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Discrete Series Representations of Unipotent p-adic Groups

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Abstract. For a certain class of locally profinite groups, we show that an irreducible smooth discrete series representation is necessarily supercuspidal and, more strongly, can be obtained by induction from a linear character of a suitable open and compact modulo center subgroup. If F is a non-Archimedean local field, then our class of groups includes the groups of F-points of unipotent algebraic groups defined over F. We therefore recover earlier results of van Dijk and Corwin.
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Introduction

Let F be a non-Archimedean local field and let G be the group of F-points of a unipotent algebraic group defined over F. Let (π, V) be an irreducible smooth discrete series representation of G. We show that (π, V) can be obtained by (compact) induction from a linear character of an open compact modulo center subgroup. In other words, a discrete series representation of G is always supercuspidal (that is, its matrix coefficients are of compact support modulo the center, not just square-integrable modulo the center) and all supercuspidal representations of G can be realized by induction from compact modulo center subgroups.

When F has characteristic zero, these observations are not new. Indeed, in this case, van Dijk showed that an irreducible discrete series representation of G is always supercuspidal [7]. (In fact, van Dijk worked with topologically irreducible discrete series representations. It is a routine matter, however, to translate his result to the smooth setting. We indicate the straightforward details in §1. below.) Later Corwin showed, independently of [7], that an irreducible discrete series representation of G is always induced from a linear character of an open compact modulo center subgroup [3].

Both authors make essential use of Kirillov theory, which carries over to this setting [5], and it is this which limits their arguments to the characteristic zero

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case (or, for suitable groups, to the case of sufficiently large positive characteristic). In contrast, our approach is valid in all characteristics. Our arguments are completely elementary, and virtually trivial: they rely only on the Schur orthogonality relations for matrix coefficients and some straightforward representation theory. In addition, we use only a simple structural property of G as a locally profinite group, namely that it can be written as a countable union of compact modulo center subgroups, each one normal in the next. For such groups, we show that each irreducible discrete series representation can be obtained by induction from a suitable compact modulo center subgroup. When G is the group of F-points of a unipotent algebraic group, it is easy to see further that such representations are monomial, as G is then nilpotent as an abstract group.

We note that the results of this paper carry over trivially to the case in which $G = \underline{G}(F)$ with $\underline{G}$ a connected nilpotent algebraic group defined over F, as G is then the product of the F-points of a central torus in $\underline{G}$ and the F-points of the unipotent radical of $\underline{G}$.

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1. Smooth versus unitary discrete series

In this section, G denotes a separable unimodular locally profinite group. We briefly recall some basic definitions and constructions. (Similar material is treated, in somewhat more detail, in [2] §1.2.) This section is not needed in the next. We include it only to clearly demonstrate that our main result fully contains that of [7].

We write Z for the center of G. Let (π, V) be a smooth irreducible representation of G and write $(\pi^\vee, V^\vee)$ for the smooth dual or contragredient of (π, V). Since G is separable, (π, V) admits a central character χ. We assume that this character is unitary. Let $v \in V, v^\vee \in V^\vee$ and set $f_{v,v^\vee}(g) = \langle \pi(g)v, v^\vee \rangle$, $g \in G$, where, as usual, $\langle\ ,\ \rangle$ denotes the canonical pairing. As χ is unitary, $|f_{v,v^\vee}|$ is a function on G/Z. Then (π, V) is a discrete series representation if $|f_{v,v^\vee}|$ belongs to $L^2(G/Z)$ for all $v \in V, v^\vee \in V^\vee$. It is easy to prove that an irreducible smooth discrete series representation is admissible (that is, for each open subgroup K of G, the space V^K of K-fixed vectors is finite-dimensional). This follows, for example, from a slight modification of the proof of a) $\Rightarrow$ b) on page 26 of [1] (which shows that a finitely generated smooth representation with compactly supported matrix coefficients is admissible).

We next fix a Haar measure $d\dot{g}$ on G/Z. Consider the space of, say, continuous functions $f : G \to \mathbb{C}$ such that

1. $f(zg) = \chi(z)f(g)$, for all $z \in Z, g \in G$,
2. $\int_{G/Z} |f(g)|^2 \, d\dot{g} < \infty$.

This carries the obvious inner product

$$(f_1, f_2) = \int_{G/Z} f_1(g)\overline{f_2(g)} \, d\dot{g}.$$

Of course, G acts on this space by right translations and the action preserves the inner product. Suppose now that (π, V) *is* a discrete series representation. Then, for any non-zero $v^\vee \in V^\vee$, the non-zero map $v \mapsto f_{v,v^\vee}$ intertwines π with the above action. It follows that (π, V) admits a G-invariant inner product, which we again denote by $(\ ,\ )$. Further, as (π, V) is admissible, the inner product is unique up to positive scalars. This leads to the following part of Schur orthogonality: there is a positive scalar $d(\pi)$, the formal degree of π, such that

$$\int_{G/Z} (\pi(g)v_1, w_1)\overline{(\pi(g)v_2, w_2)}\, d\dot{g} = \frac{1}{d(\pi)}(v_1, v_2)\overline{(w_1, w_2)}, \tag{1}$$

for all $v_1, v_2, w_1, w_2 \in V$. If G/Z is compact, then (π, V) has finite dimension and $d(\pi)$ is simply this dimension divided by the measure of G/Z.

