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Compactification of Parahermitian Symmetric Spaces and its Applications, II: Stratifications and Automorphism Groups

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Abstract. A simple parahermitian symmetric space is a symplectic symmetric space of a simple Lie group G with two invariant Lagrangian foliations. Such a symmetric space has a nice G-equivariant compactification. In this paper, we obtain the stratification of the compactification, whose strata are G-orbits. By using this, we determine the automorphism group of the double foliation for each simple parahermitian symmetric space.

Introduction

Let M be a smooth manifold. A pair $(F^{\pm}, \omega)$ is called a *parakähler structure* (or *bi-Lagrangian structure*) on M if ω is a symplectic form on M and $F^{\pm}$ are two Lagrangian foliations. A significant property of parakähler structures is that a coadjoint orbit of a semisimple Lie group is hyperbolic if and only if it admits an invariant parakähler structure ([4]). A symmetric space G/H of a Lie group G is called a *parahermitian symmetric space* (or *bi-Lagrangian symmetric space*)([10]) if G/H admits a G-invariant parakähler structure $(F^{\pm}, \omega)$. The simplest example of parahermitian symmetric spaces is the symmetric space $\mathrm{SL}(2,\mathbb{R})/\mathbb{R}^*$, realized as the one-sheeted hyperboloid $x^2+y^2-z^2=1$ in $\mathbb{R}^3 = \mathrm{Lie}\,\mathrm{SL}(2,\mathbb{R})$. The Lagrangian foliations $F^{\pm}$ are given by the two families of rulings of the hyperboloid. Parahermitian symmetric spaces of semisimple Lie groups were classified and characterized group-theoretically in [10,5]. A semisimple symmetric space G/H is parahermitian if and only if H is an open subgroup of the Levi subgroup of a parabolic subgroup with abelian nilradical. Semisimple parahermitian symmetric spaces G/H are in one-to-one correspondence (up to covering) with semisimple graded Lie algebras (shortly, GLAs) of the 1st kind $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, in such a way that $\mathfrak{g} = \mathrm{Lie}\,G$ and $\mathfrak{g}_0 = \mathrm{Lie}\,H$. For the explicit forms of simple parahermitian symmetric pairs, see the tables in 6.4 and 7.2.

Now let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA, and let $M = G/G_0$ be the parahermitian symmetric space corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$, where G is the largest possible open subgroup of the automorphism group of $\mathfrak{g}$ such that

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G/G_0 is realized as an $\mathrm{Ad}\,\mathfrak{g}$-orbit in $\mathfrak{g}$. The subgroups $U^\pm := G_0 \exp \mathfrak{g}_{\pm1}$ are the parabolic subgroups with $\mathrm{Lie}\,U^\pm = \mathfrak{g}_0 + \mathfrak{g}_{\pm1}$. The flag manifolds $M^\pm = G/U^\pm$ are symmetric R-spaces. Let r be the rank of $M^\pm$. Then there are exactly r numbers of $\mathfrak{sl}(2,\mathbb{R})$-triplets in $\mathfrak{g}$ which are pairwise commutative and whose direct sum is expressed as a graded subalgebra $\mathfrak{a}_{-1} + \mathfrak{a}_0 + \mathfrak{a}_1$ in $\mathfrak{g}$ (cf. [8]). One has the root system $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ of $\mathfrak{g}$ with respect to the abelian subspace $\mathfrak{a}_0$. $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ is of BC_r-type or C_r-type ([8,1]). We say that G/G_0 and the GLA $\mathfrak{g}$ are of BC_r-type or C_r-type, if $\Delta(\mathfrak{g}, \mathfrak{a}_0)$ is.

A fundamental problem of the geometry of parahermitian symmetric spaces $(M = G/H,\ F^\pm,\ \omega)$ is to determine the automorphism group $\mathrm{Aut}(M, F^\pm)$—the group consisting of diffeomorphisms of M leaving the double foliation $F^\pm$ invariant. The aim of this paper is to settle this problem for an arbitrary simple Lie group G. A partial answer was given by Tanaka [15] under the assumption that G is classical simple. Let us describe our procedure to determine the automorphism group. The first step is to obtain the G-orbit structure of $\widetilde{M} = M^- \times M^+$, which is the natural G-equivariant compactification of M (cf. [6] and Sections 2 and 3). The second step is to show that the G-orbit decomposition gives $\widetilde{M}$ a stratification whose strata are G-orbits. This is done in Sections 5 and 6.

The third step is concerned with BC_r-type. We now assume that M is of BC_r-type. In terms of the root system $\Delta(\mathfrak{g}, \mathfrak{a}_0)$, we construct a grading $\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(r)$ of the 2nd kind having the property that $\mathfrak{g}_{\pm1}(r)$ are expressed as the direct sum of two equi-dimensional abelian subspaces, $\mathfrak{g}^+_{\pm1}(r) + \mathfrak{g}^-_{\pm1}(r)$ (cf. (4.10),(4.14)). Such a grading is called a *pseudo-product grading* of $\mathfrak{g}$ (Tanaka [15]). Let Q_r be the isotropy subgroup of G at the base point of the lowest dimensional G-orbit M_0. The third step is to show that Q_r is the parabolic subgroup with $\mathrm{Lie}\,Q_r = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r)$ (Proposition 4.8) and that its Levi subgroup coincides with the automorphism group of the pseudo-product grading (cf. Lemma 7.4 and Remark 7.1). The flag manifold $M_0 = G/Q_r$ has the double foliation $F_0^\pm$ induced from the product structure of $\widetilde{M}$. We denote by $\mathrm{Aut}(M_0, F_0^\pm)$ the group of diffeomorphisms of M_0 leaving $F_0^\pm$ invariant. Then Tanaka's theory [15] of Cartan connections for pseudo-product manifolds, together with the third step guarantees the validity of the relation $\mathrm{Aut}(M_0, F_0^\pm) = G$.

For the case where M is of C_r-type, the minimal G-orbit M_0 coincides with $G/U^- = M^-$, and the double foliation $F_0^\pm$ becomes trivial. But, in turn, M^- has the generalized conformal structure $\mathcal{K}$, which is obtained from the cone defined as the union of singular G_0-orbits in $\mathfrak{g}_1$ ([2]). We determined in [2] the conformal automorphism group $\mathrm{Aut}(M^-, \mathcal{K})$ for each symmetric R-space M^-.

The fourth and the last step is to obtain the injective homomorphism of $\mathrm{Aut}(M, F^\pm)$ into $\mathrm{Aut}(M_0, F_0^\pm)$ or into $\mathrm{Aut}(M^-, \mathcal{K})$. For this purpose, the stratification of $\widetilde{M}$ is essential. Let $f \in \mathrm{Aut}(M, F^\pm)$. Then f extends to $\widetilde{M}$ as an automorphism $\widetilde{f}$ of the product structure of $\widetilde{M}$. It follows that $\widetilde{f}$ is an automorphism of the stratification of $\widetilde{M}$ (Corollary 6.14). In particular $\widetilde{f}$ leaves M_0 stable. It is shown that the assignment $f \mapsto \widetilde{f}|_{M_0}$ gives the injective homomorphism as desired. The main results are Theorems 8.1 and 8.4.

We want to supplement some details on the stratification of $\widetilde{M}$, since it is a rather independent topic. By a stratification of a real analytic manifold X, we mean a partition $X = \coprod_{k=0}^{s} A_k$ which satisfies the following conditions: (S1)

Each A_k is an analytic submanifold of X, (S2) the closure $\overline{A_k}$ of A_k is an analytic set of X and coincides with $A_{\leq k} := \coprod_{i=0}^{k} A_i$ $(0 \leq k \leq s)$, and (S3) the singular locus $\mathrm{Sing}(\overline{A_k})$ is given by $A_{\leq k-1}$ $(1 \leq k \leq s-1)$. Let $\widetilde{M} = \coprod_{k=0}^{r} M_k$ be the G-orbit decomposition (G acts on $\widetilde{M}$ diagonally), where $\dim M_k > \dim M_{k-1}$ and $M_r = M$ is open dense in $\widetilde{M}$. For the G-orbit decomposition of $\widetilde{M}$, the properties (S1) and (S2) were already proved in [6]. We will verify (S3) in this paper.

Suppose first that the GLA $\mathfrak{g}$ is of BC_r-type. We consider the two abelian subspaces of $\mathfrak{g}$: $\mathfrak{g}_1 = \mathfrak{g}_2(r) + \mathfrak{g}_1^-(r)$, $\mathfrak{g}_1' := \mathfrak{g}_2(r) + \mathfrak{g}_1^+(r)$. The direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is imbedded in $\widetilde{M}$ as an open subset. We identify $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ with its image in $\widetilde{M}$. Let $M_k^* := M_k \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, which is open dense in M_k. The closure $\overline{M_k^*}$ of M_k^* in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ coincides with $M_{\leq k}^* := \coprod_{i=0}^{k} M_i^*$ and it is an algebraic variety in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ (Theorem 5.14). Obviously we have $\mathrm{Sing}(M_{\leq k}) = G(\mathrm{Sing}(M_{\leq k}^*))$. Thus, in order to find the singular locus of $M_{\leq k}$, it is enough to find that of $M_{\leq k}^*$. Now we look at $\mathfrak{g}_2(r)$. The Levi subgroup L of Q_r corresponding to $\mathfrak{g}_0(r)$ acts on $\mathfrak{g}_2(r)$. We have a partition $\mathfrak{g}_2(r) = \coprod_{k=0}^{r} V_k$, where V_k is a union of equi-dimensional L-orbits with $\dim V_k > \dim V_{k-1}$. As a conclusion, the above partition is a stratification of $\mathfrak{g}_2(r)$. The validity of (S1) and (S2) was proved in [8]; as was shown in [8], $V_{\leq k}$ $(0 \leq k \leq r-1)$ is an algebraic variety in $\mathfrak{g}_2(r)$. By Proposition 6.3, the problem of finding the singular locus of $M_{\leq k}^*$ is reduced to finding that of $V_{\leq k}$. In the case where the GLA $\mathfrak{g}$ is of C_r-type, we have $\mathfrak{g}_1 = \mathfrak{g}_1' = \mathfrak{g}_2(r)$. Therefore it is enough to look at the determinantal varieties $V_{\leq k}$ in $\mathfrak{g}_1$ for the case where the GLA $\mathfrak{g}$ is of C_r-type. In the realization of $\mathfrak{g}_1$ as a matrix space, $V_{\leq k}$ is a complex or real determinantal variety of classical or exceptional type. The determination of $\mathrm{Sing}(V_{\leq k})$ will be carried out in Section 6. Levasseur-Stafford [11] is a good reference for the complex classical case. The final result on the stratification of $\widetilde{M}$ is given by Theorem 6.13.

The class of simple parahermitian symmetric spaces of C_r-type contains an interesting sub-class of symmetric spaces of Cayley type, which are causal symmetric spaces. For a symmetric space M of Cayley type, $\widetilde{M}$ is the causal compactification of M (cf. [12]). As an application of the results of the present paper, one can determine the full causal automorphism group of M. In the forthcoming paper, we will treat this topic.

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The paper is organized as follows:

1. Preliminaries on parahermitian symmetric spaces.
2. Double foliation of M.
3. Orbit structure of $\widetilde{M}$.
4. Isotropy subgroups for boundary orbits.
5. Siegel-type realization of orbits.
6. Stratification of $\widetilde{M}$.
7. Double foliation on the minimal boundary orbit.
8. Determination of automorphism groups of M.

Throughout this paper, a diffeomorphism always means a C^∞- diffeomorphism. The group of diffeomorphisms of a smooth (i.e., C^∞) manifold M is denoted by $\mathrm{Diffeo}(M)$.

1. Preliminaries on parahermitian symmetric spaces

Let M be a connected $2n$-dimensional smooth manifold, and let $F^\pm$ be two n-dimensional completely integrable distributions on M. $(F^\pm)$ is called a *paracomplex structure* on M ([10]) if the tangent bundle TM of M can be expressed as the Whitney sum $F^+ \oplus F^-$. In this case $(M, F^\pm)$ is called a *paracomplex manifold.* A paracomplex manifold $(M, F^\pm)$ is called a *parakähler manifold* ([10]) if there exists a symplectic form ω on M with respect to which $F^\pm$ are Lagrangian subbundles. For a parakähler manifold $(M, F^\pm, \omega)$, one can consider the two kinds of automorphisms: By a *paracomplex automorphism* of M we mean a diffeomorphism of M which leaves $F^\pm$ invariant. By a *paracomplex isometry* of M we mean a paracomplex automorphism leaving ω invariant. We denote by $\mathrm{Aut}(M, F^\pm)$ (resp. $\mathrm{Aut}(M, F^\pm, \omega)$) the group of paracomplex automorphisms (resp. paracomplex isometries) of M. The group $\mathrm{Aut}(M, F^\pm, \omega)$ is always a finite-dimensional Lie group, but $\mathrm{Aut}(M, F^\pm)$ is not in general.

Definition 1.1. ([10]). Let $M = G/H$ be an almost effective symmetric coset space of a Lie group G, and let $(F^\pm, \omega)$ be a parakähler structure on M. If G acts on M as paracomplex isometries with respect to $(F^\pm, \omega)$, then $(M = G/H$, $F^\pm$, $\omega)$ is called a *parahermitian symmetric space.*

For each parahermitian symmetric space $M = G/H$, the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ has the structure of a GLA of the first kind $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ ([10]). Under the assumption of semisimplicity of $\mathfrak{g}$, the assignment $M \rightsquigarrow \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ induces a bijection between the set of local isomorphism classes of parahermitian symmetric spaces and the set of isomorphism classes of effective semisimple GLA of the first kind ([5]). In this case the original parakähler structure on M can be recovered by the grading of $\mathfrak{g}$.

Let us start with a real simple GLA of the first kind

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1. \tag{1.1}$$

The automorphism group of the Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{Aut}\,\mathfrak{g}$. Let (Z, τ) be the associated pair of the GLA: Z is the *characteristic element* of the GLA, that is, Z is a unique element of $\mathfrak{g}_0$ satisfying the condition $\mathrm{ad}\, Z = k1$ on $\mathfrak{g}_k$, $k = 0, \pm 1$, and τ is a Cartan involution of $\mathfrak{g}$ satisfying $\tau(Z) = -Z$. Let $\sigma = \mathrm{Ad}\exp(\pi i Z)$. Then σ is the involutive automorphism of $\mathfrak{g}$ such that $\sigma = 1$ on $\mathfrak{g}_0$ and -1 on $\mathfrak{m} := \mathfrak{g}_{-1} + \mathfrak{g}_1$. Thus we have a symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. Let G_0 be the centralizer of Z in $\mathrm{Aut}\,\mathfrak{g}$, and let G be the open subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by G_0 and $\mathrm{Ad}\,\mathfrak{g}$. Note that $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$ and that G_0 coincides with the group of grade-preserving automorphisms of the GLA (1.1).

