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A Classification of Reductive Linear Groups with Spherical Orbits

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Abstract. We classify finite dimensional G-modules V of an algebraic reductive group G such that any G-orbit in V is spherical. It is shown that any module with this property can be realized as a spherical module after an extension of the group by a central torus.
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1. Introduction

Let G be a connected reductive algebraic group over an algebraically closed field K of zero characteristic, and G^s be the maximal connected semisimple subgroup of G. Let us recall that an irreducible G-variety Y is called *spherical* if the induced action $B : Y$ of a Borel subgroup $B \subseteq G$ has an open orbit. By Rosenlicht's theorem, an action $G : Y$ is spherical iff $K(Y)^B = K$, where $K(Y)^B$ is the field of rational B-invariants. An algebraic subgroup $H \subset G$ is said to be *spherical* if the homogeneous space G/H is a spherical G-variety.

Definition 1.1. Let X be an irreducible algebraic variety. We shall say that an action $G : X$ is *an action with spherical orbits* if there exists an open subset $X_0 \subset X$ such that for any $x \in X_0$ the orbit Gx is spherical.

Below we list some basic facts about actions with spherical orbits.

(1) Any trivial G-action is an action with spherical orbits.

(2) Suppose that for an action $G : X$ a generic isotropy subgroup exists, see [9, sec. 7.3]. (This is always the case for linear actions.) Denote this subgroup by H. The action $G : X$ is an action with spherical orbits iff H is a spherical subgroup of G.

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(3) Rosenlicht's theorem implies that an action $G : X$ is an action with spherical orbits iff $K(X)^G = K(X)^B$.

(4) It is shown in [1, Corollary 1] that for an action with spherical orbits any G-orbit is spherical.

(5) Let $G_1 : X_1$ and $G_2 : X_2$ be actions with spherical orbits. Then the action $(G_1 \times G_2) : (X_1 \times X_2)$ is an action with spherical orbits.

In this paper we classify G-modules with spherical orbits. Let V be a finite dimensional G-module and $V = V_1 \oplus \ldots \oplus V_k$ be a decomposition into simple G-submodules. Denote by T the k-dimensional algebraic torus acting by dilatations on any V_i. Then $\overline{G} = TG$ is a linear group acting on V. It is shown (Proposition 3.3) that the $\overline{G}$-module V is spherical. This result and a classification of spherical modules from [8] allow us to list all "minimal" indecomposable linear actions with spherical orbits (Tables 1-3). In Theorem 6.2 a characterization of arbitrary G-modules with spherical orbits is obtained. It is proved that for any G-module with spherical orbits the algebras of U- and G^s-invariants are free. (Here U is a maximal unipotent subgroup of G.) Finally we get a classification of G-actions with spherical orbits on projective spaces $\mathbb{P}(V)$.

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2. Indecomposable modules

Definition 2.1. A G-module V is said to be *a module with spherical orbits* if the linear action $G : V$ is an action with spherical orbits.

Definition 2.2. A G-module V is *indecomposable* if there exist no proper decompositions $G^s = G_1^s \times G_2^s$ and $V = V_1 \oplus V_2$ such that $(g_1, g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2)$ for any $g = (g_1, g_2) \in G^s$ and any $v = (v_1, v_2) \in V$.

Definition 2.3. We say that a G'-module V is obtained from a G-module V *by a torus extension* if there exists a torus T acting on V such that T- and G-actions commute and $G' = TG$.

It is clear that any G-module V is obtained by a torus extension from the G^s-module V.

