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The Variety of Lie Bialgebras

Nicola Ciccoli and Lucio Guerra

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Abstract. We define a Lie bialgebra cohomology as the total cohomology of a double complex constructed from a Lie algebra and its dual, we show that its 2–cocycles classify Lie bialgebra formal deformations and we prove the usual cohomological condition (i. e. $H^2 = 0$) for formal rigidity. Lastly we describe the results of explicit computations in low–dimensional cases.

1. Introduction

Lie bialgebras were first introduced by Drinfel'd in [9] as the algebraic structures underlying quantized enveloping algebras (*quantum groups*). They can be described as Lie algebras provided with a Lie bracket on the dual satisfying a suitable compatibility condition. Already in Drinfel'd's work many interesting examples of Lie bialgebras based on complex semisimple Lie algebras were given. A few years later De Smedt in [8] proved the existence of a non trivial Lie bialgebra based on any Lie algebra. The classification of such structures was, at that time, a natural problem to pose. The first positive result was obtained for real compact Lie algebras ([14, 15]). More recently, in [7], a complete classification of Lie bialgebras with reductive double was given (recall that the double is an orthogonal Lie algebra containing both the starting Lie algebra and its dual as maximal isotropic Lie subalgebras).

Still a classification of Lie bialgebras is out of reach, just for similar reasons as for the Lie algebra classification. As an example, classifying the subclass of triangular Lie bialgebras of semisimple complex Lie algebras contains the problem of Frobenius Lie subalgebras classification as a special case. This last problem is known to be quite hard in general (apart from the trivial $\mathfrak{sl}_2$ case) as it does not allow induction on dimension. In the nonsemisimple case only a bunch of low–dimensional examples were thoroughly studied.

In view of these obstacles, and in analogy with the Lie algebra case, an alternative approach is to understand the geometry of the algebraic variety of Lie bialgebra laws on a given vector space. For Lie brackets, for example, interesting general results, together with a quite accurate description in low dimension are given in [6].

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In this paper we show how the usual deformation and cohomology arguments can be described in this situation. More precisely we will define a Lie bialgebra cohomology as the total cohomology of a double complex constructed from a Lie algebra and its dual. We will show that its 2–cocycles classify Lie bialgebra formal deformations, the coboundaries coinciding with trivial ones. We will then prove the usual cohomological condition (i.e. $H^2 = 0$) for formal rigidity. Lastly we describe the results of explicit computations in low–dimensional cases.

2. The variety of Lie bialgebras

Let V be a finite dimensional vector space over the field $\mathbb{K}$.

Definition 2.1. A Lie bialgebra structure $D = (V, \mu, \delta)$ is a pair of operations (μ, δ) on V such that:

1. $\mu : \wedge^2 V \to V$ is a Lie bracket;
2. ${}^t\delta : \wedge^2 V^* \to V^*$ is a Lie bracket (here ${}^t\delta$ denotes the transpose map of $\delta : V \to \wedge^2 V$).
3. δ is a 1–cocycle w.r.t. the adjoint action $\mathrm{ad}^{(2)}$ of (V, μ) on $\wedge^2 V$, i.e.:

$$\delta([X, Y]) = \mathrm{ad}^{(2)}_X \delta(Y) - \mathrm{ad}^{(2)}_Y \delta(X)\,, \qquad \forall X, Y \in V. \tag{1}$$

Let us introduce some notations. Let $D = (V, \mu, \delta)$ be a Lie bialgebra. We will reserve gothic letters for Lie algebras and denote with $\mathfrak{g}$ the Lie algebra (V, μ) and with $\mathfrak{g}^*$ the Lie algebra $(V^*, {}^t\delta)$. The double of a Lie bialgebra (see [11] for its definition) is a Lie algebra structure on $V \oplus V^*$ and will be denoted with $\mathfrak{d}$. We will often denote a Lie bialgebra simply as the pair (μ, δ), the underlying vector space being fixed. The set of all Lie bialgebras on V can be thus identified with a subset of the vector space $(\wedge^2 V^* \otimes V) \oplus (\wedge^2 V \otimes V^*)$. We will denote such subset as $\mathcal{D}(V)$ (and often omit reference to V). When V is an n–dimensional vector space we will also write $\mathcal{D}_n$ for $\mathcal{D}(V)$. Let us remark that identity (1) can be rewritten as (see [11]):

$$\begin{aligned} \langle [X_1, X_2], [\psi_1, \psi_2] \rangle \;=\;& \langle X_1, [X_2 \cdot \psi_1, \psi_2] \rangle - \langle X_2, [X_1 \cdot \psi_1, \psi_2] \rangle \\ & - \langle X_2, [\psi_1, X_1 \cdot \psi_2] \rangle + \langle X_1, [\psi_1, X_2 \cdot \psi_2] \rangle \end{aligned} \tag{2}$$

for all $X_i \in V$ and for all $\psi_i \in V^*$, where $X \cdot \psi$ (resp. $\psi \cdot X$) denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ (resp. of $\mathfrak{g}^*$ on $\mathfrak{g}$).

The Jacobi identity for μ (resp. for ${}^t\delta$) consists of a set of quadratic equations identifying the subvariety of Lie algebras on V (resp. on V^*). The cocycle condition (1) fixes another set of quadratic equations identifying the *variety of Lie bialgebras* on the vector space V.

To be more explicit, once we fix a basis $\{X_1, \ldots, X_n\}$ in V we have:

$$\mu[X_i, X_j] = \sum_k c_{ij}^k X_k, \qquad \delta(X_i) = \sum_{j,k} \gamma_i^{jk} X_j \wedge X_k\,.$$

The cocycle condition (1) expressed in the $2n^3$–coordinates $(c_{ij}^k, \gamma_k^{ij})$ translates then into the family of n^4 equations:

$$\sum_{k=1}^{n} c_{pq}^k \gamma_k^{ij} = \sum_{k=1}^{n} (\gamma_q^{kj} c_{pk}^i + \gamma_p^{kj} c_{kq}^i + \gamma_q^{ik} c_{pk}^j + \gamma_p^{ik} c_{kq}^j) \tag{3}$$

for every $i, j, p, q = 1, \ldots, n$. These are the equations of the algebraic subvariety $\mathcal{D}_n$ of Lie bialgebras on an n–dimensional vector space, in the variety $Lie_n \times Lie_n$. It is easily verified that such variety is a cone with vertex in the origin. We will denote the corresponding projective variety as $\mathbb{P}(\mathcal{D})$.