Proposition 1.2. *The coset space $M = G/G_0$ is a parahermitian symmetric space corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. The group G acts on M effectively by paracomplex isometries.*

Proof. If we put $\widetilde{\sigma}(a) = \sigma a \sigma$, $a \in G$, then $\widetilde{\sigma}$ is an involutive automorphism of $\operatorname{Aut}\mathfrak{g}$. $\widetilde{\sigma}$ leaves G stable. Let $G_{\widetilde{\sigma}}$ be the subgroup of G consisting of all $\widetilde{\sigma}$-fixed elements of G. Then, from the definition of σ, it follows that G_0 is an open subgroup of $G_{\widetilde{\sigma}}$. So $M = G/G_0$ is a symmetric space. By the definition of G_0, M is realized in $\mathfrak{g}$ as the adjoint G-orbit through $Z \in \mathfrak{g}$. Therefore the Kirillov-Kostant form

$$\widetilde{\omega}(X,Y) = (Z,[X,Y]), \qquad X,Y \in \mathfrak{g} \tag{1.2}$$

induces a G-invariant symplectic form ω on M,where $(\ ,\ )$ denotes the Killing form of $\mathfrak{g}$. Let $0 \in M$ be the origin of $M = G/G_0$. We identify $\mathfrak{m}$ with the tangent space T_0M of M at 0. Then the two G_0-invariant subspaces $\mathfrak{g}_{\pm 1}$ extend to G-invariant distributions $F^{\pm}$ on M, which are Lagrangian with respect to ω by (1.2). The complete integrability of $F^{\pm}$ has been proved in two ways, one in [10] by a differential geometric argument, the other by an algebraic method using dipolarizations. To prove effectivity of the G-action on M, first note that the natural G_0-action on $\mathfrak{m}$ can be identified with the linear isotropy group at $0 \in M = G/G_0$. Let $a \in G$ and suppose that a acts on M as the identity. Then $a \in G_0$ and a acts on $\mathfrak{m}$ as the identity. Since $\mathfrak{g}$ is simple, we have $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$. So it follows that a acts on $\mathfrak{g}_0$ as the identity. This implies that a is the unit element of G. ∎

The parahermitian symmetric space $(M = G/G_0, F^{\pm}, \omega)$ thus constructed is called the *parahermitian symmetric space associated to a simple GLA* (1.1). The parahermitian symmetric space $M = G/G_0$, which is a hyperbolic $\operatorname{Ad}G$-orbit, is the bottom space with respect to the covering relation among the parahermitian symmetric spaces corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$.

2. Double foliations of M

Let $(M = G/G_0, F^{\pm}, \omega)$ be the parahermitian symmetric space associated to a simple GLA (1.1). We denote by $F^{\pm}(p)$ the leaves of $F^{\pm}$ through a point $p \in M$. It is easy to see that the leaves $F^{\pm}(0)$ through the origin $0 \in M$ are given by the orbits $(\exp \mathfrak{g}_{\pm 1}) \cdot 0$. Consider the parabolic subgroups $U^{\pm} := G_0 \exp \mathfrak{g}_{\pm 1}$ of G. Note that $\mathfrak{u}^{\pm} := \operatorname{Lie} U^{\pm} = \mathfrak{g}_0 + \mathfrak{g}_{\pm 1}$. The flag manifolds $G/U^{\pm}$ are called the *symmetric R-spaces associated with* M or with the GLA (1.1).

Lemma 2.1. *Let $M^{\pm}$ be the sets of leaves of $F^{\pm}$ on M. Then $M^{\pm}$ are identified with the flag manifolds $G/U^{\pm}$.*

Proof. Since $F^{\pm}$ are G-invariant, any element of G induces a permutation on the sets of leaves of $F^{\pm}$, which means that G acts on $M^{\pm}$. The transitivity of G on $M^{\pm}$ follows from that of G on M. Now let $g \in G$ and suppose for example $gF^{+}(0) = F^{+}(0)$. Then the point $g \cdot 0$ is in $F^{+}(0)$. One can write $g \cdot 0 = \exp X \cdot 0$ for some $X \in \mathfrak{g}_1$. Therefore $g^{-1} \exp X \in G_0$, which implies $g \in U^{+}$, and M^{+} is expressed as G/U^{+}. ∎

Lemma 2.2. *For any point $p \in M$, we have $F^{+}(p) \cap F^{-}(p) = \{p\}$.*

Proof. One can assume p to be the origin 0. Any point $q \in F^+(0) \cap F^-(0)$ can be expressed as $q = \exp X \cdot 0 = \exp Y \cdot 0$ for $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_{-1}$. This implies that $\exp X \in (\exp Y)G_0 \subset U^-$. Consequently $\exp X \in U^+ \cap U^- = G_0$ and hence $\exp X \in (\exp \mathfrak{g}_1) \cap G_0 = (1)$, which implies $X = 0$. Thus we have $q = p$. ■

We denote the points $F^\pm(0) \in M^\pm$ by $0^\pm$, and let us consider the product manifolds

$$\widetilde{M} = M^- \times M^+ \tag{2.1}$$

with the origin $(0^-, 0^+)$. $\widetilde{M}$ has a double foliation arising from the product structure. The leaves through the point $(g_1 0^-, g_2 0^+)$, g_1, $g_2 \in G$, are denoted by $M^\pm(g_1 0^-, g_2 0^+)$. $M^-(g_1 0^-, g_2 0^+)$ is called the horizontal leaf and $M^+(g_1 0^-, g_2 0^+)$ is called the vertical leaf. They are given by

$$\begin{aligned} M^-(g_1 0^-, g_2 0^+) &= G g_1 0^- \times \{g_2 0^+\} = G/g_1 U^- g_1^{-1} \times \{g_2 0^+\}, \\ M^+(g_1 0^-, g_2 0^+) &= \{g_1 0^-\} \times G g_2 0^+ = \{g_1 0^-\} \times G/g_2 U^+ g_2^{-1}. \end{aligned} \tag{2.2}$$

Let us define a map φ of M to $\widetilde{M}$ by putting

$$\varphi(p) = \big(F^-(p), F^+(p)\big), \qquad p \in M. \tag{2.3}$$

Lemma 2.3. *φ is a G-equivariant open imbedding of M into $\widetilde{M}$ and preserves the double foliations on M and $\widetilde{M}$; actually we have $\varphi\big(F^\mp(p)\big) \subset M^\pm\big(\varphi(p)\big)$, $p \in M$.*

Proof. Let $g \in G$. Then $\varphi(g \cdot 0) = \big(F^-(g\cdot 0), F^+(g \cdot 0)\big) = (g \cdot 0^-, g \cdot 0^+)$. From this and Lemma 2.2 it follows that φ is G-equivariant imbedding. The openness of φ follows from dimension counting. Now let $q \in F^-(p)$. Then $F^-(q) = F^-(p)$ and hence $\varphi(q) = \big(F^-(p), F^+(q)\big)$, which implies that $\varphi(q)$ lies on the vertical leaf through $\varphi(p)$. ■

Since $U^- \cap U^+ = G_0$, M has the structure of the double fibration over $M^\pm$. The projections $\pi^\pm \colon M \to M^\pm$ are given by

$$\pi^\pm(g \cdot 0) = g \cdot 0^\pm, \qquad g \in G. \tag{2.4}$$

Lemma 2.4. *For each point $p \in M$, we have*

$$F^\pm(p) = (\pi^\pm)^{-1}\big(\pi^\pm(p)\big).$$

Proof. It is enough to prove the assertion for the case where p is the origin. Choose a point $(\exp X)0 \in F^+(0)$, $X \in \mathfrak{g}_1$. Then we have $\pi^+\big((\exp X)0\big) = (\exp X)0^+ = 0^+$, and hence $(\exp X)0 \in (\pi^+)^{-1}(0^+) = (\pi^+)^{-1}\big(\pi^+(0)\big)$. Conversely, let $p \in (\pi^+)^{-1}(0^+)$. Then $\pi^+(p) = 0^+$. Consequently p can be written as $p = u0$, where $u \in U^+$. If we write $u = (\exp Y)h$, $Y \in \mathfrak{g}_1$, $h \in G_0$, then we have $p = (\exp Y)0 \in F^+(0)$. ■

If we denote the projections by $\varpi^\pm \colon \widetilde{M} \to M^\pm$, then we have $\varpi^\pm \cdot \varphi = \pi^\pm$ (cf. (2.4)). Therefore, under the identification of M with $\varphi(M)$, the double fibration of M is the restriction of the trivial double fibration of $\widetilde{M}$ to M. Later on we always identify M with its φ-image in $\widetilde{M}$. As was seen in the proof of Lemma 2.3, M is an orbit through the origin $(0^-, 0^+) \in \widetilde{M}$ under the diagonal G-action.

3. Orbit structure of $\widetilde{M}$

We wish to consider the orbit structure of $\widetilde{M}$ under the diagonal G-action. We start with a simple GLA (1.1). Recall the decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m}$ by σ (cf. §1). Also we have the Cartan involution τ satisfying $\tau(Z) = -Z$. The property $\tau(Z) = -Z$ means that τ is grade-reversing, i.e., $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$, $k = 0, \pm 1$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition by τ, where $\tau = 1$ on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Since σ and τ commute, we have the decomposition

$$\mathfrak{g} = \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{k}} + \mathfrak{p}_0 + \mathfrak{m}_{\mathfrak{p}}, \tag{3.1}$$

where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$, $\mathfrak{m}_{\mathfrak{k}} = \mathfrak{m} \cap \mathfrak{k}$, $\mathfrak{p}_0 = \mathfrak{g}_0 \cap \mathfrak{p}$ and $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{m} \cap \mathfrak{p}$. Note that $Z \in \mathfrak{p}_0$. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $Z \in \mathfrak{a}$. Then $\mathfrak{a}$ is contained in $\mathfrak{p}_0$. Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let $(\ ,\ )$ denote the Killing form of $\mathfrak{g}$. Then we have the partition of Δ corresponding to the grading of $\mathfrak{g}$:

$$\begin{aligned} \Delta &= \Delta_{-1} \amalg \Delta_0 \amalg \Delta_1, \\ \Delta_k &= \{\, \alpha \in \Delta : (\alpha, Z) = k \,\}, \qquad k = 0, \pm 1. \end{aligned} \tag{3.2}$$

Choose a linear order in Δ in such a way that $\Delta_1 \subset \Delta^+ \subset \Delta_0 \cup \Delta_1$, where Δ^+ denotes the positive system of Δ with respect to that order. Then choose a maximal system of strongly orthogonal roots, $\Gamma = \{\beta_1, \ldots, \beta_r\}$ in Δ_1, such that each β_i has the same length and that $\theta = \beta_1 > \beta_2 > \cdots > \beta_r$, θ being the highest root in Δ. Here the number r is the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Note that r is equal to the rank of the symmetric R-space G/U^-.

Moreover choose a root vector E_i in the root space $\mathfrak{g}^{\beta_i} \subset \mathfrak{g}_1$ $(1 \le i \le r)$ in such a way that

$$[E_i, E_{-i}] = \check{\beta}_i = \frac{2}{(\beta_i, \beta_i)} \beta_i, \qquad 1 \le i \le r,$$

where $E_{-i} = -\tau(E_i) \in \mathfrak{g}^{-\beta_i} \subset \mathfrak{g}_{-1}$. Put $X_i := E_i + E_{-i} \in \mathfrak{m}_{\mathfrak{p}}$ and $Y_i := E_i - E_{-i} \in \mathfrak{m}_{\mathfrak{k}}$ $(1 \le i \le r)$. Then $\mathfrak{c} = \sum_{i=1}^r \mathbb{R} X_i$ is a maximal abelian subspace of $\mathfrak{m}_{\mathfrak{p}}$. $\mathfrak{c}$ is a split Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Note that $\mathfrak{c}$ is also a Cartan subalgebra of the noncompact dual of the symmetric R-space G/U^-. It is well-known that the root system $\Delta(\mathfrak{g}, \mathfrak{c})$ of $\mathfrak{g}$ with respect to $\mathfrak{c}$ is of C_r-type or BC_r-type. Correspondingly we say that the GLA (1.1) and the parahermitian symmetric space $M = G/G_0$ are of C_r-type or BC_r-type, respectively. Let $\mathfrak{a}_0$ be the subspace of $\mathfrak{a}$ spanned by $\beta_1, \ldots, \beta_r$ and ϖ be the orthogonal projection of $\mathfrak{a}$ onto $\mathfrak{a}_0$ with respect to $(\ ,\ )$. Then either one of the following two cases occurs ([13,6]):

$$\begin{cases} \varpi(\Delta_1) = \{\, \frac{1}{2}(\beta_i + \beta_j) : 1 \le i \le j \le r \,\}, \\ \varpi(\Delta_0^+) - (0) = \{\, \frac{1}{2}(\beta_i - \beta_j) : 1 \le i < j \le r \,\}, \end{cases} \tag{3.3}$$

$$\begin{cases} \varpi(\Delta_1) = \{ \frac{1}{2}(\beta_i + \beta_j)\ (1 \le i \le j \le r),\ \frac{1}{2}\beta_i\ (1 \le i \le r) \}, \\ \varpi(\Delta_0^+) - (0) = \{ \frac{1}{2}(\beta_i - \beta_j)\ (1 \le i < j \le r),\ \frac{1}{2}\beta_i\ (1 \le i \le r) \}, \end{cases} \tag{3.4}$$

according as $\Delta(\mathfrak{g}, \mathfrak{c})$ is of C_r-type or BC_r-type, respectively. Here $\Delta_0^+ = \Delta_0 \cap \Delta^+$. Now let K be the subgroup of G consisting of elements which commute with τ.

Then K is the maximal compact subgroup of G with $\operatorname{Lie} K = \mathfrak{k}$. We denote the identity component of K by K^0, and define the elements $a_l \quad (0 \leq l \leq r)$ in the normalizer $N_{K^0}(\mathfrak{a})$ of $\mathfrak{a}$ in K^0 by putting

$$\begin{cases} a_l = \exp\bigl(-\frac{\pi}{2}\sum_{i=1}^{l} Y_i\bigr), & 1 \leq l \leq r, \\ a_0 = 1. \end{cases} \tag{3.5}$$

The following theorem gives the G-orbit structure of $\widetilde{M}$.

Theorem 3.1.

(i) *The points* $(0^-, a_l\, 0^+) \in \widetilde{M}$, $0 \leq l \leq r$, *are a complete set of representatives of* G*-orbits in* $\widetilde{M}$.

(ii) *Let* $M_l = G(0^-, a_{r-l}\, 0^+)$, $0 \leq l \leq r$. *Then the closure* $\overline{M_l}$ *of* M_l *in* $\widetilde{M}$ *is given by*

$$\overline{M_l} = M_l \amalg M_{l-1} \amalg \cdots \amalg M_0, \qquad 0 \leq l \leq r.$$

(iii) $M_r = M$ *is a single open* G*-orbit, and hence* $\widetilde{M}$ *is a* G*-equivariant compactification of* M.