Lemma 2.4. *Suppose that V is a G-module with spherical orbits and a G'-module V is obtained from this module by a torus extension. Then V is a G'-module with spherical orbits.*

Proof. Let H be the generic isotropy subgroup for the action $G : V$. By assumption, H is spherical in G. Then any subgroup of G' containing H is spherical in G'. Hence a generic isotropy subgroup for the G'-module V is spherical. ∎

Theorem 2.5. *All indecomposable G-modules with spherical orbits are either indicated in Tables 1-3 or are obtained from the indicated modules by a torus extension.*

Table 1

	G	weights	$\dim V$	$\mathcal{H}$	codim
0	$\{e\}$	0	1	0	1
1	$SL(n)$	ϕ_1	n	$A_{n-2}+R_{n-1}$	0
2	$\Lambda^2 SL(2n)$	ϕ_2	$2n^2-n$	C_n	1
3	$\Lambda^2 SL(2n+1)$	ϕ_2	$2n^2+n$	C_n+R_{2n}	0
4	$S^2 SL(2n)$	$2\phi_1$	$2n^2+n$	D_n	1
5	$S^2 SL(2n+1)$	$2\phi_1$	$2n^2+3n+1$	B_n	1
6	$SO(2n)$	ϕ_1	$2n$	B_{n-1}	1
7	$SO(2n+1)$	ϕ_1	$2n+1$	D_n	1
8	$Spin(7)$	ϕ_3	8	G_2	1
9	$Spin(9)$	ϕ_4	16	B_3	1
10	$Spin(10)$	ϕ_4	16	B_3+R_8	0
11	$Sp(2n)$	ϕ_1	$2n$	$C_{n-1}+R_{2n-1}$	0
12	G_2	ϕ_1	7	A_2	1
13	E_6	ϕ_1	27	F_4	1

Comments to the Tables. The column "G" contains a reductive group G. In Table 1 the linear group $\Lambda^2 SL(n)$ is the image of $SL(n)$ under the action in the second exterior power of the tautological representation, and $S^2 SL(n)$ is the same thing with respect to the second symmetric power.

In the column "weights" the highest weights of the G-module are indicated.

Table 2

	G	weights	$\dim V$	$\mathcal{H}$	codim
14	$SL(2)\times K^*$	$\phi_1\otimes\epsilon+\phi_1\otimes\epsilon^{-1}$	4	t_1	1
15	$SL(n)\times K^*,\ n>2$	$\phi_1\otimes\epsilon^a+\phi_1\otimes\epsilon^b,\ a\neq b$	$2n$	$A_{n-3}+t_1+R_{2(n-2)}$	0
16	$SL(n),\ n>2$	$\phi_1+\phi_{n-1}$	$2n$	A_{n-2}	1
17	$SL(2n+1)$	$\phi_1+\phi_2$	$(2n+1)(n+1)$	C_n	1
18	$SL(2n+1)\times K^*$	$\phi_1\otimes\epsilon^a+\phi_{2n-1}\otimes\epsilon^b, a\neq nb$	$(2n+1)(n+1)$	$C_{n-1}+t_1+R_{2(2n-1)}$	0
19	$SL(2n)$	$\phi_1+\phi_2$ $\phi_1+\phi_{2n-2}$	$n(2n+1)$	$C_{n-1}+R_{2n-1}$	1
20	$SO(8)$	$\phi_1+\phi_3$	16	G_2	2
21	$Sp(2n)\times K^*$	$\phi_1\otimes\epsilon+\phi_1\otimes\epsilon^{-1}$	$4n$	$C_{n-1}+t_1$	1
22	$SL(n)\times SL(m),\ n>m$	$\phi_1\otimes\phi_1$	nm	$A_{n-m-1}+A_{m-1}+R_{nm-m^2}$	0
23	$SL(n)\times SL(n)$	$\phi_1\otimes\phi_1$	n^2	A_{n-1}	1
24	$SL(2)\times Sp(2n)$	$\phi_1\otimes\phi_1$	$4n$	$C_{n-1}+A_1$	1
25	$SL(3)\times Sp(2n)\times K^*,\ n>1$	$\phi_1\otimes\phi_1\otimes\epsilon$	$6n$	$C_{n-2}+A_1+t_1+R_{2n-1}$	0
26	$SL(4)\times Sp(4)$	$\phi_1\otimes\phi_1$	16	C_2	1
27	$SL(n)\times Sp(4),\ n>4$	$\phi_1\otimes\phi_1$	$4n$	$A_{n-5}+C_2+R_{4(n-4)}$	0

