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Tits Geometry, Arithmetic Groups, and the Proof of a Conjecture of Siegel

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Abstract. Let $X = G/K$ be a Riemannian symmetric space of noncompact type and of rank ≥ 2. An irreducible, non-uniform lattice $\Gamma \subset G$ in the isometry group of X is arithmetic and gives rise to a locally symmetric space $V = \Gamma\backslash X$. Let $\pi : X \to V$ be the canonical projection. Reduction theory for arithmetic groups provides a dissection $V = \coprod_{i=1}^{k} \pi(X_i)$ with $\pi(X_0)$ compact and such that the restiction of π to X_i is injective for each i. In this paper we complete reduction theory by focusing on metric properties of the sets X_i. We detect subsets C_i of X_i ($\mathbb{Q}$–Weyl chambers) such that $\pi_{|C_i}$ is an isometry and such that C_i is a net in X_i. This result is then used to prove a conjecture of C.L. Siegel. We also show that V is quasi-isometric to the Euclidean cone over a finite simplicial complex and study the Tits geometry of V.

1. Introduction and survey of results

In this paper we study geometric properties of certain locally symmetric spaces $V = \Gamma\backslash X$. Here X is a Riemannian symmetric space of noncompact type (without Euclidean factor) and Γ is a non-uniform (torsion-free) lattice in the group of isometries of X, i.e., the quotient space V is not compact but has finite volume. We make the additional assumption that the rank of X is at least two and we also assume that the non-uniform lattice Γ is irreducible. In that case a fundamental theorem of G.A. Margulis asserts that Γ is *arithmetic.* Roughly this means that there is a semisimple algebraic group $\mathbf{G} \subset \mathbf{GL}(n, \mathbb{C})$ defined over $\mathbb{Q}$ such that Γ is commensurable with $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathbf{GL}(n, \mathbb{Z})$ (see [30] Ch. 6 and Section 2 below). The symmetric space X can be written as the homogeneous space $\mathbf{G}(\mathbb{R})^0/K$ for a (chosen) maximal compact subgroup K. The $\mathbf{G}(\mathbb{R})^0$-invariant Riemannian metric on X induces one on V so that the canonical projection $\pi : X \longrightarrow \Gamma\backslash X$ is a Riemannian covering.

1.1. Metric properties of locally symmetric spaces. Fundamental domains provide a basic and classical tool to investigate quotients of discrete groups acting isometrically on spaces of constant curvature. For lattices in

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isometry groups of higher rank symmetric spaces *exact* fundamental domains have a rather complicated geometry. In order to study locally symmetric spaces $V = \Gamma\backslash X$ one therefore uses only coarse fundamental domains, whose shapes reflect certain (algebraic) decompositions of the isometry group of X.

Reduction theory provides such coarse fundamental domains $\Omega \subset X$ for arithmetic groups. Its basic building blocks are so-called Siegel sets, which in turn are unions of certain Euclidean cones ($\mathbb{Q}$–Weyl chambers). A more refined version, *precise reduction theory*, yields a combinatorial dissection of V into disjoint subset: $V = \coprod_{i=0}^{l} V_i$ where V_0 is compact and the other V_i are noncompact and can be explicitely described as one-to-one images of Siegel sets, $V_i = \pi(\mathcal{S}_i)$, where $\pi : X \longrightarrow V = \Gamma\backslash X$ denotes the canonical projection map. A variant of such a dissection based on an exhaustion of $V = \Gamma\backslash X$ by compact submanifolds with corners has been created in [23] and in [24] and will be our main tool in investigating V. We shall discuss reduction theory and exhaustions in more detail in Sections 2 and 3.

Our main result (Theorem 4.1) completes precise reduction theory by revealing *metric properties* of subsets of the above V_j:

Theorem A. *There are $\mathbb{Q}$–Weyl chambers $\mathcal{C}_i \subset \mathcal{S}_i, 1 \le i \le l$, in X such that, for each i, the restriction of the $\pi : X \longrightarrow V$ to $\mathcal{C}_i$ is an isometry (with respect to the distance functions induced by the Riemannian metrics on X and V, respectively). Moreover $\bigcup_{i=1}^{l} \pi(\mathcal{C}_i)$ is a net in V.*

As a consequence of Theorem A we confirm a conjecture of C.L.Siegel.

Theorem B. *The projection map $\pi : X \longrightarrow \Gamma\backslash X$ restricted to a Siegel set is a $(1, D)-$ quasi-isometry.*

Siegel's conjecture has been proved independently by L. Ji and in the special case of $SL(n,\mathbb{Z}) \subset SL(n,\mathbb{R})$ also by J. Ding (see [18] and [11]). It has been applied recently by D. Kleinbock and G.A. Margulis to investigate logarithm laws for certain flows on locally symmetric spaces (see [21]).

1.2. Quasi-isometry invariants for locally symmetric spaces. Using the assertion of Theorem A we construct a quasi-isometric model of the locally symmetric space $V = \Gamma\backslash X$. It is obtained as the *asymptotic cone* which is defined as the Gromov-Hausdorff limit of "rescaled" pointed metric spaces:

$$\mathrm{Cone}(V) := \mathcal{H} - \lim_{n\to\infty}(V, v_0, \frac{1}{n} d_V),$$

where v_0 is an arbitrarily chosen point of V and d_V is the distance function on V induced by the Riemannian metric (see e.g. [13] Ch. 3 or [10] Ch. I). We remark that in contrast to the case considered here the definition of an asymptotic cone in general involves the use of ultrafilters, and the limit space may even depend on the chosen ultrafilter. Various aspects of asymptotic cones of general spaces are discussed in M. Gromov's essay [14]. Recently B. Kleiner and B. Leeb also used asymptotic cones to prove the rigidity of quasi-isometries of symmetric spaces (see [20]).

In some cases the asymptotic cone of $\Gamma\backslash X$ is easy to describe. For example if the rank of X is 1 (or, more generally, if V is a Riemannian manifold with finite volume and strictly negative sectional curvature), then $\mathrm{Cone}(V)$ is a "cone" over k points, i.e., k rays with a common origin, where k is the number of ends of V. For a Riemannian product $V = V_1 \times V_2$, where V_1, V_2 are as in the previous example and each has only one end, $\mathrm{Cone}(V)$ can be identified with the first quadrant in $\mathbb{R}^2$. Much more intricate are quotients of $SL(n,\mathbb{R})/SO(n)$ by congruence subgroups of $SL_n(\mathbb{Z})$. These are examples of higher rank locally symmetric spaces which are not only products of rank 1 factors; their asymptotic geometry was first studied by T. Hattori (see [15], [16]). In what follows we are concerned with the general higher rank case.

By the Cartan-Hadamard theorem a simply connected, complete Riemannian manifold X of nonpositive sectional curvature is diffeomorphic to $\mathbb{R}^n$, $n = \dim X$. One can compactify X by adjoining the *boundary at infinity*, $\partial_\infty X \cong S^{n-1}$, which is defined as the set of equivalence classes of asymptotic geodesic rays. The enlarged space $X \sqcup \partial_\infty X$ is homeomorphic to a closed ball in $\mathbb{R}^n$. On $\partial_\infty X$ there is a natural distance function, the *Tits metric*, which is also defined in terms of geodesic rays (see Section 3). In the special case where X is a symmetric space of noncompact type the boundary at infinity $\partial_\infty X$ carries the additional structure of a spherical Tits bulding given by the disjoint union of Weyl chambers and their walls "at infinity". In that case the building structure on $\partial_\infty X$ can be reconstructed using the Tits metric (see [3]).

Since the algebraic group $\mathbf{G}$ is defined over $\mathbb{Q}$ one also has the *Tits building* $\mathcal{T}$ *of* $\mathbf{G}$ *over* $\mathbb{Q}$, which is a simplical complex whose combinatorial structure reflects the incidence relations among parabolic $\mathbb{Q}$–subgroups of $\mathbf{G}$. The lattice Γ acts naturally on a geometric realization $|\mathcal{T}|$ of $\mathcal{T}$ as a spherical simplicial complex. The resulting quotient $\Gamma\backslash|\mathcal{T}|$ turns out to be a finite simplical complex. If $\mathrm{rank}_{\mathbb{Q}}\Gamma \geq 2$ the spherical complex $|\mathcal{T}|$ is equipped with a spherical distance function which we normalize such that the diameter of $|\mathcal{T}|$ is π (see Section 6). This metric induces a distance function $d_{\mathcal{T}}$ on the quotient space $\Gamma\backslash|\mathcal{T}|$. The *Euclidean cone* $C(\Gamma\backslash|\mathcal{T}|)$ *over* $\Gamma\backslash|\mathcal{T}|$ is the product $[0,\infty) \times \Gamma\backslash|\mathcal{T}|$ with $\{0\} \times \Gamma\backslash|\mathcal{T}|$ collapsed to a point and endowed with the cone metric

$$d_C^2((a,x),(b,y)) := a^2 + b^2 - 2ab\cos d_{\mathcal{T}}(x,y).$$

We remark that $\Gamma\backslash|\mathcal{T}|$ is connected if $\mathbb{Q}$–rank of Γ is ≥ 2. If $\mathrm{rank}_{\mathbb{Q}}\Gamma = 1$, $|\mathcal{T}|$ consists of countably many points and $C(\Gamma\backslash|\mathcal{T}|)$ is a cone over a finite number of points as in the first example described above. In that case we set $d_C((a,x),(b,y)) := a + b$ if $x \neq y$ and $d_C((a,x),(b,y)) := |a-b|$ if $x = y$.

