{\Omega^I}$, so that $P_I(\Omega) \subset \overline{\Omega^I}$. Hence the continuity of $P_I$ completes the proof. ∎

Making use of the algebra structures of $(V^I, \triangle)$, we define the determinant type polynomials $D_\alpha^I$ $(\alpha = 1, \dots, l)$ on $V^I$ associated to $\Omega^I$. Let $\mu(I;\alpha) = (\mu(I;\alpha)_1, \dots,\ \mu(I;\alpha)_r)$ be the element of $\mathbb{Z}^r$ given by

$$\mu(I;\alpha)_k := \begin{cases} 2^{\alpha-\beta-1} & (\text{if } k = i_\beta, \beta < \alpha), \\ 1 & (\text{if } k = i_\alpha), \\ 0 & (\text{otherwise}). \end{cases}$$

In view of (2.15), we have for $t \in H^I$

$$\begin{aligned} &D_1^I(t\cdot E^I) = (t_{i_1 i_1})^2, \\ &D_\alpha^I(t\cdot E^I) = (t_{i_1 i_1})^{2^{\alpha-1}}(t_{i_2 i_2})^{2^{\alpha-2}}\cdots(t_{i_{\alpha-1},i_{\alpha-1}})^2\cdot(t_{i_\alpha i_\alpha})^2 \\ &\hspace{18em}(\alpha = 2,\dots,l), \end{aligned} \tag{4.28}$$

so that we obtain

$$D_\alpha^I(t\cdot x) = \chi_{\mu(I;\alpha)}(t)\, D_\alpha^I(x) \quad (t \in H^I,\ x \in V^I). \tag{4.29}$$

We denote by $D^I$ the composite determinant $D_l^I$ associated to $\Omega^I$, and extend this $D^I$ to $V$ by $D^I(x) := D^I(x_I)$ $(x \in V)$.

Applying Proposition 1.3 (ii) to the cone $\Omega^I$, we see that the composite determinant $D^I(\tilde{x})$ is non-negative for $\tilde{x} \in \overline{\Omega^I}$. This together with Lemma 4.1 implies that, if $x \in \overline{\Omega}$, then $D^I(x) = D^I(x_I) \geq 0$ for all non-empty $I \subset \lfloor 1, r\rfloor$. Proposition 4.3 below states that the converse also holds. The proposition is proved by induction on the rank $r$, and the following lemma plays a substantial role in the induction.

**Lemma 4.2.** *Assume that $r \geq 2$. Let $x$ be an element of $V$ such that $x_{11} \neq 0$ and $L$ the element $-\sum_{m=2}^r jX_{m1}/x_{11}$ of $\mathfrak{h}$. Then $x' := (\exp L)\cdot x$ belongs to $x_{11}E_1 + V^{\lfloor 2,r\rfloor}$. For a non-empty subset $I$ of $\lfloor 2, r\rfloor$, one has*

$$D^{\{1\}\cup I}(x) = (x_{11})^{2^{l-1}}\, D^I(x') \quad (l := \sharp I).$$

**Proof.** Since $L\cdot V_{11} \subset \sum_{m>1}^{\oplus} V_{m1}$, $L\cdot(\sum_{m>1}^{\oplus} V_{m1}) \subset V^{\lfloor 2,r\rfloor}$ and $L\cdot V^{\lfloor 2,r\rfloor} = 0$ by (1.4), we have

$$\begin{aligned} &(\exp L)\cdot x_{11}E_1 = x_{11}E_1 + L\cdot x_{11}E_1 + (1/2)L^2\cdot x_{11}E_1, \\ &(\exp L)\cdot \textstyle\sum_{m>1} X_{m1} = \sum_{m>1} X_{m1} + L\cdot\sum_{m>1} X_{m1}, \\ &(\exp L)\cdot x_{\lfloor 2,r\rfloor} = x_{\lfloor 2,r\rfloor}. \end{aligned} \tag{4.30}$$