For the group $G_1\times G_2$ the weight $\phi\otimes\psi$ corresponds to the tensor product of simple G_1- and G_2-modules with highest weights ϕ and ψ respectively. The symbol $+$ denotes a direct sum of modules. If G^s is the product of several simple groups, then their fundamental weights are denoted successively by letters ϕ_i, ψ_i and τ_i. The fundamental weight of the central torus is denoted by ϵ (for a two-dimensional torus – by ϵ_1 and ϵ_2).

In the column "$\dim V$" the dimension of the module is shown.

In Tables 1 and 2 the column "$\mathcal{H}$" contains the type of the tangent algebra $\mathcal{H}$ of the generic isotropy subgroup H for our module. Here t_1 is the tangent algebra of the one-dimensional central torus in H, and R_k is the tangent algebra of the k-dimensional unipotent radical of H. The information of this column is taken from Elashvili's tables [4], [5].

Table 3

	G	weights	$\dim V$	codim
28	$SL(n) \times SL(n) \times K^*$	$\phi_1 \otimes \epsilon + \phi_1 \otimes \psi_1$ $\phi_1 \otimes \epsilon + \phi_{n-1} \otimes \psi_{n-1}$	$n(n+1)$	1
29	$SL(n+1) \times SL(n) \times K^*$	$\phi_1 \otimes \epsilon^n + \phi_1 \otimes \psi_1 \otimes \epsilon^{-1}$	$(n+1)^2$	1
30	$SL(n+1) \times SL(n) \times K^* \times K^*,\ n > 1$	$\phi_1 \otimes \epsilon_1 + \phi_n \otimes \psi_{n-1} \otimes \epsilon_2$	$(n+1)^2$	0
31	$SL(n) \times SL(m) \times K^*,\ n > m+1$	$\phi_1 \otimes \epsilon^a + \phi_1 \otimes \psi_1 \otimes \epsilon^b,\ a \neq b$	$n(m+1)$	0
32	$SL(n) \times SL(m) \times K^*,\ n > m+1 > 2$	$\phi_1 \otimes \epsilon^a + \phi_{n-1} \otimes \psi_{m-1} \otimes \epsilon^b,\ a \neq -b$	$n(m+1)$	0
33	$SL(n) \times SL(m) \times K^*,\ n < m$	$\phi_1 \otimes \epsilon^a + \phi_1 \otimes \psi_1 \otimes \epsilon^b,\ a \neq 0$ $\phi_1 \otimes \epsilon^a + \phi_{n-1} \otimes \psi_{m-1} \otimes \epsilon^b,\ a \neq 0$	$n(m+1)$	0
34	$SL(n) \times SL(2) \times SL(m),\ n > 2,\ m > 2$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$2(n+m)$	0
35	$SL(n) \times SL(2) \times Sp(2m),\ n > 2,\ m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$2(n+2m)$	1
36	$Sp(2n) \times SL(2) \times Sp(2m),\ n, m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$4(m+n)$	2
37	$SL(2) \times Sp(2n) \times K^*$	$\phi_1 \otimes \epsilon + \phi_1 \otimes \psi_1$	$2(2n+1)$	1

In the last column the codimension of a generic G-orbit in V is shown.

3. Spherical modules

It is natural to say that a G-module V is *spherical* if V is a spherical G-variety. (In particular, G acts on V with an open orbit.) Such modules are also known as *multiplicity free modules*. The motivation for this term is the following characterization of spherical G-modules: any simple G-module appears in the decomposition of the G-module $K[V]$ of polynomial functions on V with multiplicity ≤ 1 [7]. A classification of spherical modules was obtained in [6] (simple modules), and in [2], [3], [8] (non-simple modules).