(iv) *A single closed* G*-orbit* M_0 *has the property*: *If* M *is of* C_r*-type, then* $M_0 = M^-$ *and* $a_r U^+ a_r^{-1} = U^-$ *holds. If* M *is of* BC_r*-type, then* M_0 *is a flag manifold of the second kind.* M_0 *has the double fibration*:

$$G/U^- = M^-(0^-, a_r\, 0^+) \longleftarrow M_0 \longrightarrow M^+(0^-, a_r\, 0^+) = G/a_r U^+ a_r^{-1}.$$

Proof. Let us denote by G^0 the identity component of G. In [6] we proved the theorem for the G^0-action. But, by Theorem 4.12 in [6], $\dim M_l$ is strictly increasing, as l increases. Let $g \in G$. Since g normalizes G^0, $g(M_l)$ is still a G^0-orbit which has the same dimension as M_l. Therefore $g(M_l) = M_l$. In other words, G leaves each G^0-orbit stable. ∎

4. Isotropy subgroups for boundary orbits

We go back to a real simple GLA $\mathfrak{g}$ in (1.1). We wish to construct a certain class of gradings of $\mathfrak{g}$ of the second kind in terms of the subsets $\Gamma_l = \{\beta_1, \ldots, \beta_l\}$, $1 \leq l \leq r$, of Γ. When $\mathfrak{g}$ is of Hermitian type, this type of grading corresponds to the realizations of the bounded symmetric domain (corresponding to $\mathfrak{g}$) as a Siegel domain of the third kind for $1 \leq l \leq r-1$ and that of the second (or first) kind for $l=r$. Let $1 \leq l \leq r$, and put

$$
\begin{aligned}
&\Delta_2(l) = \left\{ \alpha \in \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}(\beta_i+\beta_j),\ 1 \leq i \leq j \leq l \right\},\\
&\Delta_1(l) = \left\{ \alpha \in \Delta : \begin{array}{l} \varpi(\alpha) = \frac{1}{2}(\beta_i \pm \beta_j),\ 1 \leq i \leq l, l+1 \leq j \leq r, \text{ or} \\ \varpi(\alpha) = \frac{1}{2}\beta_i,\ 1 \leq i \leq l \end{array} \right\},\\
&\Delta_0(l) = \left\{ \alpha \in \Delta : \begin{array}{l} \varpi(\alpha) = 0, \text{ or} \\ \varpi(\alpha) = \pm\frac{1}{2}(\beta_i - \beta_j),\ 1 \leq i < j \leq l \text{ or} \\ \qquad\qquad\qquad\qquad l+1 \leq i < j \leq r, \text{ or} \\ \varpi(\alpha) = \pm\frac{1}{2}(\beta_i + \beta_j),\ l+1 \leq i \leq j \leq r, \text{ or} \\ \varpi(\alpha) = \pm\frac{1}{2}\beta_i,\ l+1 \leq i \leq r \end{array} \right\},\\
&\Delta_{-1}(l) = -\Delta_1(l),\\
&\Delta_{-2}(l) = -\Delta_2(l).
\end{aligned}
\tag{4.1}
$$

Then, for a fixed $1 \leq l \leq r$, we have a partition of Δ:

$$
\Delta = \coprod_{k=-2}^{2} \Delta_k(l). \tag{4.2}
$$

By using (3.3) and (3.4) we easily have

Proposition 4.1. *Let $1 \leq l \leq r$, and let $\mathfrak{c}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. If we put*

$$
\begin{aligned}
\mathfrak{g}_0(l) &= \mathfrak{c}(\mathfrak{a}) + \sum_{\alpha \in \Delta_0(l)} \mathfrak{g}^\alpha,\\
\mathfrak{g}_k(l) &= \sum_{\alpha \in \Delta_k(l)} \mathfrak{g}^\alpha, \qquad k = \pm 1, \pm 2,
\end{aligned}
\tag{4.3}
$$

then we have the grading of $\mathfrak{g}$ of the second kind

$$
\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(l), \tag{4.4}
$$

whose characteristic element is $Z_l = \sum_{k=1}^{l} \check{\beta}_i$.

Remark 4.2. Gyoja and Yamashita[3] obtained the above gradings for $\mathfrak{g}$ complex simple, in which case there are no roots $\alpha \in \Delta$ such that $\varpi(\alpha) = 0$.

Let s_{β_i} be the reflection on $\mathfrak{a}$ corresponding to the root β_i $(1 \leq i \leq r)$, and let $s_l = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_l}$ $(1 \leq l \leq r)$ and $s_0 = 1$. It is known [13] that $\mathrm{Ad}_{\mathfrak{a}}\, a_l = s_l$, $0 \leq l \leq r$. Let Q_l $(0 \leq l \leq r)$ be the isotropy subgroup of G at $(0^-, a_l\, 0^+)$. Then the G-orbit M_{r-l} can be expressed as

$$
M_{r-l} = G/Q_l, \qquad 0 \leq l \leq r,
$$

where $Q_l = U^- \cap a_l\, U^+ a_l^{-1}$. The Lie algebra $\mathfrak{q}_l := \mathrm{Lie}\, Q_l$ can be written as

$$
\mathfrak{q}_l = \mathfrak{c}(\mathfrak{a}) + \sum_{\alpha \in \Psi_l} \mathfrak{g}^\alpha, \qquad 0 \leq l \leq r, \tag{4.5}
$$

where

$$
\Psi_l := \{ \alpha \in \Delta_0 \cup \Delta_{-1} : s_l(\alpha) \in \Delta_0 \cup \Delta_1 \}, \qquad 0 \leq l \leq r. \tag{4.6}
$$

Lemma 4.3.

$$\Psi_l = \Delta_0 \cap \Delta_0(l) \amalg \Delta_{-1}(l) \amalg \Delta_{-2}(l). \tag{4.7}$$

Proof. First we will show the inclusion $\supset$ in (4.7). Let $\alpha \in \Delta$. Then we have

$$\begin{aligned}\big(s_l(\alpha), Z\big) &= (\alpha, Z) - \sum_{k=1}^{l} (\alpha, \check{\beta}_k)(\beta_k, Z) \\ &= (\alpha, Z) - \sum_{k=1}^{l} 2(\varpi(\alpha), \beta_k)(\beta_k, \beta_k)^{-1}.\end{aligned} \tag{4.8}$$

Now let $\alpha \in \Delta_0 \cap \Delta_0(l)$. By using (4.8) it follows from (3.4) and (4.1) that $\big(s_l(\alpha), Z\big) = 0$, or equivalently $s_l(\alpha) \in \Delta_0$ and hence $\alpha \in \Psi_l$. Suppose next that $\alpha \in \Delta_{-1}(l)$. Then, by (4.1), there are three possibilities: $\varpi(\alpha) = -\frac{1}{2}(\beta_i + \beta_j)$ or $-\frac{1}{2}(\beta_i - \beta_j)$ for $1 \le i \le l$, $l+1 \le j \le r$, or $\varpi(\alpha) = -\frac{1}{2}\beta_i$ for $1 \le i \le l$. In view of (3.4) and (4.8), we have $\alpha \in \Delta_{-1}$ and $\big(s_l(\alpha), Z\big) = 0$ for the first case, and $\alpha \in \Delta_0$ and $\big(s_l(\alpha), Z\big) = 1$ for the second case. For the third case, there are two possibilities (cf. (3.4)): $\alpha \in \Delta_0$ or $\alpha \in \Delta_{-1}$. Then we have from (4.8) that $\big(s_l(\alpha), Z\big) = 1$ or 0, according as $\alpha \in \Delta_0$ or $\alpha \in \Delta_{-1}$, respectively. Consequently $s_l(\alpha) \in \Delta_0 \cup \Delta_1$ for $\alpha \in \Delta_{-1}(l)$. Suppose $\alpha \in \Delta_{-2}(l)$. Then by (3.4) and (4.8) we have $\alpha \in \Delta_{-1}$ and $\big(s_l(\alpha), Z\big) = 1$.

To prove the converse inclusion $\subset$ in (4.7), let $\alpha \in \Psi_l$ and suppose that α does not belong to the right-hand side of (4.7). Then the following three cases occur:

(i) $\alpha \in \Delta_2(l)$,

(ii) $\alpha \in \Delta_1(l)$ and

(iii) $\alpha \in \Delta_0(l) - \Delta_0$.

For (i), we have $\alpha \in \Delta_1$, contradicting the assumption that $\alpha \in \Psi_l$. For (ii), we have three possibilities: $\varpi(\alpha) = \frac{1}{2}(\beta_i + \beta_j)$ or $\frac{1}{2}(\beta_i - \beta_j)$ both for $1 \le i \le l$, $l+1 \le j \le r$ or $\varpi(\alpha) = \frac{1}{2}\beta_i$ for $1 \le i \le l$. For the first case we have $\alpha \in \Delta_1$, which contradicts $\alpha \in \Psi_l$. For the second case, we have $\alpha \in \Delta_0$. Consequently, by using (4.8), we have that $\big(s_l(\alpha), Z\big) = -1$, that is, $s_l(\alpha) \in \Delta_{-1}$. This contradicts the assumption $\alpha \in \Psi_l$. For the third case, we have either $\alpha \in \Delta_1$ or $\alpha \in \Delta_0$. In view of the condition $\alpha \in \Psi_l$, we have the only choice $\alpha \in \Delta_0$, in which case $\big(s_l(\alpha), Z\big) = -1$, still contradicting the assumption $\alpha \in \Psi_l$. Let us consider the case (iii) finally. Since α lies in $\Psi_l \cap \big(\Delta_0(l) - \Delta_0\big)$, we have that $\varpi(\alpha) = -\frac{1}{2}(\beta_i + \beta_j)$, $l+1 \le i \le j \le r$, or $\varpi(\alpha) = -\frac{1}{2}\beta_i$, $l+1 \le i \le r$. In particular $\alpha \in \Delta_{-1}$. Therefore, in both cases, we have $\big(s_l(\alpha), Z\big) = -1$, which contradicts the assumption $\alpha \in \Psi_l$. ∎

Proposition 4.4. *The isotropy subalgebra $\mathfrak{q}_l$ of $\mathfrak{g}$ at the point $(0^-, a_l 0^+)$ is given by*

$$\mathfrak{q}_l = \mathfrak{g}_{-2}(l) + \mathfrak{g}_{-1}(l) + \mathfrak{g}_0(l) \cap \mathfrak{g}_0, \qquad 0 \le l \le r.$$

Proof. This follows immediately from Lemma 4.3 and (4.5). ■

In this paragraph we will determine the isotropy subgroup Q_r of G at the point $(0^-, a_r\, 0^+) \in M_0$. Let us consider the special case $l = r$. Then (4.1) has the following simple form:

$$\begin{aligned} \Delta_2(r) &= \left\{ \alpha \in \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}(\beta_i + \beta_j),\ 1 \le i \le j \le r \right\}, \\ \Delta_1(r) &= \left\{ \alpha \in \Delta_0^+ \cup \Delta_1 : \varpi(\alpha) = \tfrac{1}{2}\beta_i,\ 1 \le i \le r \right\}, \\ \Delta_0(r) &= \left\{ \alpha \in \Delta : \varpi(\alpha) = 0 \text{ or } = \pm\tfrac{1}{2}(\beta_i - \beta_j),\ 1 \le i < j \le r \right\}, \\ \Delta_{-k}(r) &= -\Delta_k(r), \qquad k = 1, 2. \end{aligned} \tag{4.9}$$

We have the grading of the second kind

$$\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k(r). \tag{4.10}$$

Remark 4.5. In the case of C_r-type, we have $\Delta_1(r) = \emptyset$, $\Delta_2(r) = \Delta_1$ and $\Delta_0(r) = \Delta_0$. Hence the grading (4.10) is reduced to the grading (1.1).

Lemma 4.6. $\mathfrak{q}_r = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r)$. *In particular,* $\mathfrak{q}_r$ *is a parabolic subalgebra of* $\mathfrak{g}$ *of the second kind.*

Proof. Since $\Delta_0(r) \subset \Delta_0$, we have the inclusion $\mathfrak{g}_0(r) \subset \mathfrak{g}_0$. ■

Now we put

$$\begin{aligned} \Delta_1^+(r) &= \Delta_1(r) \cap \Delta_0^+, & \Delta_1^-(r) &= \Delta_1(r) \cap \Delta_1, \\ \Delta_{-1}^+(r) &= \Delta_{-1}(r) \cap \Delta_{-1}, & \Delta_{-1}^-(r) &= \Delta_{-1}(r) \cap \Delta_0^-. \end{aligned} \tag{4.11}$$

Then we have

$$\Delta_{\pm 1}(r) = \Delta_{\pm 1}^+(r) \amalg \Delta_{\pm 1}^-(r). \tag{4.12}$$

We define the following four subspaces of $\mathfrak{g}$:

$$\mathfrak{g}_{\pm 1}^+(r) = \sum_{\alpha \in \Delta_{\pm 1}^+(r)} \mathfrak{g}^\alpha, \qquad \mathfrak{g}_{\pm 1}^-(r) = \sum_{\alpha \in \Delta_{\pm 1}^-(r)} \mathfrak{g}^\alpha. \tag{4.13}$$

Those four subspaces are equi-dimensional and abelian ([6]). We have the decompositions

$$\mathfrak{g}_1(r) = \mathfrak{g}_1^-(r) + \mathfrak{g}_1^+(r), \qquad \mathfrak{g}_{-1}(r) = \mathfrak{g}_{-1}^+(r) + \mathfrak{g}_{-1}^-(r). \tag{4.14}$$

The original grading (1.1) of $\mathfrak{g}$ can be reconstructed as

$$\begin{aligned} \mathfrak{g}_{-1} &= \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}^+(r), \\ \mathfrak{g}_0 &= \mathfrak{g}_{-1}^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r), \\ \mathfrak{g}_1 &= \mathfrak{g}_1^-(r) + \mathfrak{g}_2(r). \end{aligned} \tag{4.15}$$

Let $C(Z_r)$ be the centralizer of Z_r in $\mathrm{Aut}\,\mathfrak{g}$. Then the normalizer $N(\mathfrak{q}_r)$ in $\mathrm{Aut}\,\mathfrak{g}$ of $\mathfrak{q}_r$ can be written as

$$N(\mathfrak{q}_r) = C(Z_r) \cdot \exp\bigl(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\bigr). \qquad \text{(semi-direct)} \tag{4.16}$$

Q_r is a subgroup of $U^- \cap N(\mathfrak{q}_r) = N_{U^-}(\mathfrak{q}_r)$, the normalizer of $\mathfrak{q}_r$ in U^-. (4.15) implies that

$$N_{U^-}(\mathfrak{q}_r) = \big(C(Z_r) \cap U^-\big) \cdot \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big). \tag{4.17}$$

Lemma 4.7. *Let $C(Z, Z_r)$ be the centralizer of both elements Z and Z_r in* $\operatorname{Aut}\mathfrak{g}$. *Then we have $C(Z_r) \cap U^- = C(Z, Z_r)$.*

Proof. Let $a \in C(Z_r) \cap U^-$. We write $a = b \exp X$, $b \in C(Z)$, $X \in \mathfrak{g}_{-1}$. Then $[X, Z_r] \in \mathfrak{g}_{-1}$ and hence $\big[X, [X, Z_r]\big] = 0$. Therefore we have

$$Z_r = (\operatorname{Ad} a)Z_r = (\operatorname{Ad} b)(\operatorname{Ad}\exp X)Z_r = (\operatorname{Ad} b)Z_r + (\operatorname{Ad} b)[X, Z_r] \subset \mathfrak{g}_0 + \mathfrak{g}_{-1}.$$

Hence $(\operatorname{Ad} b)[X, Z_r] = 0$. $\operatorname{Ad} b$ being invertible on $\mathfrak{g}_{-1}$, we have $[X, Z_r] = 0$. Consequently $X \in \mathfrak{g}_0(r) \cap \mathfrak{g}_{-1} = (0)$, and $a = b \in C(Z)$. ■

Proposition 4.8. *The isotropy subgroup Q_r of G at $(0^-, a_r 0^+) \in M_0$ is given by*

$$Q_r = C(Z, Z_r) \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big).$$

Proof. By Lemma 4.5 and (4.17) we have

$$N_{U^-}(\mathfrak{q}_r) = C(Z, Z_r) \exp\big(\mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r)\big).$$

Recall $Q_r \subset N_{U^-}(\mathfrak{q}_r)$. To prove the converse inclusion, it suffices to show that $C(Z, Z_r) \subset Q_r$. By using (4.8), one sees $s_r(Z) = Z - Z_r$, and consequently

$$\begin{aligned} a_r^{-1} C(Z) a_r &= C\big((\operatorname{Ad} a_r^{-1})Z\big) = C\big(s_r(Z)\big) = C(Z - Z_r), \\ a_r^{-1} C(Z_r) a_r &= C\big((\operatorname{Ad} a_r^{-1})Z_r\big) = C\big(s_r(Z_r)\big) = C(Z_r). \end{aligned}$$

As a result, $a_r^{-1} C(Z, Z_r) a_r = C(Z - Z_r) \cap C(Z_r) = C(Z, Z_r) \subset U^+$. Hence we have $C(Z, Z_r) \subset U^- \cap a_r U^+ a_r^{-1} = Q_r$. ■

Corollary 4.9. $G = C(Z, Z_r) G^0$.