By (1.3) we have

$$L \cdot x_{11}E_1 = -\sum_{m>1} X_{m1} \triangle E_1 = -\sum_{m>1} X_{m1},$$

so that

$$L^2 \cdot x_{11}E_1 = -L \cdot \sum_{m>1} X_{m1} \in V^{\lfloor 2,r \rfloor}.$$

Using them, we sum up the both sides of the equalities in (4.30) to obtain

$$\begin{aligned}(\exp L)\cdot x &= x_{11}E_1 + x_{\lfloor 2,r\rfloor} + (1/2)L \cdot \textstyle\sum_{m>1} X_{m1} \\ &= x_{11}E_1 + x_{\lfloor 2,r\rfloor} - (2x_{11})^{-1}(\textstyle\sum_{m>1} X_{m1}) \triangle (\textstyle\sum_{m>1} X_{m1}) \\ &\in x_{11}E_1 + V^{\lfloor 2,r\rfloor}.\end{aligned} \tag{4.31}$$

For the proof of the second assertion, we show the following claim:
*Claim: For an element $\tilde{x}$ of $V_{11} \oplus V^{\lfloor 2,r\rfloor}$, one has*

$$D(\tilde{x}) = (\tilde{x}_{11})^{2^{r-2}} D^{\lfloor 2,r\rfloor}(\tilde{x}_{\lfloor 2,r\rfloor}).$$

We first prove the claim for the case $\tilde{x} \in \Omega$. Take $t \in H$ for which $\tilde{x} = t \cdot E$ and express this $t$ as in (1.6). Since $0 = \tilde{X}_{m1} = t_{11}\tau_{m1}$ by (1.10), we have $L_1 = \sum_{m>1} j\tau_{m1} = 0$. Put $\tilde{t} := \exp T_{22} \exp L_2 \cdots \exp T_{rr} \in H^{\lfloor 2,r\rfloor}$. Since $A_1 \cdot V^{\lfloor 2,r\rfloor} = \{0\}$ and $\mathfrak{h}^{\lfloor 2,r\rfloor} \cdot V_{11} = \{0\}$ by (1.4), the elements $\exp T_{11}$ and $\tilde{t} \in H^{\lfloor 2,r\rfloor}$ act trivially on the spaces $V^{\lfloor 2,r\rfloor}$ and $V_{11} = \mathbb{R}E_1$ respectively. Thus we have

$$\begin{aligned}\tilde{x} = t \cdot E &= (\exp T_{11})\tilde{t} \cdot (E_1 + E^{\lfloor 2,r\rfloor}) = (\exp T_{11}) \cdot (E_1 + \tilde{t} \cdot E^{\lfloor 2,r\rfloor}) \\ &= (\exp T_{11}) \cdot E_1 + \tilde{t} \cdot E^{\lfloor 2,r\rfloor} = (t_{11})^2 E_1 + \tilde{t} \cdot E^{\lfloor 2,r\rfloor},\end{aligned}$$

which implies that $\tilde{x}_{11} = (t_{11})^2$ and $\tilde{x}_{\lfloor 2,r\rfloor} = \tilde{t} \cdot E^{\lfloor 2,r\rfloor} \in \Omega^{\lfloor 2,r\rfloor}$. In particular, we have

$$\tilde{x} = \tilde{x}_{11}E_1 + \tilde{x}_{\lfloor 2,r\rfloor} \in \Omega \iff \tilde{x}_{11} > 0,\ \tilde{x}_{\lfloor 2,r\rfloor} \in \Omega^{\lfloor 2,r\rfloor}. \tag{4.32}$$