Let $V = V_1 \oplus \ldots \oplus V_k$ be a decomposition of a G-module as a sum of simple submodules, and $T = T_1 \times \ldots \times T_k$ be a torus acting on V, where T_i is a one-dimensional torus acting on V_i by dilatations and trivially on the other components. We shall say that T is the *saturating* torus for the G-module V, and set $\overline{G} = GT$.

Definition 3.1. A G-module V is said to be *saturated* if $G = \overline{G}$.

We need the following well-known lemma.

Lemma 3.2. ([6, Lemma 3.11]) *If V is a simple G-module and H is a spherical subgroup of G, then for the subspace V^H of H-fixed points one has* $\dim V^H \leq 1$.

Proposition 3.3. *If V is a G-module with spherical orbits, then V is a spherical $\overline{G}$-module.*

Proof. Let H be the generic isotropy subgroup for V. By assumption, H is a spherical subgroup of G. Hence $\dim V_i^H \leq 1$ and $\dim V^H \leq k$. This implies that the torus T acts on V^H with an open orbit, and the $\overline{G}$-orbit of a point of the open T-orbit in V^H is open in V. On the other hand, this orbit is spherical by Lemma 2.4. ∎

4. The algebra of U-invariants

Let U be a maximal unipotent subgroup of G. For any G-module V the algebra $K[V]^U$ of polynomial U-invariants is the linear span of the highest weight vectors of the G-module $K[V]$. The following proposition can be found, for example, in [8, Theorem 1.2].

Proposition 4.1. *A G-module V is spherical iff $K[V]^U = K[g_1, \ldots, g_m]$, where the g_i's are algebraically independent and the weights of g_i are linearly independent.*

In particular, the algebra $K[V]^U$ is free for any spherical G-module V.

Proposition 4.2. *For any G-module V with spherical orbits the algebras $K[V]^U$ and $K[V]^{G^s}$ are free.*

Proof. The subgroup U is a maximal unipotent subgroup of the saturated linear group $\overline{G}$. Thus the first statement follows from Proposition 3.3 and Proposition 4.1. For a semisimple group the semigroup of dominant weights generates (over $\mathbb{Q}$) a strictly convex cone. Hence the algebra $K[V]^{G^s}$ is generated by G^s-invariant free generators g_i of the algebra $K[V]^U$. ∎

Finally we get an analog of Proposition 4.1 for modules with spherical orbits.

Proposition 4.3. *Let G be a connected semisimple group. The following conditions are equivalent:*

(1) V is a G-module with spherical orbits;

(2) $K[V]^U = K[f_1, \ldots, f_h, g_1, \ldots, g_m]$, where the f_i's are generators of $K[V]^G$, the g_i's are algebraically independent and their weights are linearly independent.

Proof. (1) $\Rightarrow$ (2) We may suppose that f_i, g_j are free generators of $K[V]^U$. Denote by ω_i the weight of g_i. If ω_i are linearly dependent, then (after reindexing) one has $b_1\omega_1 + \ldots + b_p\omega_p = c_{p+1}\omega_{p+1} + \ldots + c_m\omega_m$, where b_i, c_j are non-negative integers. Hence

$$\frac{g_1^{b_1} \cdots g_p^{b_p}}{g_{p+1}^{c_{p+1}} \cdots g_m^{c_m}} \in K(V)^B = K(V)^G.$$

But for a semisimple group any rational G-invariant is the ratio of two polynomial G-invariants [9, sec. 3.2]. Therefore

$$g_1^{b_1} \ldots g_p^{b_p} F_1(f_1, \ldots, f_h) = g_{p+1}^{c_{p+1}} \ldots g_m^{c_m} F_2(f_1, \ldots, f_h).$$

This contradiction concludes the proof.