Proof. We have $M_0 = G/Q_r = G^0/Q_r \cap G^0 = G^0 Q_r/Q_r$, which implies $G = Q_r G^0 = C(Z, Z_r) G^0$. ■

Corollary 4.10. *Suppose that M is of C_r-type. Then $Q_r = U^- = a_r U^+ a_r^{-1}$ and $M_0 = G/U^- = M^-$.*

Proof. By Remark 4.5 and Proposition 4.8, we see that $Q_r = C(Z) \exp \mathfrak{g}_{-1} = U^-$, and hence $U^- = Q_r = U^- \cap a_r U^+ a_r^{-1} \subset a_r U^+ a_r^{-1}$. Hence we have $U^- = a_r U^+ a_r^{-1}$. ■

5. Siegel-type realization of orbits

Lemma 5.1.

$$s_r\big(\Delta_k(r)\big) = \Delta_{-k}(r), \qquad k = 0, \pm 1, \pm 2, \tag{5.1}$$

$$s_r\big(\Delta^{\pm}_{-1}(r)\big) = \Delta^{\mp}_{1}(r). \tag{5.2}$$

Proof. Let $\alpha \in \Delta_k(r)$. Then $\big(s_r(\alpha), Z_r\big) = \big(\alpha, s_r(Z_r)\big) = -(\alpha, Z_r) = -k$, which implies that $s_r(\alpha) \in \Delta_{-k}(r)$. Let $\alpha \in \Delta^{+}_{-1}(r)$. Then, by (4.8) we have $\big(s_r(\alpha), Z\big) = 0$, and hence $s_r(\alpha) \in \Delta_0$. One can write $\varpi(\alpha) = -\frac{1}{2}\beta_i$ for some i. Hence we have $\varpi\big(s_r(\alpha)\big) = s_r\varpi(\alpha) = s_r(-\frac{1}{2}\beta_i) = \frac{1}{2}\beta_i$, proving that $s_r(\alpha) \in \Delta^{+}_{1}(r)$. ∎

Lemma 5.2. *The operator* $\operatorname{Ad} a_r$ *is grade-reversing with respect to the grading* (4.10)*. Moreover* $\operatorname{Ad} a_r$ *interchanges* $\mathfrak{g}^{\pm}_{-1}(r)$ *with* $\mathfrak{g}^{\mp}_{1}(r)$*, respectively.*

Proof. Since $\operatorname{Ad} a_r$ induces s_r on $\mathfrak{a}$ (cf. 4.1), the lemma is immediate from Lemma 5.1. ∎

Up to the present, we have expressed $\widetilde{M}$ as $M^- \times M^+$. Here $M^{\pm}$ are just the leaves of the product foliation through the origin $(0^-, 0^+)$. In order to get the Siegel-type realization of G-orbits, we choose the point $(0^-, a_r\, 0^+) \in M_0$ as the new origin of $\widetilde{M}$. Then $\widetilde{M}$ can be expressed as

$$\widetilde{M} = M^-(0^-, a_r\, 0^+) \times M^+(0^-, a_r\, 0^+) = G/U^- \times G/a_r\, U^+ a_r^{-1}. \tag{5.3}$$

For simplicity we write $(M^+)_r$ for $G/a_r\, U^+ a_r^{-1}$. We identify the tangent space $T_{0^-}(G/U^-)$ with $\mathfrak{g}_1 = \mathfrak{g}_2(r) + \mathfrak{g}^{-}_{1}(r)$ (cf. (4.15)), and $T_{0^+}(G/U^+)$ with $\mathfrak{g}_{-1} = \mathfrak{g}_{-2}(r) + \mathfrak{g}^{+}_{-1}(r)$. Then the tangent space $T_{a_r 0^+}(G/a_r\, U^+ a_r^{-1})$ can be identified with $(\operatorname{Ad} a_r)\mathfrak{g}_{-1} = \mathfrak{g}_2(r) + \mathfrak{g}^{+}_{1}(r)$ by Lemma 5.2. We will denote $(\operatorname{Ad} a_r)\mathfrak{g}_{-1}$ by $\mathfrak{g}'_1$. Let us consider the exterior direct sum of the vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}'_1$

$$\mathfrak{g}_1 \oplus \mathfrak{g}'_1 = \big(\mathfrak{g}_2(r) + \mathfrak{g}^{-}_{1}(r)\big) \oplus \big(\mathfrak{g}_2(r) + \mathfrak{g}^{+}_{1}(r)\big). \tag{5.4}$$

We define the map ξ of $\mathfrak{g}_1 \oplus \mathfrak{g}'_1$ into $\widetilde{M} = M^- \times (M^+)_r$ by

$$\xi(X, X') = \big((\exp X)0^-, (\exp X')a_r\, 0^+\big). \tag{5.5}$$

Then ξ is an open dense imbedding of $\mathfrak{g}_1 \oplus \mathfrak{g}'_1$. We will always identify $\mathfrak{g}_1 \oplus \mathfrak{g}'_1$ with its ξ-image.

Lemma 5.3. *Let* $a^{\pm}_l = \exp\big(\sum_{i=1}^{l} E_{\pm i}\big)$ $(1 \le l \le r)$*. Then*

$$a_r^{-1}(a^{-}_l)^{-1} a_r = a^{+}_l, \qquad 1 \le l \le r. \tag{5.6}$$

Proof. Consider the elements in $\mathfrak{sl}(2,\mathbb{R})$

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

By an easy computation we have

$$\exp\big(\frac{\pi}{2}(e_+ - e_-)\big)\exp(-e_-)\exp\big(-\frac{\pi}{2}(e_+ - e_-)\big) = \exp e_+.$$

Let $\varphi_i\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g} \quad (1 \le i \le r)$ be the maps defined by $\varphi_i(e_\pm) = E_{\pm i}$. By the strong orthogonality of the β_i, we have $[\varphi_i, \varphi_j] = 0 \quad (i \ne j)$. φ_i can be extended to the homomorphism of $\mathrm{SL}(2,\mathbb{R})$ to G, denoted again by φ_i.

LHS of (5.6)

$$\begin{aligned}
&= \exp\Big(\frac{\pi}{2}\sum_{i=1}^{r}(\varphi_i(e_+) - \varphi_i(e_-))\Big)\exp\Big(-\sum_{i=1}^{l}\varphi_i(e_-)\Big)\exp\Big(-\frac{\pi}{2}\sum_{i=1}^{r}(\varphi_i(e_+) - \varphi_i(e_-))\Big)\\
&= \prod_{i=1}^{r}\varphi_i\Big(\exp\big(\frac{\pi}{2}(e_+ - e_-)\big)\Big)\prod_{i=1}^{l}\varphi_i\big(\exp(-e_-)\big)\prod_{i=1}^{r}\varphi_i\Big(\exp\big(-\frac{\pi}{2}(e_+ - e_-)\big)\Big)\\
&= \prod_{i=1}^{l}\varphi_i\Big(\exp\big(\frac{\pi}{2}(e_+ - e_-)\big)\exp(-e_-)\exp\big(-\frac{\pi}{2}(e_+ - e_-)\big)\Big)\\
&= \prod_{i=1}^{l}\varphi_i(\exp e_+) = \exp\Big(\sum_{i=1}^{l}\varphi_i(e_+)\Big) = \exp\Big(\sum_{i=1}^{l}E_i\Big).
\end{aligned}$$

■

The following lemma was proved in [9]. Note that we do not use there the assumption that $\Delta(\mathfrak{g},\mathfrak{c})$ is of C_r-type.

Lemma 5.4. $a_l^- a_l^{-1} a_l^- = a_l^+$.

Lemma 5.5. $(0^-, a_l a_r\, 0^+) \equiv (a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+) \bmod G$.

Proof. First note that $a_l\, 0^\pm = a_l^{-1}\, 0^\pm$. In fact, $\mathrm{Ad}\, a_l^2$ is the identity on $\mathfrak{a}$, which implies that a_l^2 lies in the centralizer $C(Z) = U^+ \cap U^-$. Also note that $a_l^{-1} a_r = a_r a_l^{-1}$, since a_l and a_r commute. Consequently, in view of Lemmas 5.4 and 5.3, we have

$$\begin{aligned}
(0^-, a_l a_r\, 0^+) &= (0^-, a_r a_l\, 0^+) = (0^-, a_r a_l^{-1}\, 0^+) = (0^-, a_l^{-1} a_r\, 0^+)\\
&\equiv (a_l^+ a_l^-\, 0^-, a_l^+ a_l^- a_l^{-1} a_r\, 0^+) = (a_l^+\, 0^-, (a_l^+)^2 (a_l^-)^{-1} a_r\, 0^+)\\
&= (a_l^+\, 0^-, (a_l^+)^2 a_r a_l^+\, 0^+) = (a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+) \bmod G.
\end{aligned}$$

■

Let us put $0_l = \sum_{i=1}^{l} E_i \in \mathfrak{g}_2(r)$, $\ 1 \le l \le r$, $\ 0_0 = 1$.

Proposition 5.6. *The point* $(0_l, 2\,0_l) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$ *(identified with its* ξ*-image) is a representative of the* G*-orbit* M_l *for* $0 \le l \le r$.

Proof. We have

$$\begin{aligned} M_l &= G(0^-, a_{r-l}\, 0^+) = G(0^-, a_l a_r\, 0^+) = G(a_l^+\, 0^-, (a_l^+)^2 a_r\, 0^+) \\ &= G\big((\exp 0_l)\, 0^-, (\exp 0_l)^2 a_r\, 0^+\big) = G\big((\exp 0_l)\, 0^-, (\exp 2\, 0_l) a_r\, 0^+\big) \\ &= G\big(\xi(0_l, 2\, 0_l)\big). \end{aligned}$$

■

A preliminary step for Siegel-type realization of orbits was done by Tanaka [15], which is needed for later consideration. Let $\widehat{Q_r}$ be the parabolic subgroup of G opposite to Q_r, that is, $\widehat{Q_r} = C(Z, Z_r) \cdot N$, where $N = \exp \mathfrak{n}$ and $\mathfrak{n} = \mathfrak{g}_2(r) + \mathfrak{g}_1(r) = \mathfrak{g}_2(r) + \mathfrak{g}_1^+(r) + \mathfrak{g}_1^-(r)$. The group G acts on the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ birationally through ξ. But the subgroup $\widehat{Q_r}$ acts on it as affine transformations.

Proposition 5.7. *(Tanaka [15]). Let a, x, $y \in \mathfrak{g}_2(r)$, b^+, $v^+ \in \mathfrak{g}_1^+(r)$, b^-, $u^- \in \mathfrak{g}_1^-(r)$ and $h \in C(Z, Z_r)$. Then the ξ-equivariant action of $\widehat{Q_r}$ is given by*

$$\begin{aligned} &\exp(a + b^+ + b^-)\big((x, u^-) \oplus (y, v^+)\big) \\ &\quad = (x + a + [b^+, u^-] + \tfrac{1}{2}[b^+, b^-], u^- + b^-) \\ &\qquad \oplus (y + a + [b^-, v^+] + \tfrac{1}{2}[b^-, b^+], v^+ + b^+), \end{aligned} \tag{5.7}$$

$$\begin{aligned} &h\big((x, u^-) \oplus (y, v^+)\big) \\ &\quad = \big((\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)x, (\mathrm{Ad}_{\mathfrak{g}_1^-(r)}\, h)u^-\big) \oplus \big((\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)y, (\mathrm{Ad}_{\mathfrak{g}_1^+(r)}\, h)v^+\big). \end{aligned} \tag{5.8}$$

Definition 5.8. We define a surjective submersion Φ of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ onto $\mathfrak{g}_2(r)$ as follows: For $X = (x, u^-) \in \mathfrak{g}_1$, $Y = (y, v^+) \in \mathfrak{g}_1'$,

$$\Phi(X \oplus Y) = y - x + [v^+, u^-].$$

Φ has the following $\big(\widehat{Q_r}, C(Z, Z_r)\big)$-equivariance property.

Proposition 5.9. *([15]). Φ is invariant under the action of N. Moreover let $h \in C(Z, Z_r)$ and let $X' \oplus Y' = h(X \oplus Y)$. Then*

$$\Phi(X' \oplus Y') = (\mathrm{Ad}_{\mathfrak{g}_2(r)}\, h)\Phi(X \oplus Y).$$

We restate Lemma 3.8 [15] as follows:

Lemma 5.10. *N acts on $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ freely. Moreover let $X \oplus Y \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$ and let $X = (x, u^-)$, $Y = (y, v^+)$. Then we have*

$$\exp(-v^+) \exp(-x - u^-)(X \oplus Y) = (0, 0) \oplus \big(\Phi(X, Y), 0\big). \tag{5.9}$$

Note that the group $C(Z_r)$ is the group of grade-preserving automorphisms with respect to the grading (4.10) and $\mathrm{Lie}\, C(Z_r) = \mathfrak{g}_0(r)$. $C(Z, Z_r)$ is an open subgroup of $C(Z_r)$.