As an example consider the two–dimensional case. If $\mathfrak{g} = (V, \mu)$ is any 2–dimensional Lie algebra it is easily verified that every linear map $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ defines a Lie bialgebra on V. Therefore $\mathcal{D}_2 \simeq \mathbb{K}^2 \times \mathbb{K}^2$.

Definition 2.2. Given two Lie bialgebras, (V_1, μ_1, δ_1) and (V_2, μ_2, δ_2) a vector space isomorphism $f : V_1 \to V_2$ is said to be a Lie bialgebra isomorphism if it is a Lie algebra isomorphism between (V_1, μ_1) and (V_2, μ_2) such that $\delta_2(f(X)) = (f \otimes f)\delta_1(X)$.

Bialgebra isomorphisms define an action of $GL(V)$ on the algebraic variety $\mathcal{D}(V)$ given by

$$f \cdot (\alpha_1, \delta_1) = (\alpha_2, \delta_2)$$

and such that

$$\alpha_2(f(X), f(Y)) = f\alpha_1(X, Y), \qquad \delta_2(f(X)) = (f \otimes f)\delta_1(X).$$

The orbits of this action can be identified with isomorphism classes of Lie bialgebras on V. The space of orbits is non Hausdorff. In fact, the trivial $(0, 0)$ Lie bialgebra has isomorphism class lying in the closure of every orbit. Even more also in $\mathbb{P}(\mathcal{D})$ orbits are not necessarily closed. The already cited two–dimensional case provides an easy example of an orbit space equivalent to a 5–point space with non Hausdorff topology. More explicitly one can find the following equivalence classes: $P_0 = (0, 0, 0, 0)$, $P_1 = \{(a, b, 0, 0) \,|\, (a, b) \neq (0, 0)\}$, $P_2 = \{(0, 0, \alpha, \beta) \,|\, (\alpha, \beta) \neq (0, 0)\}$, $P_3 = \{(a, b, \alpha, \beta) \,|\; a\alpha + b\beta = 0\}$, $P_4 = \{(a, b, \alpha, \beta) \,|\; a\alpha + b\beta \neq 0\}$.

Let us note that for our definition of isomorphism the Lie bialgebras P_1 and P_2 are not isomorphic, although one can be obtained by the other simply by interchanging the role of $\mathfrak{g}$ and $\mathfrak{g}^*$. It is of course possible to give a weaker notion of Lie bialgebra isomorphism which takes into account this duality.

We finally remark that Lie bialgebras, at present, are classified up to isomorphisms over real or complex numbers only in dimension ≤ 3 ([10]).

3. Deformations

Definition 3.1. A formal deformation of a Lie bialgebra $D = (\mu, \delta)$ on V is a family of Lie bialgebra structures $D_t = (\mu_t, \delta_t)$ on V such that

$$\begin{aligned} \mu_t &= \mu + t\mu_1 + \ldots + t^n\mu_n + \ldots \\ \delta_t &= \delta + t\delta_1 + \ldots + t^n\delta_n + \ldots \end{aligned}$$

where for every $n \in \mathbb{N}$, $\mu_n \in \mathrm{Hom}(\wedge^2 V, V)$ and $\delta_n \in \mathrm{Hom}(V, \wedge^2 V)$.

Examples: Let μ_t be a formal deformation of a Lie product μ_0. Then $(\mu_t, 0)$ is a formal deformation of the Lie bialgebra $(\mu_0, 0)$. Let μ_t be a formal deformation of a Lie product μ_0 and let $\delta_0 : V \to \wedge^2 V$ satisfy the cocycle identity for every $t \in \mathbb{C}$. Then(μ_t, δ_0) is a formal deformation of (μ_0, δ_0). An explicit three–dimensional example of this form can be found in [2].

Given a family $\alpha_t = (\mu_t, \delta_t)$ on V such that $\mu_i \in \mathrm{Hom}(\wedge^2 V, V)$ and $\delta_i \in \mathrm{Hom}(V, \wedge^2 V)$ let us impose that it is a Lie bialgebra deformation. Then three conditions have to be satisfied: μ_t is a Lie bracket, δ_t is a Lie cobracket, and δ_t is a μ_t–1–cocycle with values in $\wedge^2 V$. These conditions have to be identically satisfied for coefficients of powers of t. In t^0 this simply amounts to require that (μ, δ) is a Lie bialgebra. At every degree in t one gets three equations which we will call *deformation equations* for Lie bialgebras. To write them explicitly let us fix the following notation: for every $X \in V$ the operation $X \cdot_\mu y$ on a 2-vector $y \in \wedge^2 V$ is defined by linearity from $X \cdot_\mu (Y_1 \wedge Y_2) = \mu(X, Y_1) \wedge Y_2 + Y_1 \wedge \mu(X, Y_2)$. The deformation equations in degree 1 are, then, given by

$$\sum\nolimits^{\circlearrowleft} \mu_1(\mu_0(X,Y),Z) + \mu_0(\mu_1(X,Y),Z) = 0 \tag{4}$$

$$\sum\nolimits^{\circlearrowleft} {}^t\delta_1({}^t\delta_0(x,y),z) + {}^t\delta_0({}^t\delta_1(x,y),z) = 0 \tag{5}$$

$$\begin{aligned} \delta_1(\mu_0(X,Y)) - X \cdot_{\mu_0} \delta_1(Y) + Y \cdot_{\mu_0} \delta_1(X) &= \\ = -\delta_0(\mu_1(X,Y)) + X \cdot_{\mu_1} \delta_0(Y) - Y \cdot_{\mu_1} \delta_0(X)& \end{aligned} \tag{6}$$

where $\sum^{\circlearrowleft}$ stands for the sum over cyclic permutations of arguments, for any $X, Y, Z \in V$, $x, y, z \in V^*$.