(2) $\Rightarrow$ (1) Any rational B-invariant is the ratio of two polynomial B-semi-invariants of the same weight [9, sec. 3.2]. Thus any element of $K(V)^B$ is the ratio of $g_1^{d_1} \ldots g_m^{d_m} F_1(f_1, \ldots, f_h)$ and $g_1^{d_1} \ldots g_m^{d_m} F_2(f_1, \ldots, f_h)$. This shows that $K(V)^B = K(V)^G$. ∎

5. A technical lemma

Let T be an algebraic torus and $X(T)$ be the character group of T. Consider a field $K_1 = K(a_1, \ldots, a_n)$ with a T-action (by automorphisms) such that $t * a_i = \lambda_i(t)a_i$ for some $\lambda_i \in X(T)$. The subfield of T-fixed elements of K_1 is denoted by K_1^T. By tr.deg F we denote the transcendency degree of a field F over K. Consider the subfield $K_2 = K(a_1, \ldots, a_q)$, $q \le n$. Denote by $S_1 =< \lambda_1, \ldots, \lambda_n >$ and $S_2 =< \lambda_1, \ldots, \lambda_q >$ the corresponding sublattices in $X(T)$. For any sublattice $L \subset X(T)$ define $L_{\mathbb{Q}}$ as the vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$.

A sublattice L determines the subgroup

$$T_L = \{t \in T \mid \lambda(t) = 1 \ \ \forall \lambda \in L\}.$$

Lemma 5.1. *Suppose that $K_1^T = K$. Then the following conditions are equivalent:*

(1) tr.deg $K_1^{T_L}$ = tr.deg $K_2^{T_L}$*;*

(2) $L \cap S_1 \subset (S_2)_{\mathbb{Q}}$.

Proof. Set $A_1 = K[a_1, \ldots, a_n]$ and $A_2 = K[a_1, \ldots, a_q]$. Denote by Spec A_i the affine variety corresponding to the algebra A_i, $i = 1, 2$. For the natural T-action one has tr.deg $K_i^{T_L} = \dim \mathrm{Spec}\ A_i - h_i$, where h_i is the dimension of generic T-orbits in Spec A_i. The condition $K_1^T = K$ implies $\dim \mathrm{Spec}\ A_1 = \mathrm{rk}\ S_1$ and $\dim \mathrm{Spec}\ A_2 = \mathrm{rk}\ S_2$. Moreover, $h_i = \dim T_L - \dim T_{S_i + L}$. Thus condition (1) is equivalent to $\mathrm{rk}\,(S_1 \cap L) = \mathrm{rk}\,(S_2 \cap L)$. This is a reformulation of (2). ∎

6. Modules with spherical orbits

Let V be a saturated spherical G-module. Any character of the saturating torus T can be considered as a character of G with trivial restriction to G^s. For any sublattice $L \subset X(T)$ define

$$G_L = \{g \in G \mid \lambda(g) = 1 \ \ \forall \lambda \in L\}.$$

Let B^s be a Borel subgroup of G^s. Then $B = TB^s$ and $B_L = T_L^0 B^s$ are Borel subgroups of G and G_L respectively. Set

$$S(V) = \{\lambda \in X(T) \mid \exists f \in K(V)^{B^s} : \ t * f = \lambda(t) f \ \ \forall t \in T\}$$

and

$$M(V) = \{\lambda \in X(T) \mid \exists f \in K(V)^{G^s} : \ t * f = \lambda(t) f \ \ \forall t \in T\}.$$

Proposition 6.1. *(cf. [8, Theorem 2.6]) The G_L-module V is spherical iff $L \cap S(V) = \{0\}$.*

Proof. Lemma 5.1 with $K_1 = K(V)^{B^s}$ and $K_2 = K$ shows that the condition $L \cap S(V) = \{0\}$ is equivalent to tr.deg $K(V)^{B_L} = 0$, or $K(V)^{B_L} = K$. ■

Theorem 6.2. *A G-module V is a module with spherical orbits iff the $\overline{G}$-module V is spherical and $G = G_L$ with $L \cap S(V) \subset M(V)_{\mathbb{Q}}$.*