Let $\widehat{Q_r}^0$ and $C^0(Z_r)$ be the identity components of $\widehat{Q_r}$ and $C(Z_r)$, respectively. The following proposition follows from Proposition 5.9 (cf. [15]).

Proposition 5.11. *There exists a bijection between the set of $\widehat{Q_r}^0$-orbits in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ and the set of $C^0(Z_r)$-orbits in $\mathfrak{g}_2(r)$. More precisely, the Φ-image of a $\widehat{Q_r}^0$-orbit is a $C^0(Z_r)$-orbit, and the complete inverse image by Φ of a $C^0(Z_r)$-orbit is a $\widehat{Q_r}^0$-orbit.*

We are interested in the intersection of a G-orbit with $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Let $M_l^* = M_l \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, $0 \leq l \leq r$, which is a dense open set in M_l. M_l^* is stable under $\widehat{Q_r}$, and hence it can be expressed as the union of $\widehat{Q_r}^0$-orbits contained in M_l^*. Those $\widehat{Q_r}^0$-orbits are open in M_l^* ([15]). This fact can also be proved by using Proposition 4.4. Let us consider the (reductive) graded subalgebra of $\mathfrak{g}$

$$\mathfrak{g}_{\mathrm{ev}}(r) = \mathfrak{g}_{-2}(r) + \mathfrak{g}_0(r) + \mathfrak{g}_2(r), \tag{5.10}$$

which contains the simple graded ideal

$$\mathfrak{g}_{\mathrm{ev}}'(r) = \mathfrak{g}_{-2}(r) + [\mathfrak{g}_{-2}(r), \mathfrak{g}_2(r)] + \mathfrak{g}_2(r). \tag{5.11}$$

By the table of $(\mathfrak{g}, \mathfrak{g}_{\mathrm{ev}})$ in [7], or by the property of roots forming $\Delta_2(r)$, it turns out that $\mathfrak{g}_2(r)$ has the structure of a real simple Jordan algebra and the adjoint action of $C^0(Z_r)$ on $\mathfrak{g}_2(r)$ coincides with the identity component of the structure group of this Jordan algebra. Therefore we have the rank decomposition ([2])

$$\mathfrak{g}_2(r) = V_r \amalg V_{r-1} \amalg \cdots \amalg V_0, \tag{5.12}$$

where V_l is the union of equi-dimensional $C^0(Z_r)$-orbits. V_r is open dense in $\mathfrak{g}_2(r)$, $\dim V_k > \dim V_{k-1}$, and $V_0 = (0)$.

Lemma 5.12. *Let $p = X \oplus Y \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Then $\Phi^{-1}\big(\Phi(p)\big)$ is the N-orbit through the point p.*

Proof. This is an easy consequence of Proposition 5.9 and Lemma 5.10. ∎

Proposition 5.13. $M_l^* = \Phi^{-1}(V_l)$, $0 \leq l \leq r$.

Proof. By Proposition 5.6, we have $M_l^* = M_l \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1') = G^0(0_l, 2\,0_l) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$, which contains the orbit $\widehat{Q_r}^0(0_l, 2\,0_l)$ of the same dimension. On the other hand

$$\widehat{Q_r}^0(0_l, 2\,0_l) = \Phi^{-1}\big(C^0(Z_r)\Phi(0_l, 2\,0_l)\big) = \Phi^{-1}\big(C^0(Z_r)\,0_l\big).$$

Let

$$0_{p,q} = \sum_{i=1}^{p} E_i - \sum_{j=p+1}^{p+q} E_j.$$

Note that $0_l = 0_{l,0}$. Let $V_{p,q} = C^0(Z_r)\,0_{p,q}$. It is known [8] that $V_l = \coprod_{p+q=l} V_{p,q}$. Those spaces $V_{p,q}$ in the right-hand side exhaust all $C^0(Z_r)$-orbits of the dimension equal to $\dim C^0(Z_r)\,0_l$. By Lemma 5.12, $\Phi^{-1}(V_{p,q})$, $p+q=l$, are the $\widehat{Q_r}^0$-orbits of the same dimension. Therefore, by Proposition 5.11, we have

$$\Phi^{-1}(V_l) = \Phi^{-1}\Big(\coprod_{p+q=l} V_{p,q}\Big) = \coprod_{p+q=l} \Phi^{-1}(V_{p,q}) = M_l^*. \qquad ∎$$

We say that $\Phi^{-1}(V_l)$ is the Siegel-type realization of the G-orbit M_l. Let $P\colon \mathfrak{g}_2(r) \to \operatorname{End}\mathfrak{g}_2(r)$ be the quadratic operator of the Jordan algebra $\mathfrak{g}_2(r)$. Then we have

Theorem 5.14. *The Siegel-type realization of the* G*-orbit* M_l, $0 \le l \le r$, *is given by*

$$M_l^* = \Phi^{-1}(V_l) = \{\, (x,u^-) \oplus (y,v^+) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1' : \operatorname{rk} P(y-x+[v^+,u^-]) = i_l \,\}, \quad (5.13)$$

where $i_l = \operatorname{rk} P(0_l)$. *In particular, when* $l = r$, *the Siegel-type realization of the parahermitian symmetric space* $M = G/G_0$ *is given by*

$$M_r^* = \Phi^{-1}(V_r) = \{\, (x,u^-) \oplus (y,v^+) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1' : \nu(y-x+[v^+,u^-]) \neq 0 \,\}, \quad (5.14)$$

where ν *denotes the generic norm of the Jordan algebra* $\mathfrak{g}_2(r)$.

Proof. By Proposition 5.13, we have

$$M_l^* = \Phi^{-1}(V_l) = \{\, (x,u^-) \oplus (y,v^+) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1' : y-x+[v^+,u^-] \in V_l \,\}. \quad (5.15)$$

Also we have [2] that

$$V_l = \{\, x \in \mathfrak{g}_2(r) : \operatorname{rk} P(x) = i_l \,\}. \quad (5.16)$$

Note that the condition $\operatorname{rk} P(x) = i_r$ is equivalent to the condition $\nu(x) \neq 0$. ■

Remark 5.15. (5.14) is an analogue of the Siegel domain realization of a bounded symmetric domain. In the case of C_r-type, (5.13) was obtained in [9], in which case $v^+ = u^- = 0$.

The closure $\overline{V_l}$ of V_l in $\mathfrak{g}_2(r)$, was given by $V_{\le l} := \coprod_{i=0}^{l} V_i$ ([2]). Therefore, from (5.16) it follows that $\overline{V_l} = V_{\le l}$ is an algebraic variety in $\mathfrak{g}_2(r)$, which we call a *generalized determinantal variety.* In the case of $\mathfrak{g}_2(r) = M_n(\mathbb{C})$, $\mathrm{Sym}_n(\mathbb{C})$ (resp. $\mathrm{Alt}_{2n}(\mathbb{C})$), the number l is just the rank (resp. one-half of the rank) of a matrix for $M_n(\mathbb{C})$ and $\mathrm{Sym}_n(\mathbb{C})$ (resp. $\mathrm{Alt}_{2n}(\mathbb{C})$). In those cases, $V_{\le l}$ is a usual determinantal variety. By the definition of M_l^*, we have the following decomposition

$$\mathfrak{g}_1 \oplus \mathfrak{g}_1' = \coprod_{l=0}^{r} M_l^*,$$

which is viewed as the rank decomposition of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ by Theorem 5.14.

6. Stratifications of $\widetilde{M}$

We wish to construct a polynomial map Ψ of $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ to $\mathfrak{g}_2(r) \times \mathfrak{n}$ (cf.5.3). Choose a point $p = (x,u^-) \oplus (y,v^+) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$, and consider the element $n_p = \exp(-v^+)\exp(-x-u^-) \in N$. n_p can be written as

$$n_p = \exp(-x + \frac{1}{2}[v^+,u^-] - v^+ - u^-).$$

Since $\exp : \mathfrak{n} \to N$ is diffeomorphic, one can define Ψ to be

$$\Psi(p) = \big(\Phi(p), \log n_p\big) = \big(y-x+[v^+,u^-], -x+\frac{1}{2}[v^+,u^-] - v^+ - u^-\big). \quad (6.1)$$

Lemma 6.1. *The polynomial map* Ψ *is a diffeomorphism of* $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ *onto* $\mathfrak{g}_2(r) \times \mathfrak{n}$. Ψ^{-1} *is also a polynomial map.*

Proof. Let $p,\ q \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$, and suppose $\Psi(p) = \Psi(q)$. Then we have $\Phi(p) = \Phi(q)$ and $n_p = n_q$. By Lemma 5.10, we have that $n_p(p) = (0,0) \oplus \big(\Phi(p), 0\big) = (0,0) \oplus \big(\Phi(q), 0\big) = n_q(q) = n_p(q)$, which implies that $p = q$, proving the injectivity of Ψ. Now let $(a, X) \in \mathfrak{g}_2(r) \times \mathfrak{n}$, and let $p := (\exp\ -X)\big((0,0) \oplus (a,0)\big) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Then, by Lemma 5.12, $\Phi(p) = \Phi\big((0,0) \oplus (a,0)\big) = a$, and hence, by Lemma 5.10, we have $(\exp\ X)p = (0,0) \oplus \big(\Phi(p), 0\big) = n_p(p)$. Since N acts freely on $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$, it follows that $(a, X) = (\Phi(p), \log n_p)$, proving the surjectivity of Ψ. On the other hand, Ψ^{-1} is given by $\Psi^{-1}(a, X) = (\exp -X)\big((0,0) \oplus (a,0)\big)$, which is a polynomial in a and X by (5.7). ∎

From the expression of Ψ^{-1} in the above proof, we have

Lemma 6.2. *Let V be a $C^0(Z_r)$-orbit in $\mathfrak{g}_2(r)$. Then, for the corresponding $\widehat{Q_r}^0$-orbit $\Phi^{-1}(V)$, we have $\Psi(\Phi^{-1}(V)) = V \times \mathfrak{n}$.*

Let $M_{\leq k}$ and $M^*_{\leq k}$ denote the unions $\coprod_{i=0}^k M_i$ and $\coprod_{i=0}^k M_i^*$, respectively. By Theorem 3.1 and Proposition 5.13, the closure $\overline{M_k^*}$ of M_k^* in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is given by $\overline{M_k^*} = M^*_{\leq k} = \Phi^{-1}(V_{\leq k})$, which is an algebraic variety in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ (cf. Theorem 5.14). Since $V_{\leq k}$ is an algebraic variety in $\mathfrak{g}_2(r)$ ([2]), $V_{\leq k} \times \mathfrak{n}$ is an algebraic variety in $\mathfrak{g}_2(r) \times \mathfrak{n}$. We will denote the singular locus and the regular locus of an algebraic variety A by $\mathrm{Sing}(A)$ and $\mathrm{Reg}(A)$, respectively.

Proposition 6.3. *The algebraic variety $M^*_{\leq k}$ is isomorphic to the algebraic variety $V_{\leq k} \times \mathfrak{n}$, for $0 \leq k \leq r-1$. We have*

$$\Psi(\mathrm{Sing}(M^*_{\leq k})) = \mathrm{Sing}(V_{\leq k}) \times \mathfrak{n}, \quad 0 \leq k \leq r-1.$$

In other words,

$$\mathrm{Sing}(M^*_{\leq k}) = \Phi^{-1}(\mathrm{Sing}(V_{\leq k})), \quad 0 \leq k \leq r-1.$$

Proof. The first assertion is an immediate consequence of Lemmas 6.1 and 6.2. The other assertions follow from the first one. ∎

We wish to find the singular locus $\mathrm{Sing}(V_{\leq k})$ of a generalized determinantal variety $V_{\leq k}$ in $\mathfrak{g}_2(r)$. In the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is of BC_r-type, $V_{\leq k}$ is determined by the graded subalgebra (5.10) or (5.11) of the first kind. As is seen from the classification of simple GLA's of the 2nd kind([8]), the simple GLA (5.11) is of C_r-type in this case. As for the case where $(\mathfrak{g}, \mathfrak{g}_0)$ is of C_r-type, the GLA (5.11) coincides with the original GLA (1.1), more precisely, we have $\mathfrak{g}_{\pm 2}(r) = \mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_0(r) = \mathfrak{g}_0$. Therefore one has only to consider the generalized determinantal varieties arising from a simple GLA (1.1) of C_r-type.

Consider a simple GLA (1.1) of C_r-type. In this case $\mathfrak{g}_1$ is a simple Jordan algebra on which G_0 acts as the structure group. As for (5.12) we have the rank decomposition

$$\mathfrak{g}_1 = V_r \amalg V_{r-1} \amalg \cdots \amalg V_0, \tag{6.2}$$

where V_r is an open subset and $V_0 = (0)$. If we denote by G_0^0 the identity component of G_0, then V_k is a union of the equidimensional orbits [8]:

$$V_k = \amalg_{p+q=k} G_0^0 0_{p,q}, \quad 0 \leq k \leq r, \tag{6.3}$$

where $0_{p,q}$ is the same as in the proof of Proposition 5.13. (5.16) is still valid by replacing $\mathfrak{g}_2(r)$ by $\mathfrak{g}_1$. Therefore $V_{\le k}$ is an algebraic variety in $\mathfrak{g}_1$ defined over $\mathbb{R}$. We have $V_{\le r-1} = \{x \in \mathfrak{g}_1 : \det P(x) = 0\}$. Since $\det P(x)$ is a power of the generic norm ν of the Jordan algebra $\mathfrak{g}_1$, the defining ideal $I(V_{\le r-1})$ of $V_{\le r-1}$ is generated by the irreducible polynomial ν. The variety $V_{\le k}$ is a conic variety, since G_0^0 contains the one-dimensional center acting on $\mathfrak{g}_1$ as homotheties. Therefore the defining ideal $I(V_{\le k})$ of $V_{\le k}$ is a homogeneous ideal. Let $I(V_{\le k})_m$ denote the totality of homogeneous polynomials in $I(V_{\le k})$ of degree m.