Definition 3.2. Two formal deformations (μ_t^1, δ_t^1) and (μ_t^2, δ_t^2) are said to be equivalent if there exists a family of vector space isomorphisms $f_n \in \mathrm{End}(V)$ such that if $f_t = Id + tf_1 + \ldots + t^n f_n + \ldots$ then $(\mu_t^2, \delta_t^2) = f_t \cdot (\mu_t^1, \delta_t^1)$. A trivial deformation is a deformation equivalent to the constant one.

Let us remark that for a fixed Lie bialgebra (μ, δ) and for any given $s \in \wedge^2 \mathfrak{g}$ one can ask whether $(\mu, \delta + ds)$ is again a Lie bialgebra. If it is then it is called a *twisting* of (μ, δ). In our context one can consider the family $(\mu, \delta + tds)$ and impose deformation equations on it. The first and last deformation equations are then trivially verified (respectively because the product is undeformed and a coboundary is always a cocycle). The second deformation equation gives the infinitesimal condition on s to define a twist. Lie bialgebras twisting were extensively used in [10] to classify real and complex 3–dimensional Lie bialgebras.

4. Cohomology and tangent space

Let $D_0 = (\mu_0, \delta_0)$ be a fixed Lie bialgebra. We would like to define a cohomology related to Lie bialgebra deformations. Let us consider

$$C^{p,q} = \mathrm{Hom}(\wedge^p V, \wedge^q V) \qquad \text{and} \qquad C^n = \bigoplus_{\substack{p+q=n+1 \\ p,q\neq 0}} C^{p,q}.$$

In particular $C^0 = 0$, $C^1 \simeq V \otimes V^*$ and $C^2 \simeq (\wedge^2 V \otimes V^*) \oplus (V \otimes \wedge^2 V^*)$. For any $\xi_n \in C^n$ we will denote with $\xi_n = \sum_{p+q} \xi_{p,q}$ its direct sum decomposition.

Let us consider the following coboundary operator:
for every $\xi_{p,q} \in \mathrm{Hom}(\wedge^p V, \wedge^q V)$, ∂_{μ_0} is the differential on p-cochains for the cohomology of $\mathfrak{g}$ with coefficients in $\wedge^q \mathfrak{g}$. Explicitly:

$$\partial_{\mu_0}(\xi_{p,q})(Y_1 \wedge \ldots \wedge Y_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1} Y_i \cdot \xi_{p,q}(Y_1 \wedge \ldots \hat{Y_i} \ldots \wedge Y_{p+1})$$
$$+ \sum_{j<k}(-1)^{j+k}\xi_{p,q}(\mu_0(Y_j, Y_k) \wedge Y_1 \wedge \ldots \hat{Y_j} \ldots \hat{Y_k} \ldots \wedge Y_{p+1})$$

where the action in the first sum is the adjoint action of $\mathfrak{g}$ on $\wedge^q \mathfrak{g}$ (and thus depends on μ_0). So $\partial = \partial_{\mu_0} : C^{p,q} \to C^{p+1,q}$. To define a second coboundary $\bar{\partial}_{\delta_0}$ remark that $\mathrm{Hom}(\wedge^p V, \wedge^q V)$ can be identified to $\mathrm{Hom}(\wedge^q V^*, \wedge^p V^*)$ as follows:

$$\langle \tilde{\xi}_{p,q}(f_1 \wedge \ldots \wedge f_q), X_1 \wedge \ldots \wedge X_p \rangle = \langle f_1 \wedge \ldots \wedge f_q, \xi_{p,q}(X_1 \wedge \ldots \wedge X_p)\rangle \quad (7)$$

where $\langle , \rangle$ is the usual duality pairing between V and V^*. Let $\bar{\partial}_{\delta_0}$ be the differential on q–cochains for the cohomology of $\mathfrak{g}^*$ with coefficients in $\wedge^p \mathfrak{g}^*$. Explicitly:

$$\begin{aligned}\bar{\partial}_{\delta_0}(\tilde{\xi}_{p,q})(f_1 \wedge \ldots \wedge f_{q+1}) &= \sum_{i=1}^{q+1}(-1)^{i+1} f_i \cdot \tilde{\xi}_{p,q}(f_1 \wedge \ldots \hat{f_i} \ldots \wedge f_{q+1}) + \\ &+ \sum_{j<k}(-1)^{j+k}\tilde{\xi}_{p,q}({}^t\delta_0(f_j, f_k) \wedge f_1 \ldots \hat{f_j} \ldots \hat{f_k} \ldots \wedge f_{q+1})\end{aligned}$$

where the action in the first sum is the adjoint action of $\mathfrak{g}^*$ on $\wedge^p \mathfrak{g}^*$ (and thus depends on δ_0). So $\bar{\partial} = \bar{\partial}_{\delta_0} : C^{p,q} \to C^{p,q+1}$. Let us observe that

$$\begin{aligned}\partial_{\mu_0} &: \mathrm{Hom}(\wedge^p V, \wedge^q V) \to \mathrm{Hom}(\wedge^{p+1} V, \wedge^q V), \text{ and} \\ \bar{\partial}_{\delta_0} &: \mathrm{Hom}(\wedge^p V, \wedge^q V) \to \mathrm{Hom}(\wedge^p V, \wedge^{q+1} V).\end{aligned}$$

Let us define, lastly,

$$d\xi = \sum_{p+q=n+1}(\partial_{\mu_0}\xi_{p,q} + (-1)^n \bar{\partial}_{\delta_0}\xi_{p,q}).$$

Proposition 4.1. *The map d is a coboundary map for the cochain complex C^n.*

The proof of this statement, which can be obtained with a direct, although involved computation, is postponed to the appendix. We will denote the cohomology of this complex with $H^*_{(\mu_0,\delta_0)} = H^*_D$. Let us remark that being $C^0 = 0$ we have $H^0_D = 0$, $H^1_D = Z^1_D = \{f \in \mathrm{End}(V) \big| \, \partial f = \bar{\partial}^t f = 0\}$.