Proof. Appling Lemma 5.1 with $K_1 = K(V)^{B^s}$ and $K_2 = K(V)^{G^s}$ one shows that the second condition is equivalent to

$$\text{tr.deg}\, (K(V)^{B^s})^{T_L} = \text{tr.deg}\, (K(V)^{G_s})^{T_L},$$

or $K(V)^{B_L} = K(V)^{G_L}$. ■

In Table 4 we indicate bases of the subspaces $S(V)_{\mathbb{Q}}$ and $M(V)_{\mathbb{Q}}$ for all indecomposable saturated spherical G-modules. There is a natural bijection between such modules and items of Tables 1-3, see [8, Theorem 2.5]. Our calculations are based on [8, Table 2].

7. Proof of Theorem 2.5

Suppose that V is a spherical indecomposable saturated $\overline{G}$-module. By Proposition 3.3 and Lemma 2.4, we need to find all minimal (with respect to inclusion) subgroups G, $G^s \subseteq G \subseteq \overline{G}$, such that V is a G-module with spherical orbits. By Theorem 6.2, these subgroups coincide with G_L for a maximal sublattice $L \subset X(T)$ satisfying $L \cap S(V) \subset M(V)_{\mathbb{Q}}$.

Case 1. $S(V) = M(V) = 0$. Here V is a spherical G^s-module, $G = G^s$, and we obtain items 1, 3, 10, 11, 22, 27, 34.

Table 4

items	$S(V)_{\mathbb{Q}}$	$M(V)_{\mathbb{Q}}$
1, 3, 10, 11, 22, 27, 34	0	0
0, 2, $4-9$, 12, 13, 23, 24, 26	$<\epsilon_1>$	$<\epsilon_1>$
19, 35	$<\epsilon_2>$	$<\epsilon_2>$
25, 33	$<\epsilon_1>$	0
20, 36	$<\epsilon_1,\epsilon_2>$	$<\epsilon_1,\epsilon_2>$
15, 31	$<\epsilon_1-\epsilon_2>$	0
18	$<\epsilon_1-n\epsilon_2>$	0
30	$<\epsilon_1,\epsilon_2>$	0
32	$<\epsilon_1+\epsilon_2>$	0
16	$<\epsilon_1+\epsilon_2>$	$<\epsilon_1+\epsilon_2>$
14, 21	$<\epsilon_1,\epsilon_2>$	$<\epsilon_1+\epsilon_2>$
17	$<\epsilon_1+n\epsilon_2>$	$<\epsilon_1+n\epsilon_2>$
28, 37	$<\epsilon_1,\epsilon_2>$	$<\epsilon_2>$
29	$<\epsilon_1,\epsilon_2>$	$<\epsilon_1+n\epsilon_2>$

Case 2. rk $S(V)=$ rk $M(V)\neq 0$. Here V is a G^s-module with spherical orbits. One gets items 0, 2, 4 – 9, 12, 13, 16, 17, 19, 20, 23, 24, 26, 35, 36.

Case 3. rk $S(V)=1$ and $M(V)=0$. In this case G can be obtained as an extension of G^s by any one-dimensional subtorus of T_L with the restriction $L\cap S(V)=\{0\}$. We get items 15, 18, 25, 31, 32, 33.

Case 4. rk $S(V) = 2$ and $M(V) = 0$. In this case the $\overline{G}$-module V is the only module with spherical orbits – item 30.

Case 5. rk $S(V) = 2$ and rk $M(V) = 1$. In this case G is obtained as an extension of G^s by a one-dimensional subtorus of T_L with $L_{\mathbb{Q}} = M(V)_{\mathbb{Q}}$ (this is the torus fixing generic G^s-orbits in V). One gets items 14, 21, 28, 29, 37.

This completes the proof of Theorem 2.5.

Remark 7.1. Suppose that there is a decomposable G-module V. We are going to check that all orbits in V are spherical. First we verify that for all indecomposable components V_i the G-module V_i is a module with spherical orbits (Theorem 2.5). If this is the case we apply Theorem 6.2 and use Table 4.