Proposition 6.4. *For a simple GLA (1.1) with $r \ge 2$, the singular locus* $\mathrm{Sing}(V_{\le 1})$ *of the generalized determinantal variety $V_{\le 1}$ in $\mathfrak{g}_1$ coincides with $V_0 = (0)$.*

Proof. Let $\mathfrak{a}_1$ be the linear span of $E_1, \cdots, E_r$ in $\mathfrak{g}_1$. Then it is known [1,8] that

$$\mathfrak{g}_1 = G_0^0 \mathfrak{a}_1. \tag{6.4}$$

First we claim that $I(V_{\le 1})_1 = \emptyset$. Suppose the contrary. One can then choose a nonzero linear form f on $\mathfrak{g}_1$ such that $f(V_1) = 0$. Since $r \ge 2$, there exists a point $x_0 \in \mathfrak{g}_1$ such that $f(x_0) \ne 0$. By (6.4) one can assume that x_0 lies in $\mathfrak{a}_1$. Since E_i is conjugate to E_1 under G_0^0, $E_1, \cdots, E_r$ belong to V_1. By the assumption for f we have that $f(E_i) = 0, 1 \le i \le r$, which implies that f is identically zero on $\mathfrak{a}_1$. This contradicts the fact that $f(x_0) \ne 0$, which shows the claim that $I(V_{\le 1})_1 = \emptyset$. Recall that the variety $V_{\le 1}$ is defined over $\mathbb{R}$ (cf.6.2). One can choose a generator $\{f_1, \cdots, f_s\}$ of the ideal $I(V_{\le 1})$ such that each polynomial f_i is homogeneous and defined over $\mathbb{R}$. From the above argument, it follows that $\deg f_i \ge 2$. Consequently $(df_i)_0 = 0, 1 \le i \le s$, which shows that 0 is a singularity of $V_{\le 1}$. Obviously we have that $\mathrm{Reg}(V_{\le 1}) \supset V_1$. Therefore we conclude $\mathrm{Sing}(V_{\le 1}) = V_0$. ■

In this paragraph we treat the case where the GLA (1.1) is complex simple of C_r-type. The subspace $\mathfrak{g}_1$ is then a complex simple Jordan algebra. The following is a list of complex simple Jordan algebras.

Type	$\mathfrak{g}_1$	r
I	$M_n(\mathbb{C})$	n
II	$\mathrm{Alt}_{2n}(\mathbb{C})$	n
III	$\mathrm{Sym}_n(\mathbb{C})$	n
IV	$\mathbb{C}^n$	2
VI	$H_3(\mathbb{O}^{\mathbb{C}})$	3

Here $H_3(\mathbb{O}^{\mathbb{C}})$ denotes the exceptional simple Jordan algebra of 3×3 Hermitian matrices with entries in complex octonions $\mathbb{O}^{\mathbb{C}}$.

Proposition 6.5. *For any complex simple Jordan algebra $\mathfrak{g}_1$ with $r \ge 2$, we have*

$$\mathrm{Sing}(V_{\le k}) = V_{\le k-1}, \qquad 1 \le k \le r-1. \tag{6.5}$$

Proof. (i) For the case of types I or III, $X \in V_{\le k}$ if and only if $\mathrm{rk}\, X \le k$. For the case of type II, $X \in V_{\le k}$ if and only if $\mathrm{rk}\, X \le 2k$. In those three cases, (6.5) is well-known (see for example [11]).

(ii) Consider the case of type IV. In this case we have $r = 2$ and $\mathfrak{g}_1 = V_2 \amalg V_1 \amalg V_0$. Therefore (6.5) follows from Proposition 6.4.

(iii) Now we consider the case of type VI. In this case $\mathfrak{g}_1$ can be identified with $H_3(\mathbb{O}^{\mathbb{C}})$ in a such a way that $E_i (i = 1, 2, 3)$ is sent to the diagonal matrix $(\delta_{i1}, \delta_{i2}, \delta_{i3})$. An element $x \in H_3(\mathbb{O}^{\mathbb{C}})$ is expressed as

$$x = \begin{pmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{pmatrix}, \qquad \xi_i \in \mathbb{C}, \quad a, b, c \in \mathbb{O}^{\mathbb{C}}. \tag{6.6}$$

The generic norm $\nu(x)$ of x is given by

$$\nu(x) = \xi_1\xi_2\xi_3 - \xi_1 n(a) - \xi_2 n(b) - \xi_3 n(c) + t(abc),$$

where n and t denote respectively the norm and the trace of an octonion. We express an element $a \in \mathbb{O}^{\mathbb{C}}$ as $a = \sum_{i=0}^{7} a_i e_i$, where $\{e_i\}$ is the canonical basis of $\mathbb{O}^{\mathbb{C}}$. The variety $V_{\leq 2}$ is defined by the single equation $\nu(x) = 0, x \in \mathfrak{g}_1$. We then have

$$\begin{aligned} d\nu = {} & \big(\xi_2\xi_3 - n(a)\big)\, d\xi_1 + \big(\xi_1\xi_3 - n(b)\big)\, d\xi_2 + \big(\xi_1\xi_2 - n(c)\big)\, d\xi_3 \\ & + \sum_{i=0}^{7}\Big(-2\xi_1 a_i + \frac{\partial}{\partial a_i} t(abc)\Big)\, da_i + \sum_{i=0}^{7}\Big(-2\xi_2 b_i + \frac{\partial}{\partial b_i} t(abc)\Big)\, db_i \\ & + \sum_{i=0}^{7}\Big(-2\xi_3 c_i + \frac{\partial}{\partial c_i} t(abc)\Big)\, dc_i. \end{aligned} \tag{6.7}$$

Now let $x \in V_{\leq 1}$. From (6.4) and (6.3) it follows that there exists an element $g \in G_0^0$ such that $gx \in \mathfrak{a}_1 \cap V_{\leq 1}$, in other words, gx is a diagonal matrix $\mathrm{diag}(\xi_1, \xi_2, \xi_3) \in V_{\leq 1}$, which implies that at least two of ξ_1, ξ_2, ξ_3 are zero. Therefore we have from (6.7) that

$$(d\nu)_{gx} = \xi_2\xi_3\, d\xi_1 + \xi_1\xi_3\, d\xi_2 + \xi_1\xi_2\, d\xi_3 = 0$$

Therefore, in view of the relative invariance of ν under G_0, we have $(d\nu)_x = 0$. By the Jacobian criterion, we obtain $V_{\leq 1} \subset \mathrm{Sing}(V_{\leq 2})$. On the other hand, clearly we have $V_2 \subset \mathrm{Reg}(V_{\leq 2})$. Consequently we conclude that $V_{\leq 1} = \mathrm{Sing}(V_{\leq 2})$. The equality $V_0 = \mathrm{Sing}(V_{\leq 1})$ follows from Proposition 6.4. ■

We denote the defining ideal of an algebraic variety A by $I(A)$.

Corollary 6.6. *Let $V_{\leq k} (1 \leq k \leq r-1)$ be a generalized determinantal variety in a complex simple Jordan algebra $\mathfrak{g}_1$. Then there exists a basis $\{f_1, \ldots, f_{s_k}\}$ of $I(V_{\leq k})$ such that each f_i is defined over $\mathbb{R}$ and that $df_i \in I(\mathrm{Sing}(V_{\leq k})), 1 \leq i \leq s_k$, in other words, $(df_i)_p = 0$, $1 \leq i \leq s_k$ for each point $p \in \mathrm{Sing}(V_{\leq k})$.*

Proof. Note that $V_{\leq k}$ is defined over $\mathbb{R}$ (cf.6.2). For types I and III, we choose, as a generator of $I(V_{\leq k})$, the totality of $(k+1)$-minors of a generic element of $\mathfrak{g}_1$. For Type II, we choose, as a generator of $I(V_{\leq k})$, the totality of the Pfaffians of principal $(2k+2)$-submatrices of a generic element of $\mathfrak{g}_1$. Then the assertion is well-known for those cases (cf.[11]). For the remaining two cases, the assertion was shown in the proof of Propositions 6.4 and 6.5. ■

In this paragraph we wish to show that Proposition 6.5 is valid for a real simple Jordan algebra. Let us consider a real simple but not complex simple GLA(1.1) of C_r-type: $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. r is the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. In this case $\mathfrak{g}_1$ is a real simple but not complex simple Jordan algebra. Consider the complexification of the GLA $\mathfrak{g}$:

$$\mathfrak{g}^c = \mathfrak{g}^c_{-1} + \mathfrak{g}^c_0 + \mathfrak{g}^c_1$$

Let $\bar{r}$ be the split rank of the symmetric pair $(\mathfrak{g}^c, \mathfrak{g}^c_0)$. The following is a list of real simple GLAs of C_r-type and their complexifications:

Type I

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{sl}(n,\mathbb{C}) + \mathfrak{sl}(n,\mathbb{C}) + \mathbb{C}, M_n(\mathbb{C})\big), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \big(\mathfrak{sl}(2n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) + \mathfrak{sl}(n,\mathbb{R}) + \mathbb{R}, M_n(\mathbb{R})\big), & r = n, \\ \big(\mathfrak{su}(n,n), \mathfrak{sl}(n,\mathbb{C}) + \mathbb{R}, H_n(\mathbb{C})\big), & r = n, \end{cases} \end{cases}$$

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{sl}(4n,\mathbb{C}), \mathfrak{sl}(2n,\mathbb{C}) + \mathfrak{sl}(2n,\mathbb{C}) + \mathbb{C}, M_{2n}(\mathbb{C})\big), & \bar{r} = 2n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{sl}(2n,\mathbb{H}), \mathfrak{sl}(n,\mathbb{H}) + \mathfrak{sl}(n,\mathbb{H}) + \mathbb{R}, M_n(\mathbb{H})\big), & r = n, \end{cases}$$

Type II

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{so}(4n,\mathbb{C}), \mathfrak{gl}(2n,\mathbb{C}), \mathrm{Alt}_{2n}(\mathbb{C})\big), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \big(\mathfrak{so}(2n,2n), \mathfrak{gl}(2n,\mathbb{R}), \mathrm{Alt}_{2n}(\mathbb{R})\big), & r = n, \\ \big(\mathfrak{so}^*(4n), \mathfrak{gl}(n,\mathbb{H}), H_n(\mathbb{H})\big), & r = n, \end{cases} \end{cases}$$

Type III

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{sp}(n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C}), \mathrm{Sym}_n(\mathbb{C})\big), & \bar{r} = n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}), \mathrm{Sym}_n(\mathbb{R})\big), & r = n, \end{cases}$$

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{sp}(2n,\mathbb{C}), \mathfrak{gl}(2n,\mathbb{C}), \mathrm{Sym}_{2n}(\mathbb{C})\big), & \bar{r} = 2n, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{sp}(n,n), \mathfrak{gl}(n,\mathbb{H}), \mathrm{SH}_n(\mathbb{H})\big), & r = n, \end{cases}$$

Type IV

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(\mathfrak{so}(n+2,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}) + \mathbb{C}, \mathbb{C}^n\big), & \bar{r} = 2, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \big(\mathfrak{so}(p+1,q+1), \mathfrak{so}(p,q) + \mathbb{R}, \mathbb{R}^n\big), & r = \begin{cases} 1 & (p = 0), \\ 2 & (p \geq 1), \end{cases} \\ \qquad p \leq q,\ p+q = n, & \end{cases}$$

Type VI

$$\begin{cases} (\mathfrak{g}^c, \mathfrak{g}^c_0, \mathfrak{g}^c_1) = \big(E_7^{\mathbb{C}}, E_6^{\mathbb{C}} + \mathbb{C}, H_3(\mathbb{O}^{\mathbb{C}})\big), & \bar{r} = 3, \\ (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_1) = \begin{cases} \big(E_{7(7)}, E_{6(6)} + \mathbb{R}, H_3(\mathbb{O}')\big), & r = 3, \\ \big(E_{7(-25)}, E_{6(-26)} + \mathbb{R}, H_3(\mathbb{O})\big), & r = 3. \end{cases} \end{cases}$$

In the above list, $\mathbb{H}$ denotes the quaternion algebra. $\mathbb{O}$, (resp. $\mathbb{O}'$) denotes the octonion (resp. split octonion) algebra. $H_n(\mathbb{F}), \mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{O}'$, denotes the Jordan algebra of $n \times n$ $\mathbb{F}$-Hermitian matrices. $\mathrm{SH}_n(\mathbb{H})$ denotes the Jordan algebra of $n \times n$ skew-Hermitian quaternion matrices. Note that the generalized determinantal varieties in a complex simple Jordan algebra are defined over $\mathbb{R}$.

From a result of Takeuchi [14] we have

Proposition 6.7. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_1^c$ be as above, and let $\mathfrak{g}_1 = \amalg_{k=0}^r V_k$ and $\mathfrak{g}_1^c = \amalg_{k=0}^{\bar{r}} \widetilde{V}_k$ be the rank decomposition (cf.(6.2)) of $\mathfrak{g}_1$ and $\mathfrak{g}_1^c$, respectively. Suppose that $\bar{r} = r$. Then $V_{\le k}(1 \le k \le r-1)$ coincides with the set of $\mathbb{R}$-rational points of the complex algebraic variety $\widetilde{V}_{\le k}$. Suppose that $\bar{r} = 2r$. Then $V_{\le k}(1 \le k \le r-1)$ coincides with the sets of $\mathbb{R}$-rational points of the algebraic varieties $\widetilde{V}_{\le 2k}$ and of $\widetilde{V}_{\le 2k+1}$.*

From Proposition 6.7, we obtain

Lemma 6.8. *Let $f_1, \ldots, f_{s_k}$ be real polynomials on $\mathfrak{g}_1$, and let $\widetilde{f}_i(1 \le i \le s_k)$ be the natural extension of f_i to $\mathfrak{g}_1^c$. Then $I(V_{\le k})(1 \le k \le r-1)$ is generated by $f_1, \ldots, f_{s_k}$ if and only if $I(\widetilde{V}_{\le k})$ (resp. $I(\widetilde{V}_{\le 2k})$) is generated by $\widetilde{f}_1, \ldots, \widetilde{f}_{s_k}$ for $\bar{r} = r$ (resp. $\bar{r} = 2r$.).*

Proposition 6.9. *For a real simple (not complex simple) Jordan algebra $\mathfrak{g}_1$, we have*

$$\mathrm{Sing}(V_{\le k}) = V_{\le k-1}, \qquad 1 \le k \le r-1.$$

Proof. Let θ be the conjugation of $\mathfrak{g}_1^c$ with respect to $\mathfrak{g}_1$. Since $\widetilde{V}_{\le k}$ is θ-stable, $\mathrm{Sing}(\widetilde{V}_{\le k})$ is also θ-stable. Let $\big(\mathrm{Sing}(\widetilde{V}_{\le k})\big)_\theta$ be the set of θ-fixed points in $\mathrm{Sing}(\widetilde{V}_{\le k})$. Suppose first $\bar{r} = r$. Since $\widetilde{V}_{\le k}$ is a conic variety defined over $\mathbb{R}$ (cf.6.2), one can choose a generator $\{\widetilde{f}_1, \ldots, \widetilde{f}_{s_k}\}$ of $I(\widetilde{V}_{\le k})$ such that each $\widetilde{f}_i$ is homogeneous and defined over $\mathbb{R}$. By Corollary 6.6, $d\widetilde{f}_i \in I(\mathrm{Sing}(\widetilde{V}_{\le k}))$. Let $f_i = \widetilde{f}_i|_{\mathfrak{g}_1}$. Then $\{f_1, \ldots, f_{s_k}\}$ is a generator of $I(V_{\le k})$, by Lemma 6.8. Let $p \in \big(\mathrm{Sing}(\widetilde{V}_{\le k})\big)_\theta$. Then $p \in \big(\widetilde{V}_{\le k}\big)_\theta = V_{\le k}$, by Proposition 6.7. We have $(df_i)_p = (d\widetilde{f}_i)_p = 0$, which implies that $p \in \mathrm{Sing}(V_{\le k})$. Hence, by Propositions 6.5 and 6.7, we have

$$V_{\le k-1} = (\widetilde{V}_{\le k-1})_\theta = \big(\mathrm{Sing}(\widetilde{V}_{\le k})\big)_\theta \subset \mathrm{Sing}(V_{\le k}). \tag{6.8}$$

In view of the inclusion $V_k \subset \mathrm{Reg}(V_{\le k})$, we conclude $V_{\le k-1} = \mathrm{Sing}(\widetilde{V}_{\le k})_\theta = \mathrm{Sing}(V_{\le k})$. As for the case $\bar{r} = 2r$, we should replace (6.8) by the equality

$$V_{\le k-1} = (\widetilde{V}_{\le 2k-2})_\theta = (\widetilde{V}_{\le 2k-1})_\theta = \big(\mathrm{Sing}(\widetilde{V}_{\le 2k})\big)_\theta \subset \mathrm{Sing}(V_{\le k}).$$

■

Combining Propositions 6.5 and 6.9, we have

Theorem 6.10. *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA of C_r-type, and let $\mathfrak{g}_1 = \amalg_{k=0}^r V_k$ be the rank decomposition. Then the closure $\bar{V}_k$ of V_k is the generalized determinantal variety $V_{\le k}$, and $\mathrm{Sing}(V_{\le k}) = V_{\le k-1}$ for $1 \le k \le r-1$.*

From Theorem 6.10 and Proposition 6.3 we have

Theorem 6.11. *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a simple GLA, and let r be the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Let $\mathfrak{g}_1 \oplus \mathfrak{g}_1' = \amalg_{k=0}^{r} M_k^*$ be the rank decomposition (cf. 5.4) of the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Then the closure $\overline{M_k^*}$ of M_k^* in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$ is the algebraic variety $M_{\le k}^*$, and*

$$\mathrm{Sing}(M_{\le k}^*) = M_{\le k-1}^*, \qquad \mathrm{Reg}(M_{\le k}^*) = M_k^*, \qquad 1 \le k \le r-1.$$

Now we go back to the full G-orbits M_k.