Proposition 4.2. *Let $D_t = (\mu_t, \delta_t)$ be a Lie bialgebra deformation of $D_0 = (\mu_0, \delta_0)$. Then its derivative in 0 defines a two cocycle $Z^2_{D_0}$. If the deformation is trivial then the cocycle is a coboundary.*

Proof. Let (μ_t, δ_t) be a Lie bialgebra deformation of (μ_0, δ_0). Then the deformation equation of degree one (4) says that μ_1 should be a μ_0 Lie 2–cocycle, i.e. $\partial_{\mu_0}\mu_1 = 0$, the deformation equation (5) that ${}^t\delta_1$ should be a ${}^t\delta_0$ Lie 2–cocycle, i.e. $\bar{\partial}_{\delta_0}\delta_1 = 0$. What remains to be proven is that the equation (6), which is the linear part of the compatibility condition (1), is equivalent to $\partial_{\mu_0}\delta_1 = -\bar{\partial}_{\delta_0}\mu_1$. This last statement follows easily from the following lemma.

Lemma 4.3. *The differential $\bar\partial^1_{\delta_0}(\mu)$ coincides with the map*

$$(X,Y)\mapsto X\cdot_\mu \delta_0(Y) - Y\cdot_\mu \delta_0(X) - \delta_0(\mu(X,Y))\,.$$

Proof The differential is dual to the map

$$(\varphi,\psi)\mapsto \varphi\cdot_{{}^t\delta_0}{}^t\mu(\psi) - \psi\cdot_{{}^t\delta_0}{}^t\mu(\varphi) - {}^t\mu({}^t\delta_0(\varphi,\psi))\,.$$

So what we have to prove is:

$$\begin{aligned}&\langle X\wedge Y, \varphi\cdot_{{}^t\delta_0}{}^t\mu(\psi) - \psi\cdot_{{}^t\delta_0}{}^t\mu(\varphi) - {}^t\mu({}^t\delta_0(\varphi,\psi))\rangle\\ &= \langle X\cdot_\mu \delta_0(Y) - Y\cdot_\mu \delta_0(X) - \delta_0(\mu(X,Y)), \varphi\wedge\psi\rangle\end{aligned}$$

It is easy to see that $\langle X\wedge Y, {}^t\mu({}^t\delta_0(\varphi,\psi))\rangle = \langle \delta_0(\mu(X,Y)), \varphi\wedge\psi\rangle$. In order to compute the other terms, we observe a general formula:

$$\langle X\wedge Y, \varphi\cdot_{{}^t\delta_0}\Psi\rangle = \langle \delta_0(X)\wedge Y - X\wedge\delta_0(Y), \varphi\wedge\Psi\rangle - \langle X\wedge Y, {}^t\delta_0(\Psi)\wedge\varphi\rangle$$

for an arbitrary 2-form Ψ. In a similar way one has the following:

$$\langle X\cdot_\mu y, \varphi\wedge\psi\rangle = \langle X\wedge y, {}^t\mu(\varphi)\wedge\psi - \varphi\wedge{}^t\mu(\psi)\rangle + \langle X\wedge\mu(y), \varphi\wedge\psi\rangle$$

for arbitrary 2-vector y. So the equation above becomes:

$$\begin{aligned}&\langle \delta_0(X)\wedge Y - X\wedge\delta_0(Y), \varphi\wedge{}^t\mu(\psi)\rangle\\ &-\langle X\wedge Y, {}^t\delta_0({}^t\mu(\psi))\wedge\varphi\rangle\\ &-\langle \delta_0(X)\wedge Y - X\wedge\delta_0(Y), \psi\wedge{}^t\mu(\varphi)\rangle + \langle X\wedge Y, {}^t\delta_0({}^t\mu(\varphi))\wedge\psi\rangle\\ &= \langle X\wedge\delta_0(Y), {}^t\mu(\varphi)\wedge\psi - \varphi\wedge{}^t\mu(\psi)\rangle + \langle X\wedge\mu(\delta_0(Y)), \varphi\wedge\psi\rangle\\ &-\langle Y\wedge\delta_0(X), {}^t\mu(\varphi)\wedge\psi - \varphi\wedge{}^t\mu(\psi)\rangle - \langle Y\wedge\mu(\delta_0(X)), \varphi\wedge\psi\rangle\end{aligned}$$

and then:

$$\begin{aligned}&\langle \delta_0(X)\wedge Y - X\wedge\delta_0(Y), \varphi\wedge{}^t\mu(\psi) - \psi\wedge{}^t\mu(\varphi)\rangle\\ &-\langle X\wedge Y, {}^t\delta_0({}^t\mu(\psi))\wedge\varphi - {}^t\delta_0({}^t\mu(\varphi))\wedge\psi\rangle\\ &= \langle X\wedge\delta_0(Y) - Y\wedge\delta_0(X), {}^t\mu(\varphi)\wedge\psi - \varphi\wedge{}^t\mu(\psi)\rangle\\ &+\langle X\wedge\mu(\delta_0(Y)) - Y\wedge\mu(\delta_0(X)), \varphi\wedge\psi\rangle\,.\end{aligned}$$

It is clear that the left hand halves of both terms cancel, so the equation becomes:

$$\begin{aligned}&-\langle X\wedge Y, {}^t\delta_0({}^t\mu(\psi))\wedge\varphi - {}^t\delta_0({}^t\mu(\varphi))\wedge\psi\rangle\\ &= \langle X\wedge\mu(\delta_0(Y)) - Y\wedge\mu(\delta_0(X)), \varphi\wedge\psi\rangle\,.\end{aligned}$$