We can now describe the asymptotic cone of an arithmetic quotient:

Theorem C. *Let X be a symmetric space of noncompact type (without Euclidean factor) and rank ≥ 2. Let Γ be an irreducible, non-uniform lattice in the group of isometries of X. Then the asymptotic cone of the locally symmetric space $V = \Gamma\backslash X$ is quasi-isometric to V and is isometric to the Euclidean cone over the finite simplicial complex $\Gamma\backslash|\mathcal{T}|$.*

Various invariants of the algebraic group $\mathbf{G}$ are reflected in the geometry of $X = \mathbf{G}(\mathbb{R})^0/K$ and of $\Gamma\backslash X$. For instance it is well-known that the $\mathbb{R}$–rank of $\mathbf{G}$ equals the rank of the symmetric space X, i.e., the maximal dimension of a totally geodesic, flat submanifold in X.

It is an elementary but crucial fact that bi-lipschitz invariants of the asymptotic cone of $V = \Gamma\backslash X$ are quasi-isometry invariants of V. Theorem C thus yields the following geometric interpretation of the (purely algebraic) notion of "$\mathbb{Q}$–rank" as a quasi-isometry invariant of V. Note that it follows from our assumptions that the $\mathbb{Q}$–structure of G is unique (see [28]) and we can thus set $\mathrm{rank}_{\mathbb{Q}}\Gamma := \mathrm{rank}_{\mathbb{Q}}\mathbf{G}$.

Corollary D. *Let Γ be an arithmetic lattice as in Theorem C. Then*

$$\mathrm{rank}_{\mathbb{Q}}\Gamma = \dim \mathrm{Cone}(\Gamma\backslash X).$$

The dimension of the Euclidean cone $C(\backslash|\mathcal{T}|)$ is defined as the (topological) dimension of the simplicial complex $\Gamma\backslash|\mathcal{T}|$ *plus one.* According to [20] the asymptotic cones of a globally symmetric space X are Euclidean buildings and one has $\dim \mathrm{Cone}(X) = \mathrm{rank}_{\mathbb{R}}\mathbf{G}$.

The asymptotic cone of a compact metric space is just a point. Corollary D thus reflects geometrically a classical result of A. Borel and Harish-Chandra asserting that an arithmetic lattice Γ is co-compact, i.e., $\Gamma\backslash X$ is compact, if and only if the $\mathbb{Q}$–rank of Γ (or $\mathbf{G}$) is 0 (see [4]).

Typical for Gromov-Hausdorff limits are collapsing phenomena (as for instance the just mentioned fact that the asymptotic cone of a compact space is a point). This makes the limit objects in some sense easier to understand. On the other hand Gromov-Hausdorff limits of manifolds usually degenerate into (singular) metric spaces. In particular concepts which are related to the differentiable structure, like sectional curvature, may have no meaning in the limit. It is therefore necessary to work in a more general setting using notions that are defined for (geodesic) metric spaces. "Nonpositive sectional curvature", for instance, can be generalized by the local property "curvature bounded from above by 0" (resp. by its global counterpart, "CAT(0) space"). Roughly speaking one requires that sufficiently small (resp. all) geodesic triangles are "thinner" than their comparison triangles of the same side length in the Euclidean plane. For example a locally symmetric space of non compact type has curvature bounded above by 0 and every globally symmetric space is CAT(0). Recent references for these purely metric concepts due to A.D. Alexandrov are [2] and the comprehensive [9].

Notice that $V = \Gamma\backslash X$ has curvature bounded above by 0, but that V (in contrast to X) is not a CAT(0) space since V is in general not contractible. However we have the following corollary to Theorem C.

Corollary E. *The asymptotic cone of a locally symmetric space $\Gamma\backslash X$ (as in Theorem C) is a* CAT(0) *space.*

The boundary at infinity $\partial_\infty V$ of $V = \Gamma\backslash X$ is defined as the set of equivalence classes of asymptotic geodesic rays in V. From Theorem C we deduce the following combinatorial description of the boundary at infinity of a locally symmetric space, which was first proved by L. Ji and R. MacPherson in [19] (extending methods first developed in [16] and [22]):

Theorem F. *Let $V = \Gamma\backslash X$ be as in Theorem C. The points of the boundary at infinity $\partial_\infty V$ correspond bijectively to the points of $\Gamma\backslash|\mathcal{T}|$, i.e.,*

$$\partial_\infty V \cong \Gamma\backslash|\mathcal{T}| \cong \partial_\infty C(\Gamma\backslash|\mathcal{T}|).$$

As a consequence the distance $d_\mathcal{T}$ on $\Gamma\backslash|\mathcal{T}|$ can be defined intrinsically in terms of geodesic rays in V (see also [19], Remark 11.9).

Corollary G. *Let $rank_\mathbb{Q}\Gamma \geq 2$. For any two points z_1, z_2 in the boundary at infinity $\partial_\infty V \cong \Gamma\backslash|\mathcal{T}|$ represented by unit speed geodesic rays c_1, c_2 in V, respectively, one has the formula*

$$2\sin\frac{1}{2}d_\mathcal{T}(z_1,z_2) = \lim_{t\to\infty}\frac{1}{t}d_V(c_1(t),c_2(t)).$$

Observe that this is exactly the same formula as the one for simply connected, complete Riemannian manifolds of nonpositive curvature or more generally for CAT(0) spaces (see [3] Ch. 4, [9] and Section 5 below). For symmetric spaces the (crucial) difference is that in the simply connected case $d_\mathcal{T}$ refers to the Tits distance on the spherical building associated to $\mathbf{G}(\mathbb{R})$, whereas in the *non* simply connected case $d_\mathcal{T}$ is induced by the Tits distance of the spherical building associated to $\mathbf{G}(\mathbb{Q})$. While the former (being independent of any $\mathbb{Q}$–structure on the isometry group of X) reflects geometric properties of X, the latter depends on the $\mathbb{Q}$–structure associated to the lattice Γ and encodes (large scale) geometric properties of the quotient $V = \Gamma\backslash X$.

2. Coarse fundamental domains for arithmetic groups

Let G denote the identity component of the group of isometries of the symmetric space X; it is a connected, semisimple Lie group with trivial center. In this section we discuss some classical results of reduction theory about coarse fundamental domains for arithmetic groups, which are adopted to the group structure of G.

We make two natural assumptions about the non-uniform lattice Γ in G. Firstly, we assume that Γ is *irreducible* and, secondly, that Γ is *neat* and hence in particular torsion-free (see [4] §17). Under these assumptions the arithmeticity theorem of Margulis asserts that there is a connected semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, $\mathbb{Q}$–embedded in a general linear group $\mathbf{GL}(n,\mathbb{C})$, and a Lie group isomorphism

$$\rho : G \longrightarrow \mathbf{G}(\mathbb{R})^0 = (\mathbf{G}\cap\mathbf{GL}(n,\mathbb{R}))^0$$

such that $\rho(\Gamma)$ is *arithmetic.* This means that

$$\rho(\Gamma) \subset \mathbf{G}(\mathbb{Q}) = \mathbf{G}\cap\mathbf{GL}(n,\mathbb{Q}) \subset \mathbf{GL}(n,\mathbb{C})$$

is commensurable with the group $\mathbf{G}(\mathbb{Z}) = \mathbf{G}\cap\mathbf{GL}(n,\mathbb{Z})$ (see [30] 3.1.6 and 6.1.10). Note that the symmetric space X can be recovered as the manifold of maximal compact subgroups of the identity component of the group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$–rational points of $\mathbf{G}$. For simplicity we identify G with $\mathbf{G}(\mathbb{R})^0$ and Γ with $\rho(\Gamma)$ in this paper.

Remark 2.1. For an irreducible, non-uniform lattice Γ as above the connected semisimple $\mathbb{Q}$–group $\mathbf{G}$ is unique up to isomorphism (see [28] Section 3.2). We can thus define $\mathrm{rank}_{\mathbb{Q}}\Gamma := \mathrm{rank}_{\mathbb{Q}}\mathbf{G}$.

2.1. Horocyclic coordinates and $\mathbb{Q}$–Weyl chambers. We recall some definitions and basic facts about linear algebraic groups together with geometric interpretations which are needed below.