By (4.28), we get

$$D^{\lfloor 2,r\rfloor}(\tilde{x}_{\lfloor 2,r\rfloor}) = (t_{22})^{2^{r-2}}(t_{33})^{2^{r-3}} \cdots (t_{r-1,r-1})^2 (t_{rr})^2,$$

so that by (2.15) we obtain

$$D(\tilde{x}) = (t_{11})^{2^{r-1}} \cdot (t_{22})^{2^{r-2}} \cdots (t_{r-1,r-1})^2 (t_{rr})^2 = (\tilde{x}_{11})^{2^{r-2}} D^{\lfloor 2,r\rfloor}(\tilde{x}_{\lfloor 2,r\rfloor}).$$

Namely, the claim is verified in the case that $\tilde{x}_{11} > 0$ and $\tilde{x}_{\lfloor 2,r\rfloor} \in \Omega^{\lfloor 2,r\rfloor}$. Because $(0, +\infty)$ and $\Omega^{\lfloor 2,r\rfloor}$ are open in the spaces $\mathbb{R}$ and $V^{\lfloor 2,r\rfloor}$ respectively, the claim holds for all $\tilde{x} \in V_{11} \oplus V^{\lfloor 2,r\rfloor}$.

Now we come back to the proof of the lemma. Put $J := \{1\} \cup I$. Applying the claim with $\tilde{x}$ and $V^{\lfloor 2,r\rfloor}$ replaced by $x'_J = x_{11}E_1 + x'_I$ and $V^I$ respectively, we obtain $D^J(x'_J) = (x_{11})^{2^{l-1}} D^I(x'_I)$, that is,

$$D^J(x') = (x_{11})^{2^{l-1}} D^I(x'). \tag{4.33}$$

On the other hand, replacing $x$ and $L$ by $x_J$ and $L_J := -\sum_{m\in I} jX_{m1}/x_{11}$ respectively in (4.31), we get

$$(\exp L_J) \cdot x_J = x_{11} + x_I - (2x_{11})^{-1}(\textstyle\sum_{m\in I} X_{m1}) \triangle (\textstyle\sum_{m\in I} X_{m1}).$$

Comparing this with (4.31), we have $x'_J = P_J((\exp L)\cdot x) = (\exp L_J)\cdot x_J$, so that

$$D^J(x') = D^J((\exp L_J)\cdot x_J).$$

Since $\exp L_J \in H^J$ and $\chi_{\mu(J;l+1)}(\exp L_J) = 1$, the right-hand side of the above is equal to $D^J(x_J) = D^J(x)$ by (4.29). This fact together with (4.33) completes the proof. ■

We now give an algebraic description of $\overline{\Omega}$.

**Proposition 4.3.** *One has*

$$\overline{\Omega} = \left\{ x \in V\,;\, D^I(x) \geq 0 \quad (I \subset \lfloor 1, r \rfloor,\ I \neq \emptyset) \right\}. \tag{4.34}$$

**Proof.** Thanks to the observation preceding Lemma 4.2, it is sufficient to prove that the right-hand side of (4.34) is contained in $\overline{\Omega}$. We shall show this by induction on the rank $r$. The case $r = 1$ is clear. For $r \geq 2$, assume that the claim holds for the rank $r-1$ cone $\Omega^{\lfloor 2,r \rfloor} \subset V^{\lfloor 2,r \rfloor}$, and let $x$ be an element of $V$ such that $D^I(x) \geq 0$ for all non-empty $I \subset \lfloor 1, r \rfloor$. Since $D^{\{1\}}(x) = x_{11} \geq 0$, we have the following two cases (1) and (2).
(1) The case $x_{11} = 0$.

In this case, we have for $k = 2, \ldots, r$

$$0 \leq D^{\{1,k\}}(x) = x_{11}x_{kk} - \|X_{k1}\|^2 = -\|X_{k1}\|^2,$$

so that $X_{k1} = 0$ and $x \in V^{\lfloor 2,r \rfloor}$. Therefore, since $D^I(x) \geq 0$ for all non-empty $I \subset \lfloor 2, r \rfloor$, the induction hypothesis tells us that $x \in \overline{\Omega^{\lfloor 2,r \rfloor}} \subset \overline{\Omega}$.
(2) The case $x_{11} > 0$.