This last identity can be easily verified from definitions. ■

This completes the first part of the proof. Let us start now from a trivial Lie bialgebra deformation $\alpha_t = \alpha + t\alpha_1 + \mathcal{O}(t^2)$. Then, by definition, there exists f_t such that $f_t\cdot\alpha_t = \alpha_0$. But this means exactly that $\alpha_t = (\mu_t,\delta_t)$ is such that $f_t\cdot\mu_t = \mu_0$ and $f_t\cdot{}^t\delta_t = {}^t\delta_0$ are trivial Lie algebra deformations. Therefore $\mu_1 = \partial_{\mu_0}f_1$ and $\delta_1 = \bar\partial_{\delta_0}f_1$. But this implies that $\alpha_1 = df_1$. ■

Corollary 4.4. *The space of two–cocycles $Z^2_{(\mu_0,\delta_0)}$ coincides with the tangent space of $\mathcal{D}(V)$ in the point (μ_0,δ_0).*

Proof. From the proof of the preceding proposition it is evident that the Lie bialgebra 2–cocycle condition coincide with deformation equations of degree 1. On the other deformation equations in degree 1 are the same as the linear part of the algebraic equations of the variety, which define its tangent space(as a scheme). ■

As an example consider the case of $(\mu,0)$–structures in $\mathcal{D}(V)$, where an easy computation shows that $Z^2_{(\mu,0)} = Z^2(\mathfrak{g},\mathfrak{g})\oplus Z^1(\mathfrak{g},\wedge^2\mathfrak{g})$. The first term describes Lie algebra deformations of μ, while the second term describes Lie cobrackets on the dual. Of course the same result holds true for $(0,\delta)$.

Definition 4.5. A Lie bialgebra is said to be formally rigid if all its deformations are locally trivial.

Proposition 4.6. *If $H^2_{(\mu_0,\delta_0)}=0$ the Lie bialgebra (μ_0,δ_0) is formally rigid.*

Proof Let us consider a deformation with coboundary tangent vector $\alpha_t=\alpha+tdf+\mathcal{O}(t^2)$. Let $f_t=1-tf$. Then one can prove that μ_t is equivalent, via f_t, to μ'_t, such that $\mu'_1=0$, and at the same time δ_t is equivalent to δ'_t, with $\delta'_1=0$. In fact, the equality $\mu'_t=f_t\cdot\mu_t$ (resp. $\delta'_t=f_t\cdot\delta_t$) at order 1 in t says exactly that $\mu'_1=0\Leftrightarrow\mu_1=\partial_{\mu_0}f$ (resp. that $\delta'_1=0\Leftrightarrow\delta_1=\bar\partial_{\delta_0}f$). This procedure can be iterated to build at step r a Lie bialgebra endomorphism f_r such that $\alpha^r_t=\alpha^{r-1}_t\cdot(1-t^rf_r)$ and $\alpha^r_1=\ldots=\alpha^r_r=0$. This implies that $\mu^1_r=\ldots=\mu^r_r=0$ and $\delta^1_r=\ldots=\delta^r_r=0$. Then one can formally construct $f=\Pi_r(1-t^rf_r)$ and verify that $f\cdot\alpha_t$ is a constant deformation. ■

Examples

1. Let (μ_0,δ_0) be a generic complex 2–dimensional Lie bialgebra. Non empty cochain spaces are $C^1\simeq V\otimes V^*$, $C^2\simeq V\oplus V^*$, $C^3=\mathbb{C}$, $C^p=0$, $\forall p\geq 4$. The 3–cocycle condition is trivial for dimension reasons and the 2-cocycle condition turns out to be identically verified as well. The trivial Lie bialgebra $(\mu_0,\delta_0)=(0,0)$ has trivial coboundaries and therefore $H^1=C^1=V\otimes V^*$, $H^2=C^2=V\oplus V^*$, $H^3=\mathbb{C}$. Let us consider the bialgebra $(\mu_0,\delta_0)=(\mathbf{1},0)$, where $\mathbf{1}$ denotes the non trivial 2–dimensional Lie algebra $[X,Y]=X$. Then the horizontal coboundary is trivial. An explicit computation shows that $\dim B^2(\mathbf{1},0)=2$ and therefore $H^2(\mathbf{1},0)\simeq\mathbb{C}^2$. The Lie bialgebra the bialgebra $(\mu_0,\delta_0)=(0,\mathbf{1})$ has, obviously, the same cohomology. For the two remaining cases, again by explicit computations, one can prove that $H^2(\mathbf{1},\mathbf{1})\simeq H^2(\mathbf{1},-\mathbf{1})\simeq\mathbb{C}$.

2. Similar computations, although slightly more involved, can be performed in the three–dimensional case as well. As an example let us first consider the real cohomology for the real Lie bialgebras on $\mathfrak{sl}(2;\mathbb{R})$, as listed in [1]. Once a basis is fixed, let's say the usual Cartan–Weyl basis $\{H,E,F\}$, the cobracket can be given as

$$\begin{aligned}\delta(H)&=-2\alpha H\wedge E+2\beta H\wedge F\\ \delta(E)&=\gamma H\wedge E+2\beta E\wedge F\\ \delta(F)&=\gamma H\wedge F+2\alpha E\wedge F\,.\end{aligned}$$

   In this basis 2–cochains can be represented as a pair of 3×3 matrices with real entries. The 2–cocycle condition can then be computed and results in a system of linear homogeneous equations in matrix coefficients. It then turns out that $\dim Z^2(\alpha,\beta,\gamma)=7$, for every α,β,γ (as in the cited reference) and that in all cases $H^2=0$, except for $(\alpha,\beta,\gamma)=(0,0,0)$ where $\dim H^2=1$.