Let $\mathbf{S}$ (resp. $\mathbf{T}$) be a maximal $\mathbb{Q}$–split (resp. a maximal $\mathbb{R}$–split) algebraic torus of $\mathbf{G}$, i.e. a subgroup of $\mathbf{G}$ which is isomorphic over $\mathbb{Q}$ (resp. $\mathbb{R}$) to the direct product of q (resp. $r \geq q$) copies of $\mathbb{C}^*$. All such tori are conjugate under $\mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{G}(\mathbb{R})$) and their common dimension q (resp. r) is called the $\mathbb{Q}$*–rank* (resp. $\mathbb{R}$*–rank*) of $\mathbf{G}$. The identity component of $\mathbf{S}(\mathbb{R})$ (resp. $\mathbf{T}(\mathbb{R})$) will be denoted by A (resp. by A_0), the corresponding Lie algebra by $\mathfrak{a}$ (resp. by $\mathfrak{a}_0$). The $\mathbb{R}$–rank of $\mathbf{G}$ coincides with the rank of the symmetric space X, i.e., the maximal dimension of totally geodesic flat subspaces. The choice of a maximal compact subgroup K of G is equivalent to the choice of a base point x_0 of X. We can further choose K with Lie algebra $\mathfrak{k}$ so that under the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G we have $\mathfrak{a} \subseteq \mathfrak{a}_0 \subset \mathfrak{p} \cong T_{x_0}X$. Here $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}$, i.e., the tangent space at x_0 of the (maximal $\mathbb{R}$–) flat $A_0 \cdot x_0$ in X. The pair of Lie algebras $(\mathfrak{g}, \mathfrak{a}_0)$ gives rise to the root system ${}_{\mathbb{R}}\Phi$ of the symmetric space. Similarly there is a system of $\mathbb{Q}$–roots ${}_{\mathbb{Q}}\Phi$ associated to the pair $(\mathfrak{g}, \mathfrak{a})$ (see [6] §21). It is always possible to choose orderings of ${}_{\mathbb{Q}}\Phi$ and ${}_{\mathbb{R}}\Phi$ such that the restrictions of simple $\mathbb{R}$–roots of ${}_{\mathbb{R}}\Phi$ to $\mathfrak{a}$ are either *simple* $\mathbb{Q}$–roots of ${}_{\mathbb{Q}}\Phi$, i.e., the elements of a basis $\Delta = {}_{\mathbb{Q}}\Delta$ of ${}_{\mathbb{Q}}\Phi$, or zero (see [8] 6.8). The basis ${}_{\mathbb{R}}\Delta$ defines a closed $\mathbb{R}$–Weyl chamber $\overline{\mathfrak{a}_0^+}$ in $\mathfrak{a}_0$ and Δ then determines a closed $\mathbb{Q}$*–Weyl chamber* in $\mathfrak{a}$

$$\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \geq 0, \text{ for all } \alpha \in \Delta\}.$$

We set $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$ (resp. $\overline{A_0^+} = \exp \overline{\mathfrak{a}_0^+}$). A $\mathbb{Q}$*–Weyl chamber in* X is a translate of the basic chamber $\overline{A^+} \cdot x_0 \subseteq \overline{A_0^+} \cdot x_0$ under $\mathbf{G}(\mathbb{Q})$. The elements of Δ are differentials of characters (defined over $\mathbb{Q}$) of the maximal $\mathbb{Q}$–split torus $\mathbf{S}$. It is convenient to identify the elements of Δ also with such characters. When restricted to A we denote their values by $\alpha(a)$ $(a \in A, \alpha \in \Delta)$. Notice that $\overline{A^+} = \{a \in A \mid \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta\}$.

A closed subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{Q}$ is a *parabolic* $\mathbb{Q}$*–subgroup* if $\mathbf{G}/\mathbf{P}$ is a projective variety (see [6] §11). A *parabolic* $\mathbb{Q}$*–subgroup* P *of* $G = \mathbf{G}(\mathbb{R})^0$ is by definition the intersection of G with a parabolic $\mathbb{Q}$–subgroup of $\mathbf{G}$ (see [7]). The conjugacy classes under $\mathbf{G}(\mathbb{Q})$ of parabolic $\mathbb{Q}$–subgroups are in one-to-one correspondence with the subsets Θ of the (chosen) set Δ of simple $\mathbb{Q}$–roots; they are represented by the *standard parabolic* $\mathbb{Q}$–subgroups $\mathbf{P}_\Theta$ of $\mathbf{G}$ (see [6] 21.11). The corresponding standard parabolic $\mathbb{Q}$–subgroups of G are denoted by P_Θ. Let $\mathcal{P}$ be the collection of all parabolic subgroups of $\mathbf{G}$ defined over $\mathbb{Q}$. For every $\mathbf{P} \in \mathcal{P}$ the corresponding parabolic $\mathbb{Q}$–subgroup $P = G \cap \mathbf{P}$ of G has a Langlands-decomposition $P = U_{\mathbf{P}} M_{\mathbf{P}} A_{\mathbf{P}}$, where $U_{\mathbf{P}}$ is unipotent and $M_{\mathbf{P}}$ is reductive. Furthermore $A_{\mathbf{P}}$ centralizes $M_{\mathbf{P}}$ and normalizes $U_{\mathbf{P}}$ (see [7] and [4]). This yields a (generalized) Iwasawa decomposition for G, i.e., $G = PK = U_{\mathbf{P}} M_{\mathbf{P}} A_{\mathbf{P}} K$, which in turn implies that P acts transitively on the symmetric space X. The intersection of the maximal compact subgroup K of G with $M_{\mathbf{P}}$ is maximal compact in $M_{\mathbf{P}}$ and the quotient $Z_{\mathbf{P}} = M_{\mathbf{P}}/(K \cap M_{\mathbf{P}})$ is in general the Riemannian product of a symmetric space of noncompact type by a flat Euclidean space. Let $\tau_{\mathbf{P}} : M_{\mathbf{P}} \longrightarrow Z_{\mathbf{P}}$ be the natural projection. Then the "horocyclic coordinate map"

$$\mu_{\mathbf{P}} : Y_{\mathbf{P}} = U_{\mathbf{P}} \times Z_{\mathbf{P}} \times A_{\mathbf{P}} \longrightarrow X \quad ; \quad (u, \tau_{\mathbf{P}}(m), a) \longmapsto uma \cdot x_0$$

is an isomorphism of analytic manifolds (see [7]).

2.2. Reduction theory. A subset $\Omega \subset X$ is called a *fundamental set* for an arithmetic group Γ if the following two conditions are satified: (i) $X = \Gamma \cdot \Omega$; (ii) for every $q \in \mathbf{G}(\mathbb{Q})$ the set $\{\gamma \in \Gamma \mid q\Omega \cap \gamma\Omega \neq \emptyset\}$ is finite. It is the goal of *reduction theory* to provide explicit fundamental sets.

Let $\mathbf{P}$ be the *minimal* parabolic $\mathbb{Q}$–subgroup of $\mathbf{G}$ which corresponds to the $\mathbb{Q}$–Weyl chamber $\overline{A^+} \cdot x_0$. A *generalized Siegel set* $\mathcal{S} = \mathcal{S}_{\omega,\tau}$ in X (relative to $\overline{A^+} \cdot x_0$) is a subset of X of the form $\mathcal{S}_{\omega,\tau} = \omega A_\tau \cdot x_0$ where ω is a relatively compact in $U_{\mathbf{P}} M_{\mathbf{P}}$ and, for $\tau > 0$, $A_\tau = \{a \in A = A_{\mathbf{P}} \mid \alpha(a) \geq \tau,\ \alpha \in \Delta\}$. If we define $a_0 \in A$ by $\alpha(a_0) = \tau$ for all $\alpha \in \Delta$, then $A_\tau = A_1 a_0 = \overline{A^+} a_0$ and $\mathcal{C} = A_\tau \cdot x_0 \subset \mathcal{S}$ is a (translate of a) $\mathbb{Q}$–Weyl chamber in X. Fundamental sets for arithmetic groups consist of finitely many Siegel sets (see [4] §13 and §15 for the proofs):

Proposition 2.2. (Borel, Harish-Chandra) *Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ with associated Riemannian symmetric space $X = G/K$ (of maximal compact subgroups of G). Let $\mathbf{P}$ be a minimal parabolic $\mathbb{Q}$–subgroup of $\mathbf{G}$ and let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. Then there exists a generalized Siegel set $\mathcal{S} = \mathcal{S}_{\omega,\tau}$ such that, for a fixed set $\{q_i \mid 1 \leq i \leq m\}$ of representatives of the finite set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$, the union $\Omega = \bigcup_{i=1}^m q_i \cdot \mathcal{S}$ is a fundamental set of finite volume for Γ in X.*

3. Polyhedral exhaustions of locally symmetric spaces

A crucial ingredient in the proof of the main (technical) Theorem 4.1 below is the property that a locally symmetric space $V = \Gamma\backslash X$ can be exhausted (in many ways) by polyhedra, i.e., by compact submanifolds $V(s)$ of V with corners: $V = \bigcup_{s \geq 0} V(s)$. Such exhaustions were constructed in [23] and further investigated in [24]. Similar constructions appear also in papers by J. Arthur, M.S. Osborne-G. Warner and L. Saper (see [1], [25] and [26]).

In this section we recall some facts about the geometric structure of the polyhedra $V(s)$ for $s \geq 0$. We invite the reader to consult the articles [23] and [24] for further details and complete proofs.

3.1. Exhaustions and the rational Tits building. Let d_X be the distance function induced by the Riemannian metric of the symmetric space X. A unit-speed *geodesic ray* is a curve $c : [0, \infty) \to X$ which realizes the distance between any two of its points, i.e., c is an isometric embedding of $[0, \infty) \subset \mathbb{R}$ into

X. We call two unit-speed geodesic rays $c_1, c_2 : [0,\infty) \to X$ in X *asymptotic* if there is a constant $C \geq 0$ such that $\lim_{t\to\infty} d_X(c_1(t), c_2(t)) \leq C$; the corresponding equivalence class is denoted by $c_1(\infty)(= c_2(\infty))$. The *boundary of* X *at infinity,* $\partial_\infty X$, is defined as the set of equivalence classes of asymptotic geodesic rays. A *Busemann function* h associated to a ray $c : [0,\infty) \to X$ representing the equivalence class $z = c(\infty) \in \partial_\infty X$ is defined by

$$h : X \to \mathbb{R} \ ; \quad x \longmapsto \lim_{t\to\infty} [d_X(c(t), x) - t].$$

A *horosphere* (resp. a *horoball*) centered at $z \in \partial_\infty X$ is a level set (resp. sublevel set) of h.