We set $L := -\sum_{m>1} jX_{m1}/x_{11}$ and $x' := (\exp L)\cdot x$. Then Lemma 4.2 tells us that $x' = x_{11}E_1 + x'_{\lfloor 2,r \rfloor}$, and that for all non-empty $I \subset \lfloor 2, r \rfloor$,

$$D^I(x'_{\lfloor 2,r \rfloor}) = D^I(x') = D^{\{1\}\cup I}(x)/(x_{11})^{2^{l-1}} \geq 0 \quad (l := \sharp I).$$

Therefore $x'_{\lfloor 2,r \rfloor} \in \overline{\Omega^{\lfloor 2,r \rfloor}}$ by the induction hypothesis. On the other hand, since (4.32) tells us that $x_{11}E_1 + \Omega^{\lfloor 2,r \rfloor} \subset \Omega$ for $x_{11} > 0$, we have $x_{11}E_1 + \overline{\Omega^{\lfloor 2,r \rfloor}} \subset \overline{\Omega}$. Therefore $x' = x_{11}E_1 + x'_{\lfloor 2,r \rfloor} \in \overline{\Omega}$, so that we conclude $x \in \overline{\Omega}$. ■

For non-empty $I = \{i_1, i_2, \ldots, i_l\} \subset \lfloor 1, r \rfloor$, we denote by $\Delta^I_1, \Delta^I_2, \ldots, \Delta^I_l$ the basic relative invariants on $V^I$ associated to the cone $\Omega^I$, and extend these polynomials to $V$ by $\Delta^I_\alpha(x) := \Delta^I_\alpha(x_I)$ $(x \in V)$. We write $\Delta^I$ for the reduced determinant $\Delta^I_l$. Then $\Delta^I_\alpha = \Delta^{I_\alpha}$ with $I_\alpha := \{i_1, \ldots, i_{\alpha-1}, i_\alpha\}$, and we see from (2.17) that

$$D^I = D^I_l = \Delta^I \cdot (\Delta^{I_1})^{c_1}(\Delta^{I_2})^{c_2}\cdots(\Delta^{I_{l-1}})^{c_{l-1}} \tag{4.35}$$

for some non-negative integers $c_1, c_2, \ldots, c_{l-1}$.

Now that the polynomials $\Delta^I$ are introduced, we arrive at our goal, another description of $\overline{\Omega}$ with lower degree than (4.34).

**Theorem 4.4.** *The closure $\overline{\Omega}$ of $\Omega$ is described as*

$$\overline{\Omega} = \{ x \in V \, ; \, \Delta^I(x) \geq 0 \quad (I \subset \lfloor 1, r \rfloor, \; I \neq \emptyset) \} . \tag{4.36}$$

**Proof.** Similarly to the argument preceding Lemma 4.2, we see from Proposition 2.3 and Lemma 4.1 that $\overline{\Omega}$ is contained in the right-hand side of (4.36). For the proof of the converse inclusion, let $x$ be an element of $V$ such that $\Delta^I(x) \geq 0$ for all non-empty $I \subset \lfloor 1, r \rfloor$. Then (4.35) implies that $D^I(x) \geq 0$. Therefore Proposition 4.3 tells us that $x \in \overline{\Omega}$, which completes the proof. ■

## 5. Examples

In this section, we write $a'$ for the transpose of a matrix $a$, and denote by $E_{ij}$ the matrix unit. For a symmetric matrix $x = (x_{mk})_{m,k=1}^r$, let $\underset{\sim}{x} = (\underset{\sim}{x}_{mk})$ be the lower triangular matrix defined by

$$\underset{\sim}{x}_{mk} := \begin{cases} x_{mk} & (m > k), \\ x_{kk}/2 & (m = k), \\ 0 & (m < k), \end{cases}$$

and $\hat{x}$ the upper triangular matrix $(\underset{\sim}{x})'$ (cf. [14, p. 381]).

