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New a-T-menable HNN-extensions

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Abstract. The Baumslag-Solitar groups are a-T-menable. This is proved by embedding them into topological groups and studying representation theoretic properties of the latter.
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We are mainly interested in Baumslag-Solitar groups which are discrete. However, topological groups (Lie groups and the automorphism group of a tree) arise in the proof.

Definition 1 (M. Gromov). A locally compact, second countable group G is a-T-menable iff there exists a metrically proper affine isometric action of G on some Hilbert space.

a-T-menability is often referred to as the *Haagerup approximation property.* It is equivalent to the existence in $C_0(G)$ of an approximate unit of positive definite functions on G. For ample discussion one may consult [2].

Let $G \subset \mathfrak{N}$ be a closed subgroup of a locally compact compactly generated topological group $\mathfrak{N}$. Let $i_k\colon H \to G$ $(k = 1, 2)$ be two inclusions onto finite index open subgroups, which are conjugated by an automorphism ϕ of $\mathfrak{N}$.

The main case of interest is when G is discrete and $\mathfrak{N}$ is a Lie group.

Definition 2. A $\mathfrak{N}$-BS group is the group derived from such (G, H, i_1, i_2) by the (topological) HNN construction. In other words, if G is given by the presentation $\langle S|R\rangle$, Γ has the presentation $\langle S, t|R, ti_1(g)t^{-1} = i_2(g)\ \forall g \in H\rangle$.

Through this paper let Γ be a $\mathfrak{N}$-BS group as above.

Since we are working with topological presentations, recall that the topology in the HNN extension Γ is given by the basis $\mathcal{B} = \{\gamma U\gamma'\colon U$ open in G, $\gamma, \gamma' \in \Gamma\}$. It is clear, that Γ is a topological group with respect to $\mathcal{B}$.

To see that $\mathcal{B}$ is in fact a basis consider U open in G. We can decompose U with respect to $i_1(H)$ as follows $U = \bigcup i_1(U_n)g_n$, where g_n are representatives of cosets of $i_1(H)$. Then $tU = \bigcup i_2(U_n)tg_n$. Therefore any set of $\mathcal{B}$ can be

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written uniquely in form $\bigcup U_\gamma \gamma$ where U_γ runs over open subsets of G and γ runs over the chosen set of representatives of $G \backslash \Gamma$. Thus $\mathcal{B}$ is closed under the intersections and in particular $\mathcal{B}$ is a basis. The same argument shows that the topology on G induced from Γ concides with the original one (i.e. $G \subset \Gamma$ is an open embedding).

Examples:

1. ($\mathfrak{N} = \mathbb{R}$) Baumslag-Solitar group with parameters p and q is given by a following presentation $BS^p_q = \langle x, t | x^p = tx^q t^{-1} \rangle = \mathrm{HNN}(\mathbb{Z}, \mathbb{Z}, p\cdot, q\cdot)$.
2. ($\mathfrak{N} = \mathbb{R}^n$) Torsion free, finitely generated, abelian-by-cyclic groups are exactly ascending ($i_1 = id$) HNN extensions of $\mathbb{Z}^n$ with i_2 given by a n by n matrix with nonzero determinant [1].
3. $\mathfrak{N}$ a homogeneous nilpotent group (i.e. one admitting a dilating automorphism ϕ) with a discrete subgroup $G \subset \mathfrak{N}$ such that $\phi(G) \subset G$.

An obvious adaptation of the Bass-Serre theory [6] to the topological context shows that for a topological HNN extension Γ of a group G there is a tree T with an edge-transitive Γ-action such that the vertex stabilizers are conjugated to G and edge stabilizers are conjugated to H.

Since $i_k(H)$ are of finite index in G, T is locally finite. So, the simplicial automorphism group $\mathrm{Aut}(T)$ carries the natural (compact-open) topology such that the group is locally compact. More precisely, the basis of neighborhoods of the identity is the family $\mathcal{U}_K := \{g | gv = v \ \forall v \in K\}$ where K runs over the family of the compact subsets $K \subset T$. Denote $j_T : \Gamma \to Aut(T)$ the homomorphism given by the action.

Define $\widetilde{\mathfrak{N}} = \mathbb{Z}\text{ffn}\mathfrak{N}$, to be the semidirect product given by the $\mathbb{Z}$-action on $\mathfrak{N}$ via ϕ. There is an obvious homomorphism $j_{\mathfrak{N}} : \Gamma \to \widetilde{\mathfrak{N}}$, which is the identity on G and sends t to the generator of $\mathbb{Z}$.

Theorem 1. *Let Γ be a $\mathfrak{N}$-BS group, and let j_T and $j_{\mathfrak{N}}$ be the homomorphisms defined above. Then the homomorphism $j = (j_T, j_{\mathfrak{N}}) : \Gamma \to Aut(T) \times \widetilde{\mathfrak{N}}$ is an embedding onto a closed subgroup which is a topological isomorphism onto its image.*

Proof. Observe that since $j_{\mathfrak{N}}$ restricted to G is an embedding onto a closed subgroup (this follows from the fact that G in closed subgroup in $\mathfrak{N}$) the same is true for j restricted to G. Let v be the vertex stabilized by G. Then $j_T(G) = \mathrm{Stab}_v \cap j_T(\Gamma)$.

Let $g \in \Gamma$ such that $j(g) = 1$. Since $j_T(g) = 1$, $gv = v$ i.e. $g \in G$. Since $j_{\mathfrak{N}}$ restricted to G is an embedding; this implies $g = 1$. Thus injectivity follows.

Let $\gamma \notin j(\Gamma)$. Since Γ acts transitively on T, we can multiply γ by some element of $j(\Gamma)$ and assume that $\gamma v = v$. Then $(\mathrm{Stab}_v - G) \times \mathfrak{N}$ is an open neighbourhood of γ omitting $j(\Gamma)$. So $j(\Gamma)$ is closed.

The topologies on G and $j(G)$ coincide. Since G and $j(G)$ are open in Γ and $j(\Gamma)$ respectively, the same is true for Γ and $j(\Gamma)$.

Corollary 1. *If $\mathfrak{N}$ is a-T-menable then so are $\mathfrak{N}$-BS groups.*

Proof. Since T is locally finite, the group $\mathrm{Aut}(T)$ is a-T-menable by a result of Haagerup [4].

A cyclic extension of a-T-menable group is a-T-menable by [5]. Therefore $\widetilde{\mathfrak{N}}$ is a-T-menable.

It is clear that a-T-menability is closed under direct products and under taking closed subgroups. Thus the corollary follows from Theorem 1. ∎

Remark 1. According to Chapter 4 of [2] every a-T-menable connected Lie group $\mathfrak{N}$ is locally isomorphic to a direct product of an amenable group and copies of $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$. A Lie group is amenable if it is compact-by-solvable. Thus the examples 1.-3. are a-T-menable.

Remark 2. Another proof of Corollary 1, relying on the study of a-T-menable of amalgams, is given in a forthcoming paper by the first author [3].

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