We next recall the definition of the (rational) Tits building $\mathcal{T}$ of $\mathbf{G}$. (see e.g. [29] 5.2). Let $\mathcal{P}_m$ be the set of all *maximal* parabolic $\mathbb{Q}$–subgroups of $\mathbf{G}$ (and $\neq \mathbf{G}$) (see Section 2). The *Tits building* $\mathcal{T}$ *associated to* $\mathbf{G}$ *over* $\mathbb{Q}$ is the simplicial complex whose set of vertices is $\mathcal{P}_m$, and whose simplices are the non-empty subsets I of $\mathcal{P}_m$ such that $\mathbf{P}_I = \bigcap_{\mathbf{P}\in I} \mathbf{P}$ is a parabolic $\mathbb{Q}$–subgroup of $\mathbf{G}$. A geometric realization $|\mathcal{T}|$ of $\mathcal{T}$ in $\partial_\infty X$ is obtained as follows. We define a (closed) $\mathbb{Q}$*–Weyl chamber at infinity* as

$$\begin{aligned}(\overline{A^+}x_0)(\infty) &:= \{c_H(\infty) \mid c_H(t) = \exp tH \cdot x_0,\ H \in \overline{\mathfrak{a}^+}, \|H\| = 1\} \cong \\ &\cong \{H \in \overline{\mathfrak{a}^+} \mid \|H\| = 1\}.\end{aligned}$$

Any other $\mathbb{Q}$–Weyl chamber at infinity is of the form $g \cdot (\overline{A^+}x_0)(\infty)$ with $g \in \mathbf{G}(\mathbb{Q})$. The geometric realization $|\mathcal{T}|$ is then defined as the union of all $\mathbb{Q}$–Weyl chambers and their walls in $\partial_\infty X$ partially ordered with the order relation "wall of" (see also [19]).

The following proposition is proved in [24], Theorem 3.6.

Proposition 3.1. *Let $V = \bigcup_{s\geq 0} V(s)$ be any (fixed) polyhedral exhaustion as introduced above. Any polyhedron $V(s)$ can be written as $\Gamma\backslash X(s)$, where $X(s)$ is the complement of a Γ–invariant union of countably many open horoballs:*

$$X(s) = X - \bigcup_k \mathcal{B}_k(s).$$

The "centers" in $\partial_\infty X$ of the deleted horoballs $\mathcal{B}_k(s)$ are in one-to-one correspondence with the maximal parabolic subgroups of $\mathbf{G}$ *defined over* $\mathbb{Q}$*, or, equivalently, with the vertices of a geometric realization of the Tits building $\mathcal{T}$ of* $\mathbf{G}$.

If the $\mathbb{Q}$–rank of Γ is 1, then the deleted horoballs $\mathcal{B}_k(s)$ are disjoint in X. If the $\mathbb{Q}$–rank of Γ is ≥ 2, then, in contrast, the horoballs $\mathcal{B}_k(s)$ intersect and give rise to the corners of $X(s)$ (and $V(s)$). Their local structure can be described for example by taking the (inner) unit normals of the involved horospheres $\partial\mathcal{B}_k(s)$ (see also [23] Lemma 4.1 (iv)). In fact, any of these horospheres is of the form $\gamma\{\tau_\alpha^{-1}\tilde{h}_{i\alpha} = -s\}$ with $\gamma \in \Gamma$, $i \in \{1, \ldots, m\}$, $\alpha \in \Delta$ and where the $\tilde{h}_{i\alpha}$ are Busemann functions on X associated to distinguished geodesic rays in the fundamental set $\Omega \subset X$ (see [23], Theorem 3.6).

The inner unit normal field of the horosphere $\{\tau_\alpha^{-1}\tilde{h}_{i\alpha} = -s\}$ in X is given by $-\mathrm{grad}\ \tilde{h}_{i\alpha}$. Let p be a point of $V(s)$. The *outer angle* $O(p)$ at p is defined as the set of all unit tangent vectors $v \in T_pV(s)$ such that $\langle v, w\rangle_p \leq 0$ for all w in the tangent cone of $V(s)$ at p. Let Θ be a subset of Δ. If the horospheres $\{\tau_\alpha^{-1}\tilde{h}_{i\alpha} = -s\}$, for $\alpha \in \Theta$, intersect, then the outer angle $O(p)$ at an intersection point p can be identified with all positive linear combinations (of norm 1) of the unit vectors $-\mathrm{grad}\ \tilde{h}_{i\alpha}(p)$, $\alpha \in \Theta$. From that description one can in particular see that each outer cone is isomorphic (as a cone in Euclidean space) to the face $\overline{\mathfrak{a}^+}(\Theta) := \{H \in \overline{\mathfrak{a}^+} \mid \beta(H) = 0, \quad \forall\, \beta \in_{\mathbb{Q}} \Delta - \Theta\}$ in the "model chamber" $\overline{\mathfrak{a}^+} \subset \mathfrak{g}$ resp. $\overline{A^+}x_0 \subset X$.

We define also the *outer cone* $CO(p)$ at $p \in \partial V(s)$ as the Euclidean cone in the tangent space T_pV whose vertex is 0 and whose basis is the (closed) spherical simplex $O(p)$.

3.2. Outer angles and structure of V at infinity. The outer angles $O(p)$ moreover reflect the structure at infinity of $V = \Gamma\backslash X$. More precisely, we shall see below how they correspond to the spherical simplices in a geometric realization of the quotient modulo Γ of the Tits building $\mathcal{T}$ over $\mathbb{Q}$ associated to $\mathbf{G}$.

In order to describe that quotient more explicitly we first note that the discrete, arithmetic group Γ acts on $|\mathcal{T}|$. The resulting quotient space $\Gamma\backslash|\mathcal{T}|$ turns out to be a finite simplicial complex. In fact, as mentioned in Section 2, the conjugacy classes of elements of the collection $\mathcal{P}$ of parabolic $\mathbb{Q}$–subgroups of $\mathbf{G}$ are in one-to-one correspondence with the subsets Θ of the set ${}_{\mathbb{Q}}\Delta$ of simple $\mathbb{Q}$–roots. Every conjugacy class has a standard representative denoted by $\mathbf{P}_\Theta$. One can show that the sets of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}_\Theta(\mathbb{Q})$ are *finite* for all Θ (see [4] 15.6). Let $\triangle \subset \mathbb{R}^q$ be the spherical $q-1$ simplex obtained by taking unit vectors in $\overline{\mathfrak{a}^+} \subset \mathfrak{a} \cong \mathbb{R}_q$ ($q = \mathbb{Q}$–rank of $\mathbf{G}$); note that with our notation above $\triangle \cong (\overline{A^+}x_0)(\infty)$. For a subset Θ of ${}_{\mathbb{Q}}\Delta$ we define the boundary simplex $\triangle(\Theta)$ of $\triangle$ as $\triangle(\Theta) := \triangle \cap \{\alpha = 0 \mid \alpha \in_{\mathbb{Q}} \Delta - \Theta\}$. A vertex of $\triangle$ then corresponds to a *maximal* parabolic $\mathbb{Q}$–subgroup while the entire simplex $\triangle$ corresponds to the *minimal* parabolic $\mathbb{Q}$–subgroup $\mathbf{P} := \mathbf{P}_\emptyset$ of $\mathbf{G}$. Let the set $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ be represented by $\{q_1, \dots q_m\}$ (see Proposition 2.2) and take m isometric copies $\triangle^j$ of $\triangle$ with faces $\triangle^j(\Theta)$ corresponding to Θ. The corresponding isometries $\triangle \simeq \triangle^j$ are denoted by φ_j. The simplicial complex $\Gamma\backslash|\mathcal{T}|$, which provides a geometric realization of the quotient of the Tits building $|\mathcal{T}|$ modulo Γ, is constructed from the simplices $\triangle^1, \dots, \triangle^m$ through the following *incidence relations*:

Two simplices $\triangle^j$ and $\triangle^l$ are pasted together along the faces $\triangle^j(\Theta)$ and $\triangle^l(\Theta)$ by the isometry $\varphi_j \circ \varphi_l^{-1}\ |_{\triangle^l(\Theta)}$ if and only if $\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q})$. The simplicial complex $|\mathcal{T}|$ and hence also $\Gamma\backslash|\mathcal{T}|$ is connected if the $\mathbb{Q}$–rank q is greater than or equal to two; moreover $\dim \Gamma\backslash|\mathcal{T}| = \mathrm{rank}_{\mathbb{Q}}\,\Gamma - 1$.

Having constructed the finite simplical complex $\Gamma\backslash|\mathcal{T}|$ we briefly explain how its simplices correspond to outer angles of the polyhedra $V(s)$ for an exhaustion $V = \bigcup_{s\geq 0} V(s)$ of V. Let $\mathcal{E}$ be the set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}_\Theta(\mathbb{Q}), \Theta \subset_{\mathbb{Q}} \Delta$. By [4] 15.6, $\mathcal{E}$ is a finite set.

By definition the elements of $\mathcal{E}$ are in one to one correspondence to the Γ–equivalence classes of parabolic $\mathbb{Q}$–subgroups of $\mathbf{G}$ or the simplices of

$\Gamma\backslash|\mathcal{T}|$ (see the above construction or [24], Section 4). The elements of $\mathcal{E}$ also index the boundary strata of $V(s)$. In fact, if p is some point in the interior of a boundary stratum of $V(s)$, then there is a unique set Θ of simple roots such that the outer angle at p is spanned by the gradients of the intersecting horospheres $-\text{grad}\ \tilde{h}_{i\alpha}, \alpha \in \Theta$, and hence $O(p) \cong \triangle(\Theta)$ (and $CO(p) \cong \overline{\mathfrak{a}^+}(\Theta)$).