### 5.1. The cone of real positive definite symmetric matrices

Let $V$ be the space of real symmetric matrices of size $r$, $\Omega$ the subset of $V$ consisting of the positive definite elements, and $H \subset \mathrm{GL}(r, \mathbb{R})$ the group of lower triangular matrices with positive diagonals. Define the action of $H$ on $V$ by $t \cdot x := txt'$ $(t \in H, \, x \in V)$. Then $\Omega$ is a homogeneous cone on which $H$ acts simply transitively. As the base point $E$ of $\Omega$, we choose the unit matrix. Then the multiplication of the clan $(V, \triangle)$ is given by $x \triangle y := \underset{\sim}{x} y + y \hat{x}$. The determinant type polynomials $D_k$ are calculated as

$$D_k(x) = \begin{cases} \det x_{[k]} & (k = 1, 2), \\ (\det x_{[1]})^{2^{k-3}} (\det x_{[2]})^{2^{k-4}} \cdots (\det x_{[k-2]}) \cdot (\det x_{[k]}) & (k \geq 3), \end{cases}$$

where $x_{[k]}$ $(k = 1, \dots, r)$ are the submatrices $(x_{ij})_{1 \leq i,j \leq k}$ of $x = (x_{ij}) \in V$. Hence the basic relative invariants $\Delta_k(x)$ are nothing but the principal minors $\det x_{[k]}$ by (2.17). More generally, $\Delta^I(x)$ $(I \subset \lfloor 1, r \rfloor, \, I \neq \emptyset)$ is equal to the minor $\det\big((x_{ij})_{i,j \in I}\big)$.

### 5.2. The Vinberg cone

Let $V$ be the vector space

$$\left\{ x = (x_{(1)}, x_{(2)}) \, ; \, x_{(k-1)} := \begin{pmatrix} x_{11} & x_{k1} \\ x_{k1} & x_{kk} \end{pmatrix}, \; x_{11}, x_{k1}, x_{kk} \in \mathbb{R}, \; k = 2, 3 \right\},$$

and $\Omega$ the set of elements $(x_{(1)}, x_{(2)}) \in V$ such that the both of the components are positive definite. Then $\Omega$ is an open convex cone, called the *the Vinberg cone* ([14]). Define the group $H$ to be the set

$$\left\{ t = (t_{(1)}, t_{(2)}) \, ; \, t_{(k-1)} := \begin{pmatrix} t_{11} & \\ t_{k1} & t_{kk} \end{pmatrix}, \; t_{11}, t_{kk} > 0, \; t_{k1} \in \mathbb{R}, \; k = 2, 3 \right\}$$

with the multiplication of each component. The group $H$ acts on $V$ as

$$t \cdot x := (t_{(1)} x_{(1)} t'_{(1)},\ t_{(2)} x_{(2)} t'_{(2)}) \in V \quad (t \in H,\ x \in V),$$

so that $H$ acts simply transitively on $\Omega$. Setting $E := \left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right) \in \Omega$, we obtain the clan $(V, \triangle)$ of the homogeneous cone $\Omega$:

$$x \triangle y = (\underset{\smile}{x}_{(1)} y_{(1)} + y_{(1)} \hat{x}_{(1)},\ \underset{\smile}{x}_{(2)} y_{(2)} + y_{(2)} \hat{x}_{(2)}) \quad (x, y \in V),$$

and have the normal decomposition of $V$ with respect to the idempotents $E_1 = (E_{11}, E_{11})$, $E_2 = (E_{22}, 0)$ and $E_3 = (0, E_{22})$. Using Proposition 1.4, we calculate the composite determinants $D^I$ $(I \subset \{1,2,3\}, I \neq \emptyset)$ as follows:

$$\begin{aligned}
&D^{\{1\}}(x) = x_{11}, && D^{\{2\}}(x) = x_{22}, && D^{\{3\}}(x) = x_{33},\\
&D^{\{1,2\}}(x) = x_{11}x_{22} - x_{21}^2, && D^{\{1,3\}}(x) = x_{11}x_{33} - x_{31}^2, && D^{\{2,3\}}(x) = x_{22}x_{33},\\
&D^{\{1,2,3\}}(x) = (x_{11}x_{22} - x_{21}^2)(x_{11}x_{33} - x_{31}^2),
\end{aligned}$$

so that the reduced determinants are obtained as

$$\begin{aligned}
&\Delta^{\{1\}}(x) = x_{11}, && \Delta^{\{2\}}(x) = x_{22},\\
&\Delta^{\{1,2\}}(x) = x_{11}x_{22} - x_{21}^2, && \Delta^{\{3\}}(x) = \Delta^{\{2,3\}}(x) = x_{33},\\
&\Delta^{\{1,3\}}(x) = \Delta^{\{1,2,3\}}(x) = x_{11}x_{33} - x_{31}^2.
\end{aligned} \tag{5.37}$$

Here $\Delta^{\{1\}}$, $\Delta^{\{1,2\}}$ and $\Delta^{\{1,2,3\}}$ are the basic relative invariants. These are the same as Gindikin's observed in [6, p. 98 (e)]. We note that the degree of $\Delta^I$ is sometimes less than $\sharp I$. For example, the degree of $\Delta = \Delta^{\{1,2,3\}}$ is equal to 2.

Now let us consider the dual cone of $\Omega$. We realize the dual vector space of $V$ as

$$V^* := \left\{ \xi = \begin{pmatrix} \xi_{11} & \xi_{21} & \xi_{31} \\ \xi_{21} & \xi_{22} & 0 \\ \xi_{31} & 0 & \xi_{33} \end{pmatrix} ;\ \xi_{11}, \xi_{21}, \xi_{31}, \xi_{22}, \xi_{33} \in \mathbb{R} \right\}$$

with the dual coupling $\langle \cdot, \cdot \rangle$ defined by

$$\langle (x_{(1)}, x_{(2)}), \xi \rangle := \sum_{k=1}^{3} x_{kk} \xi_{kk} + 2 \sum_{k=2}^{3} x_{k1} \xi_{k1}.$$

For $t = (t_{(1)}, t_{(2)}) \in H$ and $\xi \in V^*$, we have $t^* \cdot \xi = \tilde{t} \xi (\tilde{t})'$ with

$$\tilde{t} := \begin{pmatrix} t_{11} & t_{21} & t_{31} \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}.$$

Then the clan $(V^*, \triangle')$ of $\Omega^*$ (see Section 3) is given by

$$\xi \triangle' \eta = \hat{\xi} \eta + \eta \underset{\smile}{\xi} \quad (\xi, \eta \in V^*),$$

and the idempotents $\mathfrak{E}_k$ $(k = 1,2,3)$ are equal to $E_{4-k,4-k}$. We obtain the composite and the reduced determinants on $V^*$ as

$$\begin{aligned}
&D^*_{\{1\}}(\xi) = \xi_{33}, && D^*_{\{2\}}(\xi) = \xi_{22}, && D^*_{\{3\}}(\xi) = \xi_{11},\\
&D^*_{\{1,2\}}(\xi) = \xi_{22}\xi_{33}, && D^*_{\{1,3\}}(\xi) = \xi_{11}\xi_{33} - \xi_{31}^2, && D^*_{\{2,3\}}(\xi) = \xi_{11}\xi_{22} - \xi_{21}^2,\\
&D^*_{\{1,2,3\}}(\xi) = \xi_{11}\xi_{22}\xi_{33}^2 - \xi_{22}\xi_{33}\xi_{31}^2 - \xi_{33}^2\xi_{21}^2,
\end{aligned}$$