3. Analogously, in the *standard* Heisenberg case $H_{1,\lambda}$ ([5]), where Lie bialgebras are classified by a scalar parameter λ such that in the usual $[X,Y]=H$ basis the cobracket is

$$\delta(X)=X\wedge Y-\lambda H\wedge Y,\qquad \delta(Y)=0,\qquad \delta(H)=H\wedge Y \tag{8}$$

similar computations allow to show that $\dim Z^2(\lambda) = 10$, and $H^2(\lambda) = \mathbb{C}^3$ if $\lambda \neq 0$ and $H^2(0) = \mathbb{C}^4$.

Note: In [3, 4] a different bialgebra cohomology was introduced to classify Lie bialgebra extensions. It is an analogue of Lie bialgebra cohomology with coefficients on the base field and it seems plausible that a suitable generalization of Lie bialgebra cohomology with coefficients in a module should include both our definition and Benayed's definition as special cases.

5. Appendix

We simply want to prove that $C^{p,q}$ is a double complex and d is the boundary on the total complex. We then have to verify that the two differentials ∂_{μ_0} and $\bar\partial_{\delta_0}$ commute. This will be done by explicit computation. First we will need the following lemma, which can be easily verified.

Lemma 5.1. *Let $X_i \in V$ and $\phi, \psi_i \in V^*$, $i = 0, \dots, p$. Let $\phi \cdot (\psi_0 \wedge \dots \wedge \psi_p)$ denote the wedge adjoint action of V^* on $\wedge^p V^*$ and $\phi \cdot (X_0 \wedge \ldots \wedge X_p)$ denote the wedge coadjoint action. Then:*

$$\langle X_0 \wedge \dots \wedge X_p, \phi \cdot (\psi_0 \wedge \dots \wedge \psi_p) \rangle = -\langle \phi \cdot (X_0 \wedge \dots \wedge X_p), \psi_0 \wedge \dots \wedge \psi_p \rangle .$$

Let now $f : \wedge^p V \to \wedge^q V$. Then we will consider the following maps:

$$\begin{array}{ll} {}^t f : \wedge^q V^* \to \wedge^p V^* & \partial f : \wedge^{p+1} V \to \wedge^q V \\ \bar\partial({}^t f) : \wedge^{q+1} V^* \to \wedge^p V^* & {}^t\partial f : \wedge^q V^* \to \wedge^{p+1} V^* \\ {}^t\bar\partial({}^t f) : \wedge^p V \to \wedge^{q+1} V & \bar\partial({}^t\partial f) : \wedge^{q+1} V^* \to \wedge^{p+1} V^* \\ \partial({}^t\bar\partial({}^t f)) : \wedge^{p+1} V \to \wedge^{q+1} V & {}^t\bar\partial({}^t\partial f) : \wedge^{p+1} V \to \wedge^{q+1} V \end{array}$$

We want to prove that ${}^t\bar\partial({}^t\partial f) - \partial({}^t\bar\partial({}^t f)) = 0$. Let $X_0, \dots, X_p \in V$ and $\psi_0, \dots, \psi_q \in V^*$ and prove that:

$$\langle {}^t\bar\partial({}^t\partial f) - \partial({}^t\bar\partial({}^t f))\, (X_0 \wedge \dots \wedge X_p), \psi_0 \wedge \dots \wedge \psi_q \rangle = 0 . \tag{9}$$

We introduce some simplified notation suitable for long computations. If $X = X_0 \wedge \ldots \wedge X_p$ then we define

$$X^i = (-1)^i X_0 \wedge \ldots \widehat{X_i} \ldots \wedge X_p \quad \text{and} \quad X^{ij} = (-1)^{i+j} X_0 \wedge \ldots \widehat{X_i\, X_j} \ldots \wedge X_p$$

so that the definition of a Lie coboundary is written as

$$\partial f(X) = \sum_i X_i \cdot f(X^i) + \sum_{i<j} f([X_i, X_j] \wedge X^{ij})$$

and similarly a dual notation may be introduced in order to write a shorter formula for a dual coboundary. (As a rule of notation in what follows we will omit the $\wedge$ operator and use the standard *hat* notation for omitted terms.) Computing explicitly the l.h.s. of (9) we obtain

$$\begin{aligned} &\sum_i -\langle (\partial f)(\psi_i \cdot X), \psi^i \rangle \;+\; \sum_{i<j} \langle \partial f(X), [\psi_i, \psi_j] \wedge \psi^{ij} \rangle \\ &+ \sum_h \langle X^h, \bar\partial({}^t f)(X_h \cdot \psi) \rangle \;-\; \sum_{h<k} \langle [X_h, X_k] \wedge X^{hk}, \bar\partial({}^t f)(\psi) \rangle \end{aligned} \tag{10}$$

Now we want to render $\partial f(\psi_i \cdot X)$ and $\bar{\partial}({}^t f)(X_h \cdot \psi)$ explicit. The first element equals

$$\partial f(\psi_i \cdot (X_0 \dots X_p)) \;=\; \sum_k \partial f(X_0 \dots (\psi_i \cdot X_k) \dots X_p)$$

where

$$\begin{aligned}
&\partial f(X_0 \dots (\psi_i \cdot X_k) \dots X_p) = \\
&+\sum_{h\neq k}(-1)^h X_h \cdot f(X_0 \dots \widehat{X_h} \dots \psi_i \cdot X_k \dots X_p) + (-1)^k(\psi_i \cdot X_k)\cdot f(X_0 \dots \widehat{X_k} \dots X_p)\\
&\quad+\sum_{s<t,\neq k}(-1)^{s+t} f([X_s, X_t]X_0 \dots \widehat{X_s\, X_t} \dots \psi_i \cdot X_k \dots X_p)\\
&\quad+\sum_{s<k}(-1)^{k+s} f([X_s, \psi_i \cdot X_k]X_0 \dots \widehat{X_s\, X_k} \dots X_p)\\
&\quad+\sum_{s>k}(-1)^{k+s} f([\psi_i \cdot X_k, X_s]X_0 \dots \widehat{X_k\, X_s} \dots X_p)
\end{aligned}$$