In [24], Section 4, we showed that every point in the complement of the interior $V(s)^o$ of $V(s)$ in V is contained in the image under the Riemannian exponential map of the outer cone $CO(p)$ at some point of $\partial V(s)$. More precisely we have

Proposition 3.2. *Let $V = \bigcup_{s\geq 0} V(s)$ be any (fixed) polyhedral exhaustion as described above. Then there is a disjoint union*

$$V - V(s)^0 = \coprod_{p\in\partial V(s)} \text{Exp}_p CO(p).$$

Moreover, the points at infinity of those unit speed geodesic rays whose initial vectors are in $O(p)$ span a (closed) spherical simplex σ_E, $E \in \mathcal{E}$, of the finite complex $\Gamma\backslash|\mathcal{T}|$.

3.3. The fine structure of the boundary strata. We will need some further details about the structure of the boundary strata of a polyhedral exhaustion. For proofs we refer the reader to [24], Section 4. Pick $E = E(\Theta) \in \mathcal{E}$ and let σ_E be the corresponding simplex of the finite complex $\Gamma\backslash|\mathcal{T}|$. The pointwise isotropy group of σ_E is isomorphic to the group of real points, say P_E, of a conjugate under Γ of the standard parabolic subgroup $\mathbf{P}_E$ of $\mathbf{G}$ (see [23], Lemma 1.2). Recall from Section 2 that there is a Langlands decomposition $P_E = U_E M_E A_E$ and an associated decomposition $X \cong U_E \times Z_E \times A_E$. The arithmetic group $\Gamma \cap P_E$ has trivial A_E factor and has a finite index subgroup which is the semidirect product of $\Gamma \cap U_E$ with $\Gamma \cap M_E$. A *truncated locally symmetric space* $\Gamma \cap M_E\backslash Z_E(s)$ is a compact submanifold with corners in a totally geodesic subspace of V isometric to $\Gamma \cap M_E\backslash Z_E$.

Proposition 3.3. *Let $V = \bigcup_{s\geq 0} V(s)$ be a polyhedral exhaustion of V. Then, for each s, the boundary $\partial V(s)$ consists of a finite number of strata $V_E(s)$, $E = E(\Theta) \in \mathcal{E}$, each of which is a fibre bundle over a truncated locally symmetric space with a compact nilmanifold as fibre:*

$$0 \to \Gamma \cap U_E\backslash U_E \to V_E(s) \to \Gamma \cap M_E\backslash Z_E(s) \to 0.$$

The combinatorics of these strata is "dual" to that of the simplicial complex $\Gamma\backslash|\mathcal{T}|$, i.e., $(n-k)$-dimensional strata of $\partial V(s)$ correspond to $(k-1)$-simplices of $\Gamma\backslash|\mathcal{T}|$.

Moreover, the outer cones $CO(p)$ at the interior points of $V_E(s)$ are all isomorphic to some face $\overline{\mathfrak{a}^+}(\Theta)$ (uniquely determined by the index $E = E(\Theta)$) of the $\mathbb{Q}$–Weyl chamber $\overline{\mathfrak{a}^+}$.

We now consider the universal covering space $X(0) \subset X$ of the polyhedron $V(0)$ of some exhaustion of V. The distance function d_V on V is defined by $d_V(\pi(x), \pi(y)) = \inf_{\gamma\in\Gamma} d_X(x, \gamma\cdot y)$, where $\pi : X \to V = \Gamma\backslash X$ denotes the canonical projection. The next lemma will be used in the proof of Theorem 4.1.

Lemma 3.4. *Let $CO(z) \subset X$ be the outer cone of the submanifold with corners $X(0) \subset X$. For $x \in CO(z)$ the geodesic segment $[\pi(x)\pi(z)]$ realizes the shortest distance (in V) from $\pi(x)$ to the compact manifold with corners $V(0) = \Gamma \backslash X(0)$; and, in particular, $d_V(\pi(x), \pi(y)) = d_X(x, z)$.*

Proof. Assume that there is a point $\pi(y) \in V(0)$ such that

$$d_V(\pi(x), \pi(y)) < d_V(\pi(x), \pi(z)) = d_X(x, z);$$

where the equality comes from the fact (see [24], Corollary 3.5.) that $\pi(x)$ lies on a ray in V starting at $\pi(z)$. Since $d_V(\pi(x), \pi(y)) = \inf_{\gamma \in \Gamma} d_X(x, \gamma y)$ the assumption implies that for some $\gamma \in \Gamma$, $d_X(x, \gamma y) < d_X(x, z)$. Since $X(0)$ is Γ-invariant and $y \in X(0)$ we have $\gamma y \in X(0)$ and hence $d_X(x, X(0)) < d_X(x, z)$. This is a contradiction since the segment $[xz]$ clearly realizes the shortest distance in X from x to $X(0)$. ■

We conclude this section with an interesting metric property of $\Gamma \backslash |\mathcal{T}|$ which will be used in Section 5. The definition of a "CAT(1) space" can be found for example in [9] or [2].

Lemma 3.5. *The finite spherical simplical complex $(\Gamma \backslash |\mathcal{T}|, d_{\mathcal{T}})$ is a* CAT(1) *space.*

Proof. The model simplex $\triangle$ and all its faces are convex subsets of the unit sphere in $\mathbb{R}^q$ and therefore CAT(1) spaces. Pasting CAT(1) spaces along complete (π)–convex subspaces yields again CAT(1) spaces (see e.g. [9], Ch.II. 4.4). ■

4. Metric properties of arithmetic quotients of symmetric spaces

4.1. Isometric images of Weyl chambers. In this section we derive the crucial estimates. We will prove that the canonical projection map $\pi : X \longrightarrow V$ is an isometry (with respect to the distance function d_X and d_V) when restricted to certain closed $\mathbb{Q}$–Weyl chambers. The following theorem improves previous results of Hattori, Ding and Ji on metric properties of Siegel sets.

Theorem 4.1. *Let $V = \bigcup_{s \geq 0} V(s)$ be an exhaustion of V by manifolds with corners (as in Section 3) and let $CO(p), p \in V(0)$, be any outer cone of $V(0)$. Then the restriction of the canonical projection $\pi : X \longrightarrow V$ to any lifted outer cone $CO(z) \subset X$, with $\pi(z) = p$, is an isometry with respect to the distance functions of X and V, respectively.*

In particular, there is a $\mathbb{Q}$–Weyl chamber $\mathcal{C}_* \subset \mathcal{C}$ such that for every $i \in \{1, \ldots, m\}$ the chamber $q_i \mathcal{C}_* \subset \Omega \subset X$ is isometrically mapped to V under π.

Proof. We consider an arbitrary exhaustion $V = \bigcup_{s \geq 0} V(s)$. For a point $\pi(z) \in V(0)$ let $CO(\pi(z))$ be the outer cone. In the lift of $CO(z)$ of $CO(\pi((z))$ to X we pick two arbitrary points x, y. We wish to show that $d_V(\pi(x), \pi(y)) \geq d_X(x, y)$ (the opposite inequality holds by definition). Let c be a curve of minimal length in V between $\pi(x)$ and $\pi(y)$, i.e., $L(c) = d_V(\pi(x), \pi(y))$. We distinguish two cases: (1) c intersects the compact polyhedron $V(0)$ and (2) c does not intersect $V(0)$.

Case (1): Decompose c into three segments $c = c_1 \cup c_2 \cup c_3$, such that c_1 connects $\pi(x)$ to the first intersection point of c with $V(0)$ and c_3 connects the last intersection point of c with $V(0)$ to $\pi(y)$. By Lemma 3.4. we have

$$d_V(\pi(x), V(0)) = d_V(\pi(x), \pi(z)) = d_X(x, z) \quad \text{and}$$

$$d_V(\pi(y), V(0)) = d_V(\pi(y), \pi(z)) = d_X(y, z).$$

Hence

$$d_V(\pi(x), \pi(y)) = L(c) = L(c_1) + L(c_2) + L(c_3) \geq d_X(x, z) + d_X(y, z) \geq d_X(x, y).$$

Case (2): According to Propositions 3.2 and 3.3 we can decompose the curve

$$c : [0, d_V(\pi(x), \pi(y))] \longrightarrow V \setminus V(0)$$

into a finite number of segments $c_i, 1 \leq i \leq l$, say, each of which is contained in a subset of V diffeomorphic to $W_E(0) \times \overline{\mathfrak{a}^+}(\Theta)$, where $W_E(0)$ is a nilmanifold bundle over truncated locally symmetric and compact and the $\mathbb{Q}$-Weyl chamber face $\overline{\mathfrak{a}^+}(\Theta) \cong \overline{A^+}(\Theta)x_0$ corresponds to the (congruent) outer cones with $E = E(\Theta)$. The Riemannian metric on X with respect to horocyclic coordinates corresponding to A_E supporting $\overline{A^+}(\Theta)$ is given by a formula of Borel (see [5], Proposition 1.6). From the Γ-invariance of the exhaustion in X (compare the proof of Theorem 4.2 in [23]) we deduce that the Riemannian metric on V at the point $(w, a) \in W_E(0) \times \overline{A^+}(\Theta)$ can be written in the form

$$ds^2_{(w,a)} = dw^2_{(w,a)} + da^2_a.$$

For the length of the segment $c_i(s) = (w_i(s), a_i(s)), s \in [s_i, s_{i+1}]$, we thus have the estimate $L(c_i) \geq L(a_i), 1 \leq i \leq l$. Hence

$$d_V(\pi(x), \pi(y)) = L(c) = \sum_{i=1}^{l} L(c_i) \geq \sum_{i=1}^{l} L(a_i).$$

Now consider the continuous curve $a(s) := \bigcup_{i=1}^{l} a_i(s)$ in the chamber $\overline{A^+} \subset A$. If we write $x = ax_0$ and $y = bx_0$, the curve $a(s)x_0$ connects x and y in $\overline{A^+}x_0$ and satisfies

$$\sum_{i=1}^{l} L(a_i) \geq d_A(a, b) = d_X(x, y).$$

Together with the estimate above this proves the claim also in case (2).