8. Actions with spherical orbits on projective spaces

Any linear action $G : V$ determines an action $G : \mathbb{P}(V)$ on the projective space $\mathbb{P}(V)$. Let T' be a one-dimensional torus acting on V by dilatations and $G' = T'G$ be the torus extension. It is clear that the action $G : \mathbb{P}(V)$ is an action with spherical orbits iff the action $G' : V$ has this property. Let us recall that T is the saturating torus for the G-module V. Denote by F the sublattice in $X(T)$ corresponding to T'. (In standart coordinates F is given as $x_1 + \ldots + x_k = 0$.) The following results are immediate consequences of Theorem 6.2 and Theorem 2.5.

Proposition 8.1. *An action $G : \mathbb{P}(V)$ is an action with spherical orbits iff the $\overline{G}$-module V is spherical and $G = G_L$ with $(L \cap F \cap S(V)) \subset M(V)_{\mathbb{Q}}$.*

Proposition 8.2. *Suppose that G is semisimple and a G-module V is indecomposable. The action $G : \mathbb{P}(V)$ is an action with spherical orbits iff either the G-module V is a module with spherical orbits or V is one of the modules indicated in Table 5.*

Table 5

	G	weights	$\dim V$
1	$SL(3) \times Sp(2n),\ \ n > 1$	$\phi_1 \otimes \psi_1$	$6n$
2	$SL(2n+1),\ \ n > 1$	$\phi_1 + \phi_{2n-1}$	$(2n+1)(n+1)$
3	$SL(n) \times SL(m),\ n > m+1 > 2$	$\phi_1 + \phi_{n-1} \otimes \psi_{m-1}$	$n(m+1)$
4	$SL(n) \times SL(m),\ n < m$	$\phi_1 + \phi_1 \otimes \psi_1$ $\phi_1 + \phi_{n-1} \otimes \psi_{m-1}$	$n(m+1)$

References

[1] Arzhantsev, I. V., *On SL_2-actions of complexity one*, Izvestiya RAN, Ser. Mat.**61(4)** (1997), 3–18 (in Russian); English transl.: Izvestiya Mathematics **61(4)** (1997), 685–698.

[2] Brion, M., *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. École Norm. Sup. (4), **18** (1985), 345–387.

[3] Benson, C., and G. Ratcliff, *A classification of multiplicity free actions*, Journal of Algebra, **181** (1996), 152–186.

[4] Elashvili, A. G., *Canonical forms and stationary subalgebras of points of general position for simple linear Lie groups*, Funkts. Analiz i ego Prilog. **6(1)** (1972), 51–62 (in Russian); English transl.: Func. Anal. and its Appl. **6(1)** (1972), 44–53.

[5] —, *Stationary subalgebras of points of the common state for irreducible linear Lie groups*, Funkts. Analiz i ego Prilog. **6(2)** (1972), 65–78 (in Russian); English transl.: Func. Anal. and its Appl. **6(2)** (1972), 139–148.

[6] Kac, V. G., *Some remarks on nilpotent orbits*, Journal of Algebra, **64** (1980), 190–213.

[7] Kimel'feld, B. N., and E. B. Vinberg, *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funkts. Analiz i ego Prilog. **12(3)** (1978), 12–19 (in Russian); English transl.: Func. Anal. and its Appl. **12(3)** (1978), 168–174.

[8] Leahy, A. S., *A classification of multiplicity free representations*, Journal of Lie Theory, **8** (1998), 367–391.

[9] Popov, V. L., and E. B. Vinberg, *Invariant theory*, Itogi Nauki i Tekhniki, Sovr. Probl. Mat. Fund. Napravl., vol. 55, VINITI, Moscow 1989, pp. 137–309 (in Russian); English transl.: Algebraic Geometry IV, Encyclopaedia of Math. Sciences, vol. 55, Springer-Verlag, Berlin 1994, pp. 123–278.

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