Here we introduce some additional notation; we define:

$$\psi \cdot_k X = X_0 \wedge \dots (\psi \cdot X_k) \dots \wedge X_p$$

and we also define $X^{ij} = -X^{ji}$ for $i > j$, so the previous formula is written as:

$$\begin{aligned}
\partial f(\psi_i \cdot_k X) \;&=\; \sum_{h\neq k} X_h \cdot f((\psi_i \cdot_k X)^h) + (\psi_i \cdot X_k)\cdot f(X^k)\\
&\quad+\sum_{s<t,\neq k} f([X_s, X_t] \wedge (\psi_i \cdot_k X)^{st}) + \sum_{s\neq k} f([\psi_i \cdot X_k, X_s] \wedge X^{ks})
\end{aligned}$$

Similarly, the dual formula is obtained:

$$\begin{aligned}
\bar{\partial}({}^t f)(X_h \cdot_i \psi) \;&=\; \sum_{j\neq i} \psi_j \cdot {}^t f((X_h \cdot_i \psi)^j) + (X_h \cdot \psi_i)\cdot {}^t f(\psi^i)\\
&\quad+\sum_{s<ts,t\neq i} {}^t f([\psi_s, \psi_t] \wedge (X_h \cdot_i \psi)^{st}) + \sum_{t\neq i} {}^t f([X_h \cdot \psi_i, \psi_t] \wedge \psi^{it})
\end{aligned}$$

Then (10) can be rewritten as:

$$\begin{gathered}
-\sum_i \sum_k \Big\{ \sum_{h\neq k} \langle X_h \cdot f((\psi_i \cdot_k X)^h), \psi^i\rangle + \langle (\psi_i \cdot X_k)\cdot f(X^k), \psi^i\rangle \\
+\sum_{s<ts,t\neq k} \langle f([X_s, X_t] \wedge (\psi_i \cdot_k X)^{st}), \psi^i\rangle + \sum_{s\neq k} \langle f([\psi_i \cdot X_k, X_s] \wedge X^{ks}), \psi^i\rangle \Big\}\\
+\sum_{i<j} \sum_h \langle X_h \cdot f(X^h), [\psi_i, \psi_j] \wedge \psi^{ij}\rangle\\
+\sum_{i<j} \sum_{h<k} \langle f([X_h, X_k] \wedge X^{hk}), [\psi_i, \psi_j] \wedge \psi^{ij}\rangle\\
+\sum_h \sum_i \Big\{ \sum_{j\neq i} \langle X^h, \psi_j \cdot {}^t f((X_h \cdot_i \psi)^j)\rangle + \langle X^h, (X_h \cdot \psi_i)\cdot {}^t f(\psi^i)\rangle\\
+\sum_{s<ts,t\neq i} \langle X^h, {}^t f([\psi_s, \psi_t] \wedge (X_h \cdot_i \psi)^{st})\rangle + \sum_{t\neq i} \langle X^h, {}^t f([X_h \cdot \psi_i, \psi_t] \wedge \psi^{it})\rangle \Big\}\\
-\sum_{h<k} \sum_i \langle [X_h, X_k] \wedge X^{hk}, \psi_i \cdot {}^t f(\psi^i)\rangle\\
-\sum_{h<k} \sum_{i<j} \langle [X_h, X_k] \wedge X^{hk}, {}^t f([\psi_i, \psi_j] \wedge \psi^{ij})\rangle.
\end{gathered}$$

Note that the summation on the right side of line 3 and the summation on the right side of line 6 cancel each other. So the total summation is equal to

$$\begin{aligned}
&-\sum_i \sum_k \Big\{\sum_{h\neq k}\langle X_h \cdot f((\psi_i \cdot_k X)^h), \psi^i\rangle + \sum_{s<t,s,t\neq k}\langle f([X_s, X_t] \wedge (\psi_i \cdot_k X)^{st}), \psi^i\rangle \\
&\qquad + \langle(\psi_i \cdot X_k)\cdot f(X^k), \psi^i\rangle + \sum_{s\neq k}\langle f([\psi_i \cdot X_k, X_s] \wedge X^{ks}), \psi^i\rangle\Big\} \\
&\qquad\qquad + \sum_{i<j}\sum_h \langle X_h \cdot f(X^h), [\psi_i, \psi_j] \wedge \psi^{ij}\rangle \\
&+\sum_h \sum_i \Big\{-\sum_{j\neq i}\langle f(\psi_j \cdot X^h), (X_h \cdot_i \psi)^j\rangle + \sum_{s<t,s,t\neq i}\langle f(X^h), [\psi_s, \psi_t] \wedge (X_h \cdot_i \psi)^{st}\rangle \\
&\qquad - \langle f((X_h \cdot \psi_i)\cdot X^h), \psi^i\rangle + \sum_{t\neq i}\langle f(X^h), [X_h \cdot \psi_i, \psi_t] \wedge \psi^{it}\rangle\Big\} \\
&\qquad\qquad + \sum_{h<k}\sum_i \langle f(\psi_i \cdot ([X_h, X_k] \wedge X^{hk})), \psi^i\rangle.
\end{aligned}$$

Lemma 5.2. *The sum on the left side of line* 1 *and the sum on the left side of line* 4 *cancel.*

Proof $\sum_i \sum_k \sum_{h\neq k} \langle X_h \cdot f((\psi_i \cdot_k X)^h), \psi^i\rangle + \sum_h \sum_i \sum_{j\neq i} \langle f(\psi_j \cdot X^h), (X_h \cdot_i \psi)^j\rangle$

$$\begin{aligned}
&= -\sum_i \sum_k \sum_{h\neq k} \sum_{j\neq i}\langle f((\psi_i \cdot_k X)^h), (X_h \cdot_j \psi)^i\rangle \\
&\quad + \sum_h \sum_i \sum_{j\neq i} \sum_{h\neq k}\langle f((\psi_j \cdot_k X)^h), (X_h \cdot_i \psi)^j\rangle
\end{aligned}$$

and simply exchanging indices i and j in the second sum one has the thesis. ■

Lemma 5.3. *The right side summation on line* 1*, plus the sum on line* 3*, plus the right side summation on line* 4*, plus the sum on line* 6*, can be rewritten as:*
$-\sum_h \sum_{i<j}\langle f(X^h), (X_h \cdot [\psi_i, \psi_j]) \wedge \psi^{ij}\rangle \;+\; \sum_i \sum_{h<k}\langle f((\psi_i \cdot [X_h, X_k]) \wedge X^{hk}), \psi^i\rangle.$