Finally, the chamber $\mathcal{C}_*$ can be chosen as follows. Consider the preimages in the fundamental set Ω of the minimal boundary strata of $\partial V(0)$. They intersect the $\mathbb{Q}$–Weyl chambers $q_i\mathcal{C}$, $i = 1, \ldots, m$, in well defined points x_i (see [23] Lemma 3.5). The lifted (maximal) outer cones based at the points $\pi(x_i)$ of $\partial V(0)$ are again $\mathbb{Q}$–Weyl chambers $q_i\mathcal{C}_* \subseteq q_i\mathcal{C}$, $i = 1, \ldots, m$, and in finite Hausdorff-distance from $q_i\mathcal{C}$. ∎

4.2. A net in V. Theorem 4.1 yields a metric space which is quasi-isometric to V. Recall that a subset N of a metric space (S,d) is called a ($\varepsilon-$)*net* if there is some positive constant ε such that $d(s,N) \leq \varepsilon$ for all $s \in S$; in particular the Hausdorff-distance between N and S is at most ε.

Corollary 4.2. *There is a net N in V consisting of finitely many isometrically embedded $\mathbb{Q}$–Weyl chambers.*

Proof. Let $\mathcal{C}_*$ be as in Theorem 4.1. The definition of a fundamental set as a finite union of Siegel sets $\Omega = \bigcup_{i=1}^{m} q_i \cdot \mathcal{S}$ (see Proposition 2.2) implies that $\bigcup_{i=1}^{m} q_i \cdot \mathcal{C}$ and hence also $\bigcup_{i=1}^{m} q_i \cdot \mathcal{C}_*$ is a net in $\Omega \subset X$. Since $V = \pi(\Omega)$ and $d_V(\pi(x), \pi(y)) \leq d_X(x,y)$ for all $x, y \in X$ the claim follows. ■

Remark 4.3. From Proposition 3.1 (see also [23], Lemma 2.4, 2.5) it follows that the $q_i\mathcal{C}_*$, $1 \leq i \leq m$, can be chosen to be disjoint in X and such that $\pi : X \to V$ restricted to $\bigcup_{i=1}^{m} q_i\mathcal{C}_*$ is bijective.

5. The asymptotic cone and the proof of a conjecture of Siegel

5.1. A quasi-isometric model for a locally symmetric space. The goal of this section is to show that the asymptotic cone of a non-compact locally symmetric space $V = \Gamma\backslash X$ of finite volume is isometric to the Euclidean cone over a finite simplicial complex. The latter is given as the quotient of the Tits bulding $\mathcal{T}$ of $\mathbf{G}$ modulo Γ.

If $\mathrm{rank}_{\mathbb{Q}}\Gamma \geq 2$ the geometric realization $|\mathcal{T}|$ of $\mathcal{T}$ as a spherical complex carries a natural distance function which we normalize such that the diameter of $|\mathcal{T}|$ is π. This "spherical" metric induces a distance function $d_{\mathcal{T}}$ on the finite simplical complex $\Gamma\backslash|\mathcal{T}|$. We define the *Euclidean cone* $C(\Gamma\backslash|\mathcal{T}|)$ *over* $\Gamma\backslash|\mathcal{T}|$ to be the product $[0,\infty) \times \Gamma\backslash|\mathcal{T}|$ with $\{0\} \times \Gamma\backslash|\mathcal{T}|$ collapsed to a point $\mathcal{O}$ and endowed with the cone metric

$$d_C^2((a,x),(b,y)) := a^2 + b^2 - 2ab\cos d_{\mathcal{T}}(x,y).$$

If $\mathrm{rank}_{\mathbb{Q}}\Gamma = 1$ we set $d_C((a,x),(b,y)) := a+b$ if $x \neq y$ and $d_C((a,x),(b,y)) := |a-b|$ if $x = y$.

In the same way one defines the Euclidean cone over an arbitrary metric space of diameter $\leq \pi$ (see e.g. [9]).

There is an alternative way to build up $(C(\Gamma\backslash|\mathcal{T}|), d_C)$ which parallels the construction of $\Gamma\backslash|\mathcal{T}|$ in Section 3: We there realized the spherical simplices $\triangle^j$ for $j = 1, \ldots, m$ on the unit sphere in $\mathbb{R}^q$; thus we can take the cones $C(\triangle^j)$ in the Euclidean space $\mathbb{R}^q$ with vertex 0. We endow these simplicial cones with the induced Euclidean metric and glue them together using the same incidence relations as for $\Gamma\backslash|\mathcal{T}|$.

We recall the notion of Hausdorff-convergence of (unbounded) pointed metric spaces (see [13] or [10]). The *distortion* of a map $f : A \to B$ of metric spaces A and B is defined as

$$\mathrm{dis}(f) := \sup_{a,b} |d_A(a,b) - d_B(f(a), f(b))|.$$

The *uniform distance* between metric spaces A and B is defined as $|A,B|_u = \inf_f \operatorname{dis}(f)$ where the infimum is taken over all bijections $f : A \to B$. A sequence of metric spaces M_n *Hausdorff-converges* to a metric space M iff for every $\varepsilon > 0$ there is an ε–net M_ε in M which is the uniform limit of ε–nets $(M_n)_\varepsilon$ in M_n. We say that a sequence (M_n, p_n) of unbounded, pointed metric spaces Hausdorff-converges to a pointed metric space (M,p) if for every $r > 0$ the balls $B_r(p_n)$ in M_n Hausdorff-converge to the ball $B_r(p)$ in M.

Let v_0 be an (arbitrary) point of the locally symmetric space $V = \Gamma\backslash X$. We define the *asymptotic cone* of V as the Hausdorff-limit of pointed metric spaces:

$$\operatorname{Cone}(V) := \mathcal{H} - \lim_{n\to\infty}(V, v_0, \frac{1}{n} d_V).$$

By Corollary 4.2 there is a net N in V consisting of disjoint closed Euclidean cones $\pi(q_i\mathcal{C}_*))$ in V. We identify the abelian Lie algebra $\mathfrak{a}$ with $\mathbb{R}^q$. For all $n \in \mathbb{N}$ we define a map

$$f_n : N \subset (V, \frac{1}{n} d_V) \to C(\Gamma\backslash|\mathcal{T}|);\ \text{by}$$

$$f_n(\pi(q_j(\exp H)\cdot x_0)) := \frac{1}{n} H \in \log \mathcal{C}_* \cong C(\triangle^j) \subset \mathbb{R}^q.$$

By Remark 4.3 and the second construction of $C(\Gamma\backslash|\mathcal{T}|)$ f_n is a bijection from the interior of N onto its image in $C(\Gamma\backslash|\mathcal{T}|)$; note that this image is open and dense in $C(\Gamma\backslash|\mathcal{T}|)$ for all n.

Lemma 5.1. *There is a constant $D \geq 0$ such that $\operatorname{dis}(f_n) \leq \frac{1}{n}D$ for all $n \in \mathbb{N}$, i.e., f_n is a $(1, \frac{1}{n}D)$–quasi-isometry. In particular, V is quasi-isometric to $C(\Gamma\backslash|\mathcal{T}|)$.*

Proof. We consider some polyhedral exhaustion $V = \bigcup_{s\geq 0} V(s)$. The intersection of the net N with $V\setminus V(0)$ is still a net in V since $V(0)$ is compact. Let u, v be two points in the interior of $N \cap (V\setminus V(0))$. We take a path $c([0,L])$ in V between u and v of minimal length and parametrized by arc-length. By Proposition 2.2 we can write $V = \pi(\Omega) = \bigcup_{j=1}^m q_j\mathcal{S}$ for a fundamental set Ω. We can thus find $t_i \in [0,L]$, $0 \leq i \leq n+1$ such that $t_0 = 0, t_{n+1} = L$ and $[0,L] = \bigcup_{i=0}^n [t_i, t_{i+1}]$ with $c([t_i,t_{i+1}]) \subset \pi(q_{j_i}\mathcal{S})$ for $j_i \in \{1,\dots,m\}$. Associated to the path c we thus get a well-defined string $(j_0, j_1, \dots, j_n)$. We next replace c by a path $\overline{c}$ whose associated string contains an element $j_k \in \{1,\dots,m\}$ at most once: Start with j_0 and assume that $0 \leq l \leq n$ is the greatest index such that $j_l = j_0$. Geometrically this means that the path c returns to $\pi(q_{j_0}\mathcal{S})$ at $c(t_l)$. Using Theorem 4.1 and the definition of a Siegel set $\mathcal{S}$ we can replace $c([t_0, t_{l+1}])$ by a geodesic segment, say $\overline{c}([s_{j_0}, s_{j_0+1}])$, in $\pi(q_{j_0}\mathcal{C})$ whose length L satisfies

$$L(\overline{c}([s_{j_0}, s_{j_0+1}])) \leq L(c[t_0, t_{l+1}]) + 2D_1,$$

for $D_1 := \operatorname{diam}(\omega)$. Repeating this procedure with j_{l+2} etcetera we eventually get a sequence of at most m segments $\overline{c}([s_k, s_{k+1}])$ in N of total length $\leq L(c) + 2mD_1$. We next show that these segments can be chosen in such a way that they are mapped by f_1 to a continous path $\tilde{c}$ in $C(\Gamma\backslash|\mathcal{T}|)$ from $f_1(u)$ to $f_1(v)$.