and

$$\begin{aligned} \Delta^*_{\{1\}}(\xi) &= \xi_{33}, \qquad \Delta^*_{\{3\}}(\xi) = \xi_{11}, \quad \Delta^*_{\{1,3\}}(\xi) = \xi_{11}\xi_{33} - \xi_{31}^2, \\ \Delta^*_{\{2\}}(\xi) = \Delta^*_{\{1,2\}}(\xi) &= \xi_{22}, \quad \Delta^*_{\{2,3\}}(\xi) = \xi_{11}\xi_{22} - \xi_{21}^2, \\ \Delta^*_{\{1,2,3\}}(\xi) &= \xi_{11}\xi_{22}\xi_{33} - \xi_{22}\xi_{31}^2 - \xi_{33}\xi_{21}^2 = \det \xi, \end{aligned} \tag{5.38}$$

respectively. The basic relative invariants are $\Delta^*_{\{1\}}$, $\Delta^*_{\{1,2\}}$ and $\Delta^*_{\{1,2,3\}}$ (see also [6, p. 98 (f)]). By Proposition 2.3 and (5.38), the dual cone $\Omega^*$ is described as

$$\begin{aligned} \Omega^* &= \left\{ \xi \in V^* \,;\, \xi_{33} > 0,\ \xi_{22} > 0,\ \xi_{11}\xi_{22}\xi_{33} - \xi_{22}\xi_{31}^2 - \xi_{33}\xi_{21}^2 > 0 \right\} \\ &= \{ \xi \in V^* \,;\, \xi \text{ is positive definite} \}. \end{aligned}$$

The expressions in (5.38) are quite different from the ones in (5.37), which reflects the non-symmetry of the Vinberg cone $\Omega$.

## References

[1] D'Atri, J. E., and I. D. Miatello, *A characterization of bounded symmetric domains by curvature*, Trans. Amer. Math. Soc. **276** (1983), 531–540.

[2] Dorfmeister, J., and M. Koecher, *Relative Invarianten und nicht-assoziative Algebren*, Math. Ann. **228** (1977), 147–186.

[3] Faraut, J., and A. Korányi, "Analysis on symmetric cones," Clarendon, Oxford, 1994.

[4] Gindikin, S. G., *Analysis in homogeneous domains*, Russian Math. Surveys **19** (1964), 1–89.

[5] —, *Invariant generalized functions in homogeneous domains*, Funct. Anal. Appl. **9** (1975), 50–52.

[6] —, "Tube domains and the Cauchy problem," Transl. Math. Monogr. **111**, Amer. Math. Soc., 1992.

[7] Ishi, H., *Positive Riesz distributions on homogeneous cones*, J. Math. Soc. Japan **52** (2000), 161–186.

[8] —, *Representations of the affine transformation groups acting simply transitively on homogeneous Siegel domains*, J. Funct. Anal. **167** (1999), 425–462.

[9] Kaneyuki, S., *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo Sect. I **14** (1967), 89–130.

[10] Piatetskii-Shapiro, I. I., "Automorphic functions and the geometry of classical domains," Gordon and Breach, New York, 1969.

[11] Rossi, H., and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal. **13** (1973), 324–389.

[12] Sato, M., T. Shintani, and M. Muro, *Theory of prehomogeneous vector spaces (algebraic part)—English translation of Sato's lecture from Shintani's notes*, Nagoya Math. J. **120** (1990), 1–34.

[13] Vinberg, E. B., *Homogeneous cones*, Soviet Math. Dokl. **1** (1960), 787–790.

[14] —, *The theory of convex homogeneous cones*, Trans. Moscow Math. Soc. **12** (1963), 340–403.

[15] —, *The structure of the group of automorphisms of a homogeneous convex cone*, Trans. Moscow Math. Soc. **13** (1965), 63–93.

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