Proof $-\sum_i \sum_k \sum_{s<t,s,t\neq k}\langle f([X_s, X_t] \wedge (\psi_i \cdot_k X)^{st}), \psi^i\rangle$

$$\begin{aligned}
&\quad + \sum_{i<j}\sum_h \langle X_h \cdot f(X^h), [\psi_i, \psi_j] \wedge \psi^{ij}\rangle \\
&\quad + \sum_h \sum_i \sum_{s<t,s,t\neq i}\langle f(X^h), [\psi_s \psi_t] \wedge (X_h \cdot_i \psi)^{st}\rangle \\
&\quad + \sum_{h<k}\sum_i \langle f(\psi_i \cdot ([X_h, X_k] \wedge X^{hk})), \psi^i\rangle \\
&= -\sum_i \sum_k \sum_{s<t,s,t\neq k}\langle f([X_s, X_t] \wedge (\psi_i \cdot_k X)^{st}), \psi^i\rangle \\
&\quad + \sum_i \sum_{h<k}\langle f((\psi_i \cdot [X_h, X_k]) \wedge X^{hk}), \psi^i\rangle \\
&\quad + \sum_i \sum_{h<k} \sum_{t\neq h,k}\langle f([X_h, X_k] \wedge (\psi_i \cdot_t X)^{hk}), \psi^i\rangle \\
&\quad - \sum_{i<j}\sum_h \langle f(X^h), (X_h \cdot [\psi_i, \psi_j]) \wedge \psi^{ij}\rangle \\
&\quad - \sum_{i<j}\sum_h \sum_{s\neq i,j}\langle f(X^h), [\psi_i, \psi_j] \wedge (X_h \cdot_s \psi)^{ij}\rangle \\
&\quad + \sum_h \sum_i \sum_{s<t,s,t\neq i}\langle f(X^h), [\psi_s, \psi_t] \wedge (X_h \cdot_i \psi)^{st}\rangle
\end{aligned}$$

Remarking that the first sum cancels with the third and the fifth with the sixth one gets the claimed result. ■

Applying the lemma, then, we obtain that the total summation is

$$\begin{aligned}
&= -\sum_h \sum_{i<j}\langle f(X^h), (X_h \cdot [\psi_i, \psi_j]) \wedge \psi^{ij}\rangle + \sum_i \sum_{h<k}\langle f((\psi_i \cdot [X_h, X_k]) \wedge X^{hk}), \psi^i\rangle \\
&\quad - \sum_i \sum_k \langle(\psi_i \cdot X_k)\cdot f(X^k), \psi^i\rangle - \sum_i \sum_k \sum_{s\neq k}\langle f([\psi_i \cdot X_k, X_s] \wedge X^{ks}), \psi^i\rangle
\end{aligned}$$

$$
\begin{aligned}
&-\sum_h\sum_i\langle f((X_h\cdot\psi_i)\cdot X^h),\psi^i\rangle+\sum_h\sum_i\sum_{t\neq i}\langle f(X^h),[X_h\cdot\psi_i,\psi_t]\wedge\psi^{it}\rangle\\
=&-\sum_h\sum_{i<j}\langle f(X^h),(X_h\cdot[\psi_i,\psi_j])\wedge\psi^{ij}\rangle+\sum_h\sum_i\sum_{j\neq i}\langle f(X^h),([X_h\cdot\psi_i,\psi_j])\wedge\psi^{ij}\rangle\\
&+\sum_i\sum_h\langle f(X^h),(\psi_i\cdot X_h)\cdot\psi^i\rangle-\sum_i\sum_k\sum_{h\neq k}\langle f([\psi_i\cdot X_k,X_h]\wedge X^{kh}),\psi^i\rangle\\
&-\sum_h\sum_i\langle f((X_h\cdot\psi_i)\cdot X^h),\psi^i\rangle+\sum_i\sum_{h<k}\langle f((\psi_i\cdot[X_h,X_k])\wedge X^{hk}),\psi^i\rangle\\
=&-\sum_h\sum_{i<j}\langle f(X^h),T_{hij}\wedge\psi^{ij}\rangle+\sum_i\sum_{h<k}\langle f(S_{ihk}\wedge X^{hk}),\psi^i\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
T_{hij}&=X_h\cdot[\psi_i,\psi_j]+(\psi_i\cdot X_h)\cdot\psi_j-(\psi_j\cdot X_h)\cdot\psi_i-[X_h\cdot\psi_i,\psi_j]-[\psi_i,X_h\cdot\psi_j],\\
S_{ihk}&=\psi_i\cdot[X_h,X_k]+(X_h\cdot\psi_i)\cdot X_k-(X_k\cdot\psi_i)\cdot X_h-[\psi_i\cdot X_h,X_k]-[X_h,\psi_i\cdot X_k].
\end{aligned}
$$

Now it is enough to consider that the cocycle identity implies $T_{hij}=0$, see (2), and that the dual cocycle identity implies $S_{ihk}=0$, to deduce that both sums are zero.

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Nicola Ciccoli
Dipartimento di Matematica
Università di Perugia
Via Vanvitelli 1
06123 Perugia Italy
ciccoli@dipmat.unipg.it

Lucio Guerra
Dipartimento di Matematica
Università di Perugia
Via Vanvitelli 1
06123 Perugia Italy
guerra@dipmat.unipg.it

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