In fact, each $\pi(q_j\mathcal{C}_*) \subset \pi(q_j\mathcal{S}) \subset V$ is isometric to a simplicial cone in $C(\Gamma\backslash|\mathcal{T}|)$. The map f_1 thus yields well defined images of (the interior of) the segments $\overline{c}([s_k, s_{k+1}]) \in \pi(q_k\mathcal{C}_*)$. By construction the endpoint $\overline{c}(s_{k-1}) \in \pi(q_{k-1}\mathcal{C}_*)$ and the initial point $\overline{c}(s_k) \in \pi(q_k\mathcal{C}_*)$ are on the same levelset, say $\partial V(s)$, of the exhaustion of V we have chosen above and by construction are at most the distance $2D_1$ apart. By the (second) construction of $C(\Gamma\backslash|\mathcal{T}|)$ we can join their images by segments in $C(\Gamma\backslash|\mathcal{T}|)$ of uniformly bounded length $2D_1$ to obtain the path $\tilde{c}$. This argument yields the estimate

$$d_C(f_1(u), f_1(v)) \leq L_C(\tilde{c}) \leq 4mD_1 + L(c) = 4mD_1 + d_V(u, v).$$

On the other hand, given a geodesic path $\tilde{c}$ in $C(\Gamma\backslash|\mathcal{T}|)$ joining two points x and y in the image of the interior of $N \cap (V \setminus V(0))$ under f_1, we can lift the segments contained in the simplicial cones $C(\triangle^j)$ to the corresponding chambers $\pi(q_j\mathcal{C}_*)$ via f_1^{-1}. By the same arguments as before the endpoints of the lifted segments can be joined in V to form a continuous path c between $f_1^{-1}(x)$ and $f_1^{-1}(y)$ of length

$$\text{(2)} \qquad d_V(f_1^{-1}(x), f_1^{-1}(y)) \leq L(c) \leq 2mD_1 + L(\tilde{c}) \leq 2mD_1 + d_C(x, y).$$

Combining (1) and (2) and setting $D := 4mD_1$ we get for all u, v in (the interior of) $N \cap (V \setminus V(0))$ that

$$|d_V(u, v) - d_C(f_1(u), f_1(v))| \leq D.$$

Finally since for any $n \in \mathbb{N}$

$$\begin{aligned} |\frac{1}{n}d_V(u,v) - d_C(f_n(u), f_n(v))| &= |\frac{1}{n}d_V(u,v) - \frac{1}{n}d_C(f_1(u), f_1(v))| = \\ &= \frac{1}{n}|d_V(u,v) - d_C(f_1(u), f_1(v))| \leq \frac{1}{n}D, \end{aligned}$$

the Lemma follows. ∎

We are now prepared to state a key result of this paper. The following theorem has been proved independently by T. Hattori. His approach is not intrinsic and uses equivariant Hausdorff limits with respect to suitable finite index subgroups of Γ and an embedding into a space of positive definite matrices (see [17]).

Theorem 5.2. *Let X be a Riemannian symmetric space of rank ≥ 2, let Γ be an irreducible, non-uniform lattice in the group of isometries of X and let V be the locally symmetric space $\Gamma\backslash X$. Then the asymptotic cone* Cone(V) *is isometric to the Euclidean cone $C(\Gamma\backslash|\mathcal{T}|)$ over the finite simplicial complex $\Gamma\backslash|\mathcal{T}|$.*

Proof. Given $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$, such that $N' := N \cap (V \setminus V(0)) \subset V$ is an ε–net in $(V, \frac{1}{n}d_V)$ for all $n \geq n_\varepsilon$. Since $f_1(N')$ is dense in $C(\Gamma\backslash|\mathcal{T}|)\backslash f_1(N \cap V(0))$ it is an ε–net in the latter for all $\varepsilon > 0$. Then there is r_0 such that for $r \geq r_0$ the same assertions are true for the subsets of balls with radius r:

$$N' \cap B_r(v_0) \subset (V, \frac{1}{n}d_V) \text{ and } f_n(N') \cap B_r(\mathcal{O}) \subset C(\Gamma\backslash|\mathcal{T}|).$$

From Lemma 5.1 and its proof we obtain the following uniform estimate:

$$|N' \cap B_r(v_0), f_n(N') \cap B_r(\mathcal{O})|_u \leq \frac{1}{n} D,$$

which implies the theorem. ■

The definition of the asymptotic cone of a metric space S implies that bi-lipschitz invariants of $\mathrm{Cone}(S)$ are quasi-isometry invariants of S (see [14] for a discussion). It is well-known that the $\mathbb{R}$–rank of $\mathbf{G}$ equals the rank of the symmetric space X, i.e., the dimension of a maximal totally geodesic submanifold of X with sectional curvature zero. The dimension of a simplicial complex is equal to the dimension of a maximal simplex. We thus define

$$\dim \mathrm{Cone}(\Gamma\backslash X) := \dim(\Gamma\backslash|\mathcal{T}|) + 1.$$

Theorem 5.2 yields the following geometric interpretation of the $\mathbb{Q}$–rank and the cohomological dimension, $\mathrm{cd}\,\Gamma$, of Γ:

Corollary 5.3. *Let X be a symmetric space of noncompact type and rank ≥ 2 and let Γ be an irreducible (arithmetic) lattice in the isometry group of X. Then*

(i) $$\mathrm{rank}_{\mathbb{Q}}\Gamma = \dim \mathrm{Cone}(\Gamma\backslash X)$$

(ii) $$\mathrm{cd}\,\Gamma = \dim X - \dim \mathrm{Cone}(\Gamma\backslash X).$$

Proof. By a well-known result of Borel and Harish-Chandra V is compact if and only if $\mathrm{rank}_{\mathbb{Q}}\Gamma = 0$ (see [4]). If $V = \Gamma\backslash X$ is compact then $\mathrm{Cone}(V)$ is a point, i.e., $\dim \mathrm{Cone}(V) = 0$. On the other hand if V is not compact there is a geodesic ray in V and thus $\dim \mathrm{Cone}(V) \geq 1$. This proves the formula in the compact case. If V is not compact, i.e., if $\mathrm{rank}_{\mathbb{Q}}\Gamma \geq 1$, we have by Theorem 5.2

$$\dim \mathrm{Cone}(\Gamma\backslash X) = \dim(\Gamma\backslash|\mathcal{T}|) + 1 = \dim|\mathcal{T}| + 1 = \mathrm{rank}_{\mathbb{Q}}\mathbf{G} = \mathrm{rank}_{\mathbb{Q}}\Gamma$$

and hence (i). Equality (ii) follows from (i) and [7] 11.4.3. ■

Remark 5.4. In [20] B. Kleiner and B. Leeb showed that the asymptotic cone (with respect to any fixed ultrafilter) of a globally symmetric space X is a Euclidean building. The rank of this building is equal to the rank of X, i.e., to the $\mathbb{R}$–rank of $\mathbf{G}$. The latter is therefore a quasi-isometry invariant of X, whereas the $\mathbb{Q}$–rank is a quasi-isometry invariant of $\Gamma\backslash X$.

Corollary 5.5. *The asymptotic cone of a locally symmetric space $V = \Gamma\backslash X$ is a* CAT(0) *space and thus in particular contractible.*

Proof. By Theorem 5.2 $\mathrm{Cone}(V)$ is isometric to the Euclidean cone $C(\Gamma\backslash|\mathcal{T}|)$. A theorem of V.N. Berestovski says that for a geodesic metric space Y the Euclidean cone $C(Y)$ is a CAT(0) space iff Y is a CAT(1) space (see [9], Ch. II.4). The claim then follows from Lemma 3.4. ■

3 $q_j x$

2

$q_i x$ $1 = 1'$

$3'$ $2'$ $q_i y$

Figure 5.1

5.2. A conjecture of Siegel and its proof.

Remark 5.6. Theorem 4.1 could be deduced from an estimate about the distance function d_X of X when restricted to a fundamental set Ω, which is given in a paper by A. Borel (see [5], Theorem 2.3). But unfortunately the second part of the proof given there only works in the $\mathbb{Q}$–rank 1 case and is not complete for $\mathbb{Q}$–rank ≥ 2 (this was also observed in the preprint version of [19]). More precisely, in [5], pp. 550-552, inequality (12) does not imply inequality (14) and this in turn does not imply inequality (5) as is claimed there. Nevertheless the assertion of the theorem, which in particular answers a question of C.L. Siegel (see [27], Section 10), is correct and we next present a complete proof based on Theorem 4.1 and Lemma 5.1.

Theorem 5.7. *For $q_k \in \mathbf{G}(\mathbb{Q})$ $(k = 1, \ldots, m)$ and a Siegel set $\mathcal{S}$ as in Proposition 2.2 there is a constant D' such that*

$$d_X(\gamma q_i x, q_j y) \geq d_X(x, y) + D',$$

for all $i, j \in \{1, \ldots, m\}$, all $\gamma \in \Gamma$ and all $x, y \in \mathcal{S}$. In particular the canonical projection $\pi : X \longrightarrow \Gamma\backslash X$ restricted to each Siegel set $q_i\mathcal{S}$ is a $(1, D)$–quasiisometry.