Lemma 6.12. *$M_{\le k}$ $(0 \le k \le r-1)$ is a real analytic set in $\widetilde{M}$.*

Proof. Choose a point $p_0 \in M_{\le k}$. Then one can find an element $g \in G$ such that $g(p_0) \in \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Choose a neighborhood U of p_0 in $\widetilde{M}$ in such a way that $U' := g(U) \subset \mathfrak{g}_1 \oplus \mathfrak{g}_1'$. Let $p \in U$. Then we have that $p \in U \cap M_{\le k}$ if and only if $g(p) \in U' \cap M_{\le k}^*$. Let $\{f_1, \ldots, f_{s_k}\}$ be a basis of the ideal $I(M_{\le k}^*)$. Then $M_{\le k}$ is expressed in U as

$$U \cap M_{\le k} = \{ p \in U : (f_i \circ g)(p) = 0,\ 1 \le i \le s_k \},$$

which implies that $M_{\le k}$ is a real analytic set of $\widetilde{M}$. ■

A point $p \in M_{\le k}$ is a regular point of $M_{\le k}$, if there exists a neighborhood U of p in $\widetilde{M}$ such that $U \cap M_{\le k}$ is a smooth manifold of dimension $d_k := \dim M_k$. Otherwise we say that p is a singular point of $M_{\le k}$. We denote by $\mathrm{Reg}(M_{\le k})$ (resp. $\mathrm{Sing}(M_{\le k})$) the regular (resp. singular) locus of $M_{\le k}$. Finally we get the following theorem which gives the stratification of $\widetilde{M}$ by G-orbits.

Theorem 6.13. *For $1 \le k \le r-1$, we have $\mathrm{Reg}(M_{\le k}) = M_k$ and $\mathrm{Sing}(M_{\le k}) = M_{\le k-1}$.*

Proof. Let $p \in \mathrm{Reg}(M_{\le k})$. Choose an element $g \in G$ such that $g(p) \in M_{\le k}^*$. Then, since p is a regular point of $M_{\le k}$, $g(p)$ lies in $\mathrm{Reg}(M_{\le k}^*) = M_k^*$ by Theorem 6.11. This implies $p \in M_k$, or equivalently, $\mathrm{Reg}(M_{\le k}) = M_k$. Similarly we can show $\mathrm{Sing}(M_{\le k}) = M_{\le k-1}$. ■

Corollary 6.14. *Suppose that a diffeomorphism f of $\widetilde{M}$ leaves the open orbit M_r stable. Then f leaves all other orbits M_k $(0 \le k \le r-1)$ stable.*

Proof. When $r = 1$, the assertion is trivial. Assume that $r \ge 2$. By the assumption, f leaves $M_{\le r-1}$ stable. Let k be an integer, $1 \le k \le r-1$. Then it is enough to prove that if $f(M_{\le k}) = M_{\le k}$, then $f(M_k) = M_k$. Put $f(M_k)^* := f(M_k) \cap (\mathfrak{g}_1 \oplus \mathfrak{g}_1')$. First we want to show $f(M_k)^* \subset M_k^*$. Let $p \in f(M_k)^*$. Since $f(M_k)$ is an open C^∞-submanifold of $M_{\le k}$, $f(M_k)^*$ is expressed, in a neighborhood of p in $\mathfrak{g}_1 \oplus \mathfrak{g}_1'$, by the same polynomial equations as for the algebraic variety $M_{\le k}^*$. Consequently the tangent spaces $T_p(M_{\le k}^*)$ and $T_p(f(M_k)^*)$ are identical, which implies that

$$\dim T_p(M_{\le k}^*) = \dim T_p(f(M_k)^*) = \dim f(M_k) = \dim M_k = \dim M_{\le k}^*.$$

Therefore it follows that p is a regular point of $M^*_{\le k}$. By Theorem 6.11, we have $p \in M^*_k$, and hence $f(M_k)^* \subset M^*_k$. Now suppose that $f(M_k) \not\subset M_k$. Then we have $f(M_k) \cap M_{\le k-1} \ne \emptyset$. Since $M^*_{\le k-1}$ is open dense in $M_{\le k-1}$, we have $f(M_k)^* \cap M^*_{\le k-1} = f(M_k) \cap M^*_{\le k-1} \ne \emptyset$. This contradicts the inclusion $f(M_k)^* \subset M^*_k$. We have thus proved $f(M_k) \subset M_k$. The converse inclusion can be proved by replacing f by f^{-1} in the assumption $f(M_{\le k}) = M_{\le k}$. ■

7. Double foliation on the minimal boundary orbits

In this section, we always assume that M is of BC_r-type. In §2, we considered the double foliation on $\widetilde{M}$, $\mathcal{M}^{\pm} = \{\, M^{\pm}(g_1 0^-, g_2 0^+) : g_1, g_2 \in G \,\}$. $\mathcal{M}^{\pm}$ naturally induce a double foliation $F_0^{\pm}$ on the minimal boundary orbit M_0. The leaves $F_0^{\pm}(p)$ of $F_0^{\pm}$ through a point $p \in M_0$ are given by the intersection $M^{\mp}(p) \cap M_0$.

Lemma 7.1. *The leaves of $F_0^{\pm}$ through the origin $(0^-, a_r\, 0^+) \in M_0$ are given by*

$$F_0^-(0^-, a_r\, 0^+) = U^-(0^-, a_r\, 0^+) = U^-/Q_r,$$
$$F_0^+(0^-, a_r\, 0^+) = a_r\, U^+ a_r^{-1}(0^-, a_r\, 0^+) = a_r\, U^+ a_r^{-1}/Q_r.$$

Proof. By the definition, $F_0^{\pm}(0^-, a_r\, 0^+) = M^{\mp}(0^-, a_r\, 0^+) \cap G(0^-, a_r\, 0^+)$. Let $(g0^-, ga_r\, 0^+) \in F_0^+(0^-, a_r\, 0^+)$, $g \in G$. Then $(g0^-, ga_r\, 0^+) \in M^-(0^-, a_r\, 0^+)$, which implies that $ga_r\, 0^+ = a_r\, 0^+$, or equivalently, $g \in a_r\, U^+ a_r^{-1}$. Conversely, let $u \in U^+$. Then $a_r u a_r^{-1}(0^-, a_r\, 0^+) = (a_r u a_r^{-1} 0^-, a_r\, 0^+) \in G(\, 0^-, a_r\, 0^+) \cap M^-(0^-, a_r\, 0^+)$. ■

Lemma 7.2. *The double foliation $F_0^{\pm}$ arises from the subspaces $\mathfrak{g}_1^{\pm}(r)$ of the GLA* (4.10).

Proof. Let $\mathfrak{u}^{\pm} = \mathrm{Lie}\, U^{\pm}$. By Lemma 7.1, the tangent spaces at $(0^-, a_r\, 0^+)$ to the leaves $F_0^{\pm}(0^-, a_r\, 0^+)$ are identified with the factor spaces $\mathfrak{u}^-/\mathfrak{q}_r$ and $(\mathrm{Ad}\, a_r)\mathfrak{u}^+/\mathfrak{q}_r$. By (4.15) and Lemma 4.6, we have

$$\mathfrak{u}^- = \mathfrak{g}_{-1} + \mathfrak{g}_0 = \mathfrak{g}_{-2}(r) + \mathfrak{g}_{-1}(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r) = \mathfrak{q}_r + \mathfrak{g}_1^+(r). \tag{7.1}$$

Also, by (4.15) and Lemma 5.2, we have

$$\begin{aligned}(\mathrm{Ad}\, a_r)\mathfrak{u}^+ &= (\mathrm{Ad}\, a_r)\big(\mathfrak{g}_{-1}^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_1^+(r) + \mathfrak{g}_1^-(r) + \mathfrak{g}_2(r)\big) \\ &= \mathfrak{g}_1^-(r) + \mathfrak{g}_0(r) + \mathfrak{g}_{-1}^+(r) + \mathfrak{g}_{-1}^-(r) + \mathfrak{g}_{-2}(r) \\ &= \mathfrak{q}_r + \mathfrak{g}_1^-(r).\end{aligned} \tag{7.2}$$

Therefore $\mathfrak{u}^-/\mathfrak{q}_r$ and $(\mathrm{Ad}\, a_r)\mathfrak{u}^+/\mathfrak{q}_r$ can be identified with $\mathfrak{g}_1^+(r)$ and $\mathfrak{g}_1^-(r)$, respectively. ■

Let $E := Z_r - 2Z$. Then E is a central element of $\mathfrak{g}_0(r)$. It follows from (4.14) and (4.15) that

$$\mathrm{ad}\, E = \begin{cases} \ \ 0 & \text{on } \mathfrak{g}_{\mathrm{ev}}(r), \\ \ \ 1 & \text{on } \mathfrak{g}_{\pm 1}^+(r), \\ -1 & \text{on } \mathfrak{g}_{\pm 1}^-(r). \end{cases} \tag{7.3}$$

Lemma 7.3. *Let $g \in C(Z_r)$ and let $I = \mathrm{ad}_{\mathfrak{g}_1(r)} E$. Then the following three conditions are equivalent*:

(i) $g\big(\mathfrak{g}_1^{\pm}(r)\big) = \mathfrak{g}_1^{\pm}(r)$,

(ii) $gI = Ig$ *on* $\mathfrak{g}_1(r)$,

(iii) $g(E) = E$.

Proof. The only non-trivial assertion is the implication (ii) $\to$ (iii). Suppose (ii). Since Z, $Z_r \in \mathfrak{a}$, we have $\tau(Z_r) = -Z_r$ and $\tau(Z) = -Z$, and hence $\tau(E) = -E$. This implies that

$$\tau\big(\mathrm{ad}_{\mathfrak{g}_1(r)} E\big)\tau = -\,\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E. \tag{7.4}$$

Consider the inner product $<\ ,\ >$ on $\mathfrak{g}_1(r)$ defined by $< X, Y >= -(X, \tau Y)$. Let us denote by $g_{\pm}$ the restrictions of the actions of g to $\mathfrak{g}_{\pm 1}(r)$, and denote by g_+^* the adjoint operator of g_+ with respect to $<\ ,\ >$. Then we have that I is self-adjoint with respect to $<\ ,\ >$, and hence, by (ii) we have $(g_+^*)^{-1} I = I (g_+^*)^{-1}$. We also have $g_- = \tau(g_+^*)^{-1}\tau$. Therefore it follows from (7.4) that

$$\begin{aligned} g_-(\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E)(g_-)^{-1} &= \tau(g_+^*)^{-1}\tau(\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E)\tau(g_+^*)\tau \\ &= -\tau(g_+^*)^{-1}(\mathrm{ad}_{\mathfrak{g}_1(r)} E)(g_+^*)\tau = -\tau I \tau = \mathrm{ad}_{\mathfrak{g}_{-1}(r)} E, \end{aligned}$$

which implies that g_- commutes with $\mathrm{ad}_{\mathfrak{g}_{-1}(r)} E$. Combining this with (7.3) and (ii), we have that g commutes with $\mathrm{ad}\, E$ on the whole $\mathfrak{g}$. This implies (iii). ■

Lemma 7.4. $C(Z, Z_r) = \big\{\, g \in C(Z_r) : g\big(\mathfrak{g}_1^{\pm}(r)\big) = \mathfrak{g}_1^{\pm}(r) \,\big\}$.

Proof. Let $g \in C(Z_r)$. Then $g \in C(Z, Z_r)$ if and only if $g(E) = E$. Hence the assertion follows from Lemma 7.3. ■

Remark 7.5. Lemmas 7.2, 7.4, Proposition 4.8 and Corollary 4.9 imply that our flag manifold ($M_0 = G/Q_r$, $F_0^{\pm}$) with double foliation $F_0^{\pm}$ is a so-called pseudo-product manifold associated to the simple GLA (4.10) with decomposition (4.14), in the sense of Tanaka [15].

We give a list of simple parahermitian symmetric spaces M of BC_r-type and the corresponding minimal boundary orbits M_0. The list is obtained by extracting those ones satisfying (4.14) among all simple GLAs of the second kind classified in [7].