Proof. The definition of the Siegel set $\mathcal{S}$ implies that the $\mathbb{Q}$–Weyl chamber $\mathcal{C}_*$ is a net in $\mathcal{S}$. Hence it suffices to prove the estimate for $x, y \in \mathcal{C}_*$. By definition of d_V and Lemma 5.1 we have

$$d_X(\gamma q_i x, q_j y) \geq d_V(\pi(q_i x), \pi(q_j y)) \geq \\ \geq d_C(f_1(\pi(q_i x)), f_1(\pi(q_j y))) - D.$$

We can further decompose a shortest path c in $C(\Gamma\backslash|\mathcal{T}|)$ into segments contained in the simplicial cones $C(\triangle^l) \subset C(\Gamma\backslash|\mathcal{T}|)$. Since all these Euclidean cones are isometric (for $l = 1, \ldots, m$) we can take copies of successive segments to build a continuous path c' of the same length as c in a single cone, say $C(\triangle^i)$. In Fig. 5.1 is indicated the "shadow" of c and c' in the basis $\Gamma\backslash|\mathcal{T}|$ of the cone $C(\Gamma\backslash|\mathcal{T}|)$. It then follows that

$$d_C(f_1(\pi(q_j x)), f_1(\pi(q_i y))) \geq d_C(f_1(\pi(q_i x)), f_1(\pi(q_i y))) \geq \\ \geq d_V(\pi(q_i x), \pi(q_i y)) - D,$$

where for the last inequality we used Lemma 5.1 again. Finally, by Theorem 4.1, we have that $d_V(\pi(q_i x), \pi(q_i y)) = d_X(q_i x, q_i y) = d_X(x, y)$; this proves the theorem. ■

Theorem 5.7 has independently been proved by Ji and in a special case by Ding (see Section 1).

6. On the Tits geometry of locally symmetric spaces

We consider a locally symmetric space $V = \Gamma\backslash X$ as in the previous section 5, i.e., X is a symmetric space of non-compact type and rank ≥ 2 and Γ is an irreducible, non-uniform (arithmetic) lattice in the isometry group of X. We remark in passing that the boundary at infinity $\partial_\infty X$ of the *globally* symmetric space X can be identified with a geometric realization of the spherical Tits building associated to $\mathbf{G}(\mathbb{R})$; its dimension is equal to $\operatorname{rank} X - 1 = \operatorname{rank}_{\mathbb{R}} \mathbf{G} - 1$. The following combinatorial description of the boundary at infinity of the *locally* symmetric space V in terms of the Tits building $\mathcal{T}$ associated to $\mathbf{G}(\mathbb{Q})$ is due to Ji and MacPherson (see [19]). As an application of Theorem 5.2 we obtain a new proof of this result.

Theorem 6.1. *The equivalence classes of asymptotic geodesic rays in the locally symmetric space V correspond bijectively to the points of $\Gamma\backslash|\mathcal{T}|$, i.e.,*

$$\partial_\infty V \cong \Gamma\backslash|\mathcal{T}|.$$

Before proving this theorem we describe the geodesic rays in the Euclidean cone $C(\Gamma\backslash|\mathcal{T}|)$.

Lemma 6.2. *Any geodesic ray c in $C(\Gamma\backslash|\mathcal{T}|)$ is of the form $c(t) = (t_0 + t, z)$ for some $t_0 \geq 0$ and $z \in \Gamma\backslash|\mathcal{T}|$. Moreover two rays c_1 and c_2 are asymptotic* *iff there are* $t_1, t_2 \geq 0$ *with* $c_1(t) = (t_1 + t, z)$ *and* $c_2(t) = (t_2 + t, z)$*, i.e.,* $\partial_\infty C(\Gamma\backslash|\mathcal{T}|) \cong \Gamma\backslash|\mathcal{T}|$.

Proof. Let $c : [0, \infty) \to C(\Gamma\backslash|\mathcal{T}|)$ be a geodesic ray, i.e., a curve which realizes the distance between any two of its points. If c eventually lies in a single chamber $C(\triangle^k)$ considered as a subset of $C(\Gamma\backslash|\mathcal{T}|)$, then it is clearly of the claimed form. We may thus assume, that the ray c does not eventually stay in a single chamber. It then has to return to a fixed chamber, say $C(\triangle^j)$, i.e., there are $0 \leq t' < t''$ and $1 \leq j \leq m$ such that $c(t'), c(t'') \in C(\triangle^j)$. Since the simplicial cones $C(\triangle^k) \subset C(\Gamma\backslash|\mathcal{T}|)$, $k = 1, \ldots, m$, are all isometric we can take copies of the segments of $c([t', t''])$ in different chambers to build a continuous path of the same length $d_C(c(t'), c(t'')) = t'' - t'$ in $C(\triangle^j)$ (compare the proof of Theorem 5.7). This is a contradiction, because by construction (and the assumption about c) this path is strictly longer than the straight line in $C(\triangle^j)$ between the points $c(t')$ and $c(t'')$ on the geodesic ray c. ■

Proof. (of Theorem 6.1). Let $z \in \partial_\infty V$ be represented by a unit-speed geodesic ray $c : [0, \infty) \to V$, i.e., $z = c(\infty)$. Then c is also a geodesic ray in the rescaled space $(V, \frac{1}{n} d_V)$ and hence c converges to a ray $\hat{c}$ in $\mathrm{Cone}(V) = \lim_{n\to\infty}(V, v_0, \frac{1}{n} d_V)$. If one also has $z = \tilde{c}(\infty)$ then the rays $\tilde{c}$ and c are in finite Hausdorff distance in V and thus converge to the same ray $\hat{c}$ in $\mathrm{Cone}(V)$. By Theorem 5.2 $\mathrm{Cone}(V)$ is the Euclidean cone $C(\Gamma\backslash|\mathcal{T}|)$. Using Lemma 6.2 we thus get a well defined map

$$R : \partial_\infty V \longrightarrow \Gamma\backslash|\mathcal{T}|;\ c(\infty) \longmapsto \hat{c}(\infty),$$

which we claim to be bijective.

We first show that R is *onto*: Let $z \in \Gamma\backslash|\mathcal{T}|$. The simplicial complex $\Gamma\backslash|\mathcal{T}|$ was constructed by pasting together finitely many simplices $\triangle^k$. Thus there is at least one simplex, say $\triangle^j$, such that $z \in \triangle^j \subset \Gamma\backslash|\mathcal{T}|$. We have a natural identification $q_j\mathcal{C}_*(\infty) \cong \triangle^j$ (see Section 3) and by Theorem 4.1 there is a ray of the form $\pi(q_j c(t)) \subset \pi(q_j\mathcal{C}_*) \subset V$ with $R(\pi(q_j c(\infty))) = \hat{c}(\infty) = z$.

Next we show that R is *one-to-one*: Assume that for two rays c_1 and c_2 in the net $N \subset V$ one has

$$\hat{c}_1(\infty) = R(c_1(\infty)) = z = R(c_2(\infty)) = \hat{c}_2(\infty).$$

By Theorem 5.2 and Lemma 6.2 the rays c_1 (resp. c_2) Hausdorff-converge to $\hat{c}_1(t) = (t_1 + t, z)$ (resp. $\hat{c}_2(t) = (t_2 + t, z)$). This implies that there is $n_0 \in \mathbb{N}$ and some constant κ, such that for all $n \geq n_0$ and all $t \in [0, 1]$ one has (see proof of Lemma 5.1)

$$\frac{1}{n} d_V(c_1(tn), c_2(tn)) \leq \frac{\kappa}{n}.$$

Consequently the rays c_1 and c_2 are asymptotic in V. ■

Let X be a simply connected, complete Riemannian manifold of non-positive curvature and let $z_1, z_2 \in \partial_\infty X$ represented by geodesic rays c_1 and c_2, respectively. The *Tits metric* on $\partial_\infty X$ is defined by the formula

$$2 \sin \frac{1}{2} d_{\mathcal{T}}(z_1, z_2) := \lim_{t\to\infty} \frac{1}{t} d_X(c_1(t), c_2(t)).$$

This generalizes the corresponding formula for a globally symmetric space of non-compact type where $d_{\mathcal{T}}$ *is* the Tits distance of the spherical Tits building of $\mathbf{G}(\mathbb{R})$. See the book [12] for details. The next result asserts that exactly the same intrinsic formula also holds in the case of non simply connected, arithmetic quotients.

Corollary 6.3. *For two points z_1, z_2 in $\partial_\infty V \cong \Gamma\backslash|\mathcal{T}|$ represented by unit speed geodesic rays $c_1(t), c_2(t)$ in V, respectively, one has the formula*

$$2\sin\frac{1}{2}d_{\mathcal{T}}(z_1,z_2) = \lim_{t\to\infty}\frac{1}{t}d_V(c_1(t),c_2(t)).$$

Proof. By Theorem 5.2 and the definitions the balls of radius 1 in $(V, \frac{1}{t}d_V)$ centered at v_0 Hausdorff-converge to the ball of radius 1 centered at $\mathcal{O}$ in $C(\Gamma\backslash|\mathcal{T}|)$. Hence using Theorem 6.1 we have

$$\lim_{t\to\infty}\frac{1}{t}d_V(c_1(t),c_2(t)) = d_C((1,z_1),(1,z_2)) =$$
$$= [2-2\cos d_{\mathcal{T}}(z_1,z_2)]^{1/2} = 2\sin\frac{1}{2}d_{\mathcal{T}}(z_1,z_2).$$

Alternatively, one can identify the rays c_1, c_2 with rays in the cone $C(\Gamma\backslash|\mathcal{T}|)$ and then use Corollary 5.5 and the fact the formula holds in any CAT(0) space (see [9], Ch.III 3.6). ■

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