Type I ($r = p$).

$$M = \mathrm{SL}(n, \mathbb{F})/\,\mathrm{S}\big(\mathrm{GL}(p, \mathbb{F}) \times \mathrm{GL}(n-p, \mathbb{F})\big), \qquad \mathbb{F} = \mathbb{R}, \mathbb{H}, \mathbb{C}, \quad 1 \le p < n-p,$$

$$M_0 = \begin{cases} \mathrm{SO}(n)/\,\mathrm{S}\big(\mathrm{O}(p) \times \mathrm{O}(n-2p) \times \mathrm{O}(p)\big), & \mathbb{F} = \mathbb{R}, \\ \mathrm{Sp}(n)/\,\mathrm{Sp}(p) \times \mathrm{Sp}(n-2p) \times \mathrm{Sp}(p), & \mathbb{F} = \mathbb{H}, \\ \mathrm{SU}(n)/\,\mathrm{S}\big(\mathrm{U}(p) \times \mathrm{U}(n-2p) \times \mathrm{U}(p)\big), & \mathbb{F} = \mathbb{C}. \end{cases}$$

Type II $(r = n)$.

$$\begin{cases} \ M = \mathrm{SO}^0(2n+1, 2n+1)/\,\mathrm{GL}^0(2n+1, \mathbb{R}), \\ M_0 = \mathrm{SO}(2n+1) \times \mathrm{SO}(2n+1)/\,\mathrm{SO}(2n), \end{cases}$$
$$\begin{cases} \ M = \mathrm{SO}(4n+2, \mathbb{C})/\,\mathrm{GL}(2n+1, \mathbb{C}), \\ M_0 = \mathrm{SO}(4n+2)/\,\mathrm{U}(2n) \cdot \mathbb{T}^1. \end{cases}$$

Type V

$$\begin{cases} \ M = E_{6(6)}/\,\mathrm{Spin}(5,5) \cdot \mathbb{R}^+, & (r = 2), \\ M_0 = \mathrm{Sp}(4)/\,\mathrm{Spin}(4) \times \mathrm{Spin}(4), \end{cases}$$
$$\begin{cases} \ M = E_{6(-26)}/\,\mathrm{Spin}(1,9) \cdot \mathbb{R}^+, & (r = 1), \\ M_0 = F_4/\,\mathrm{Spin}(8), \end{cases}$$
$$\begin{cases} \ M = E_6^{\mathbb{C}}/\,\mathrm{Spin}(10, \mathbb{C}) \cdot \mathbb{C}^*, & (r = 2), \\ M_0 = E_6/\,\mathrm{Spin}(8) \cdot \mathbb{T}^2. \end{cases}$$

8. Determination of the automorphism groups of M

Let $(M = G/G_0, F^{\pm})$ be the parahermitian symmetric space associated with a simple GLA (1.1). In this paragraph we assume M to be of BC_r-type. For the minimal boundary orbit $(M_0 = G/Q_r, F_0^{\pm})$ with double foliation $F_0^{\pm}$, we define the automorphism group by

$$\mathrm{Aut}(M_0, F_0^{\pm}) = \{\, g \in \mathrm{Diffeo}(M_0) : g_* F_0^{\pm} = F_0^{\pm} \,\}.$$

Tanaka [15] determined this group by establishing a Cartan connection on M_0 and by showing that $(M_0 = G/Q_r, F_0^{\pm})$ is the model space for the Cartan connection. Therefore, taking Remark 7.5 into account, we have

$$\mathrm{Aut}(M_0, F_0^{\pm}) = G. \tag{8.1}$$

Theorem 8.1. *Let $(M = G/G_0, F^{\pm})$ be a parahermitian symmetric space of BC_r-type associated with a simple GLA (1.1). Then*

$$\mathrm{Aut}(M, F^{\pm}) = \mathrm{Aut}(M_0, F_0^{\pm}) = G.$$

Proof. We identify M with its φ-image in $\widetilde{M}$. Since G acts on M effectively and $F^{\pm}$ are G-invariant, the inclusion $G \subset \mathrm{Aut}(M, F^{\pm})$ is clear. Now let $f \in \mathrm{Aut}(M, F^{\pm})$. Then, by Lemma 2.4, f preserves the fibers of the double fibration $M^- \xleftarrow{\pi^-} M \xrightarrow{\pi^+} M^+$. Hence f induces the diffeomorphisms $f^{\pm}$ of $M^{\pm}$ such that $\pi^{\pm} \circ f = f^{\pm} \circ \pi^{\pm}$. Let $\widetilde{f} := f^- \times f^+$. Clearly, the diffeomorphism $\widetilde{f}$ preserves the product structure of $\widetilde{M}$. We claim that $\widetilde{f}|_M = f$. In fact, let $p \in M$, and let $q = f(p) \in M$. We write $p = (p^-, p^+)$ and $q = (q^-, q^+)$, where $p^{\pm}$, $q^{\pm} \in M^{\pm}$. The relation $\varpi^{\pm} \cdot \varphi = \pi^{\pm}$ (cf. §2) implies that $q^{\pm} = \varpi^{\pm}(q) = \varpi^{\pm}\big(f(p)\big) = f^{\pm}\big(\pi^{\pm}(p)\big) = f^{\pm}(p^{\pm})$. Hence $f(p) = (q^-, q^+) = \big(f^-(p^-), f^+(p^+)\big) = (f^- \times f^+)(p^-, p^+) = \widetilde{f}(p)$. Since $\widetilde{f}$ leaves M invariant, by Corollary 6.14 $\widetilde{f}$ leaves M_0 invariant. Obviously

$f_0 := \widetilde{f}|_{M_0}$ belongs to $\mathrm{Aut}(M_0, F_0^{\pm})$. We wish to show that $\widetilde{f}$ can be uniquely recovered by its restriction f_0. Corresponding to the expression $\widetilde{M} = M^- \times (M^+)_r$, one can express $\widetilde{f}$ as $\widetilde{f} = f_1 \times f_2$, where f_1 and f_2 are diffeomorphisms of M^- and $(M^+)_r$, respectively. It follows from Lemma 7.1 that the leaves of its double foliation $F_0^{\pm}$ arise as the fibers of the double fibration $M^- \xleftarrow{\pi_0^-} M_0 \xrightarrow{\pi_0^+} (M^+)_r$ given in Theorem 3.1 (iv). Moreover this double fibration of M_0 is just the restriction of the trivial double fibration $M^- \xleftarrow{\varpi_0^-} \widetilde{M} \xrightarrow{\varpi_0^+} (M^+)_r$ (cf. (5.3)). Therefore, if we denote by f_0^- and f_0^+ the diffeomorphisms of M^- and $(M^+)_r$ induced by f_0, then it follows that $f_0^- = f_1$ and $f_0^+ = f_1$. We have thus shown that $\widetilde{f}$ is uniquely recovered from f_0. As a result, the correspondence $f \mapsto f_0$ is an injective homomorphism of $\mathrm{Aut}(M, F^{\pm})$ into $\mathrm{Aut}(M_0, F_0^{\pm})$. Consequently, in view of (8.1) we have that $\mathrm{Aut}(M, F^{\pm}) = G$. ∎

In this paragraph we are concerned with C_r-type. Under this assumption, for the GLA (4.10) we have $\mathfrak{g}_{\pm 1}(r) = (0)$, $\mathfrak{g}_{\pm 2}(r) = \mathfrak{g}_{\pm 1}$, $\mathfrak{g}_0(r) = \mathfrak{g}_0$, $Z_r = Z$ and hence $C(Z_r) = C(Z) = G_0$. The rank decomposition (5.12) becomes $\mathfrak{g}_1 = \coprod_{k=0}^{r} V_k$, where V_k is a union of equi-dimensional G_0-orbits. Now consider the G_0-stable conic algebraic set $V_{\le r-1}$, which is the boundary ∂V_r of V_r. One can extend the cone ∂V_r to a cone field on the whole M^- by using the G-action on M^-. We call the cone field a generalized conformal structure $\mathcal{K}$ ([2]). One can consider the automorphism group $\mathrm{Aut}(M^-, \mathcal{K})$, the totality of diffeomorphisms leaving the cone field $\mathcal{K}$ invariant. This group was determined for each symmetric R-space M^- ([2]):

$$\mathrm{Aut}(M^-, \mathcal{K}) = \begin{cases} G, & r \ge 2, \\ \mathrm{Diffeo}(M^-), & r = 1. \end{cases} \tag{8.2}$$

Recall that $U^- = a_r U^+ a_r^{-1}$ for C_r-type (cf. Theorem 3.1). This is equivalent to the condition $a_r 0^+ = 0^-$, and the new origin $(0^-, a_r 0^+)$ becomes $(0^-, 0^-)$. By (5.3), $\widetilde{M}$ takes the form $\widetilde{M} = M^-(0^-, 0^-) \times M^+(0^-, 0^-) = M^- \times M^-$. Further the minimal boundary orbit M_0 becomes $M_0 = G(0^-, 0^-) = G/U^- = M^-$, the diagonal set of $\widetilde{M} = M^- \times M^-$.

Now let $f \in \mathrm{Aut}(M, F^{\pm})$, and let $\widetilde{f} = f^- \times f^+$ be the extension of f to $\widetilde{M}$ given in 8.1. Let us express $\widetilde{f}$ as $\widetilde{f} = f_1 \times f_2$ corresponding to the expression $\widetilde{M} = M^- \times M^-$. By Corollary 6.14, $\widetilde{f}$ leaves M_0, the diagonal of $\widetilde{M}$, invariant, from which we have $f_1 = f_2$, that is, $\widetilde{f} = f_1 \times f_1$. Thus it follows that the correspondence $f \mapsto f_1$ is an injective homomorphism of $\mathrm{Aut}(M, F^{\pm})$ into $\mathrm{Diffeo}(M^-)$. The following lemma is essentially due to Tanaka [15].

Lemma 8.2. *Suppose that $(0^-, 0^-)$ is a fixed point of $\widetilde{f}$. Then the differential $(f_1)_*$ at 0^- leaves the cone ∂V_r stable.*

Proof. Let $Y \in \partial V_r$. Then $(0, tY)$ is a path in $M^*_{\le r-1}$. By the assumption, the curve $\widetilde{f}(0, tY)$ lies in $M^*_{\le r-1}$ for $|t|$ sufficiently small. Therefore $\Phi\big(\widetilde{f}(0, tY)\big) = \Phi\big(f_1(0), f_1(tY)\big) = f_1(tY) - f_1(0) = f_1(tY)$ lies in ∂V_r. Hence $(f_1)_{*0^-}(Y) = \lim_{t \to 0} \frac{1}{t} f_1(tY) \in \partial V_r$. ∎

Lemma 8.3. *Let* $\widetilde{f} = f_1 \times f_1$ *be the extension of* $f \in \mathrm{Aut}(M, F^{\pm})$ *to* $\widetilde{M} = M^- \times M^-$. *Then* $f_1 \in \mathrm{Aut}(M^-, \mathcal{K})$.

Proof. Let $\mathcal{K} = \{(\partial V_r)_p\}_{p \in M^-}$, where $(\partial V_r)_p$ denotes the cone at a point $p \in M^-$ belonging to the field $\mathcal{K}$. Note that, if $p = b \cdot 0^-$, $b \in G$, then $(\partial V_r)_p$ is just the cone $b_*(\partial V_r)$. We have to show that $(f_1)_*(\partial V_r)_p = (\partial V_r)_{f_1(p)}$. Choose an element $a \in G$ such that $a^{-1} f_1 b(0^-) = 0^-$. Then the transformation $a^{-1}\widetilde{f}b$ on $\widetilde{M}$ is the extension of $a^{-1}fb \in \mathrm{Aut}(M, F^{\pm})$. Decompose $a^{-1}fb$ as $a^{-1}f_1 b \times a^{-1} f_1 b$ corresponding to the decomposition $\widetilde{M} = M^- \times M^-$. By Lemma 8.3, we see that $(a^{-1} f_1 b)_{*0^-}$ leaves ∂V_r invariant. Consequently we have

$$(f_1)_{*p}(\partial V_r)_p = (f_1)_{*p}(\partial V_r)_{b \cdot 0^-} = (f_1)_{*p} b_{*0^-}(\partial V_r) = a_{*0^-}(\partial V_r) = (\partial V_r)_{a \cdot 0^-} = (\partial V_r)_{f_1(p)}.$$

This implies that $f_1 \in \mathrm{Aut}(M^-, \mathcal{K})$. ∎

Theorem 8.4. *Let* $(M = G/G_0, F^{\pm})$ *be the parahermitian symmetric space of* C_r*-type associated with a simple GLA* (1.1), *and let* $\mathcal{K}$ *be the above generalized conformal structure on the symmetric R-space* $M^- = G/U^-$. *Then*

$$\mathrm{Aut}(M, F^{\pm}) = \mathrm{Aut}(M^-, \mathcal{K}) = \begin{cases} G, & r \geq 2, \\ \mathrm{Diffeo}(M^-), & r = 1. \end{cases}$$

Proof. Suppose $r \geq 2$. As we noted before Lemma 8.2, the correspondence $\mathrm{Aut}(M, F^{\pm}) \ni f \mapsto f_1 \in \mathrm{Diffeo}(M^-)$ is injective. But, by Lemma 8.3, the image f_1 lies in $\mathrm{Aut}(M^-, \mathcal{K})$. Hence we have the injective homomorphism $\mathrm{Aut}(M, F^{\pm}) \hookrightarrow \mathrm{Aut}(M^-, \mathcal{K})$. Since G is a subgroup of $\mathrm{Aut}(M, F^{\pm})$, it follows from (8.2) that $G \subset \mathrm{Aut}(M, F^{\pm}) \simeq \mathrm{Aut}(M^-, \mathcal{K}) = G$. Suppose next $r = 1$. Then the G-orbit decomposition of $\widetilde{M}$ leaves $\widetilde{M} = M \amalg M^-$. So, for any diffeomorphism f_1 of M^-, $(f_1 \times f_1)|_M$ is an element of $\mathrm{Aut}(M, F^{\pm})$. Therefore we have $\mathrm{Aut}(M, F^{\pm}) \simeq \mathrm{Aut}(M^-, \mathcal{K}) = \mathrm{Diffeo}(M^-)$ (cf. (8.27)). ∎

Remark 8.5. As is seen in the table in 6.4, the parahermitian symmetric space M of C_1-type is $\mathrm{SO}^0(1, q+1)/\mathrm{SO}(q) \cdot \mathbb{R}^+$, and the corresponding symmetric R-space M^- is the conformal q-sphere.

Remark 8.6. In case where $\mathrm{Aut}(M, F^{\pm}) = G$ in Theorems 8.1 and 8.4, we also have

$$\mathrm{Aut}(M, F^{\pm}) = \mathrm{Aut}(M, F^{\pm}, \omega).$$

Remark 8.7. A parahermitian symmetric space $M = G/H$ associated to a simple GLA (1.1) is diffeomorphic to the cotangent bundle of the associated symmetric R-space $M^- = G/U^-$. Let M^- be the quaternionic Grassmannian $\mathrm{Gr}_2(\mathbb{H}^4)$ of quaternionic 2-planes in $\mathbb{H}^4$. There are two parahermitian symmetric spaces:

$$\mathrm{SL}(4, \mathbb{H})/\mathrm{SL}(2, \mathbb{H}) \times \mathrm{SL}(2, \mathbb{H}) \times \mathbb{R}^+ \quad \text{and} \quad E_{6(6)}/\mathrm{Spin}(5,5) \cdot \mathbb{R}^+,$$

which have $\mathrm{Gr}_2(\mathbb{H}^4)$ as the associated symmetric R-spaces. The first one is of C_2-type; the second one is of BC_2-type. Theorems 8.1 and 8.4 tell us that the cotangent bundle of $\mathrm{Gr}_2(\mathbb{H}^4)$ has two different paracomplex structures which are both homogeneous.

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