Suppose next that (τ, H) is a unitary representation of G. Thus H is a Hilbert space and, for all $g \in G$, $\tau(g)$ is a unitary operator on H such that, for all $u, v \in H$,

$$g \mapsto (\tau(g)u, v) : G \to \mathbb{C}$$

is continuous, where $(\ ,\ )$ now denotes the inner product on H. The representation (τ, H) is irreducible if it is non-zero and has no non-trivial closed G-invariant subspace. An irreducible unitary representation admits a central character. Then (τ, H) is a discrete series representation if it is irreducible and if, for all $u, v \in H$, $g \mapsto |(\tau(g)u, v)|$ belongs to $L^2(G/Z)$.

A unitary discrete series representation is admissible (see, for example, [4] Theorem 2).

We briefly recall the connection between the two notions of the discrete series. We again let (π, V) be an irreducible smooth discrete series representation of G. We write $\widehat{V}$ for the Hilbert space completion of V with respect to a G-invariant inner product on V. Then the action of G on V via π extends to $\widehat{V}$ to yield a unitary representation $(\widehat{\pi}, \widehat{V})$. Using the admissibility of (π, V), it is easy to check that this unitary representation is irreducible and hence discrete series.

In the other direction, let (τ, H) be a unitary (irreducible) discrete series representation of G. Let H_∞ denote the space of smooth vectors, that is, all vectors fixed by some open subgroup of G. Then H_∞ is G-invariant and so defines a smooth representation (τ_∞, H_∞) of G. Again, one checks readily, using the admissibility of (τ, H), that (τ_∞, H_∞) is (algebraically) irreducible. It is obviously also a discrete series representation.

These procedures, $(\pi, V) \mapsto (\widehat{\pi}, \widehat{V})$ and $(\tau, H) \mapsto (\tau_\infty, H_\infty)$, are mutually inverse, in the sense that $\widehat{V}_\infty \cong V$ in the category of smooth representations and $\widehat{H_\infty} \cong H$ in the category of unitary representations. They therefore induce bijections between the set of equivalence classes of smooth irreducible discrete series representations of G and the set of equivalence classes of irreducible unitary discrete series representations of G.

The paper [7] works in the unitary setting. Its main result is the following. Let G be the group of F-points of a unipotent algebraic group defined over F, where F has characteristic zero, and let (τ, H) be an (irreducible) unitary discrete series representation of G. Then for all $u, v \in H_\infty$, the function $g \to (\tau(g)u, v)$ has compact support modulo Z. In the next section, we will prove a sharper version of the analogous statement for an irreducible *smooth* discrete series representation, in fact for a slightly wider class of groups. By the bijections above, van Dijk's result is a simple formal consequence of ours.

2. Discrete series implies supercuspidal

We assume now that G contains a sequence of open subgroups (K_i) such that

1. each K_i contains the center Z of G and is compact modulo Z;
2. K_i is normal in K_{i+1}, for all i;
3. $G = \bigcup_{i=1}^{\infty} K_i$.

We note first that the group of F-points of a unipotent algebraic group defined over F always contains such a sequence. Indeed, any such group embeds as a closed subgroup of some group U of unipotent upper triangular matrices. It can therefore be expressed as a union $\bigcup_{i=1}^{\infty} H_i$ with each H_i an open and compact modulo center subgroup such that $H_i \subset H_{i+1}$ for all i. Further, as U is nilpotent as an abstract group, each H_i is nilpotent. Thus each subgroup of H_i is strictly contained in its normalizer, for all i. Since $[H_i : H_{i-1}]$ is finite (for $i \geq 2$), it follows that we can refine the sequence (H_i) to obtain a sequence (K_i) as above.

We now state the main result.

Theorem 2.1. *Let G be a locally profinite group as above and let (π, V) be an irreducible smooth discrete series representation of G. Then there is an open compact modulo center subgroup K of G and an irreducible smooth representation ρ of K such that $\pi \cong ind_K^G \rho$.*

Proof. Fix a sequence of subgroups (K_i) as above. Let ρ_1 be an irreducible component of $\pi|_{K_1}$. For each $i \geq 2$, we inductively choose an irreducible component ρ_i of $\pi|_{K_i}$ such that $\rho_i|_{K_{i-1}}$ contains a subrepresentation isomorphic to ρ_{i-1}. Let m_i denote the multiplicity of ρ_i in $\pi|_{K_i}$. Further, for each irreducible representation τ of K_{i+1}, we write $m(\tau)$ for the multiplicity of τ in $\pi|_{K_{i+1}}$ and $[\tau|_{K_i} : \rho_i]$ for the multiplicity of ρ_i in $\tau|_{K_i}$. By considering how the irreducible components of $\pi|_{K_{i+1}}$ restrict to K_i, we see that

$$m_i = \sum_{\tau} m(\tau)\, [\tau|_{K_i} : \rho_i]$$

where the sum is over all irreducible components τ of $\pi|_{K_{i+1}}$. In particular,

$$m_i \geq m_{i+1}\, [\rho_{i+1}|_{K_i} : \rho_i] \geq m_{i+1}. \tag{2}$$

Thus (m_i) is a decreasing sequence of positive integers and so is eventually constant. Let $m = \lim_{i\to\infty} m_i$ and reindex the groups K_i, if necessary, so that $m_i = m$ for all $i \geq 1$. Of course, (2) then implies that

$$[\rho_{i+1}|_{K_i} : \rho_i] = 1, \quad \forall i. \tag{3}$$

Let $(\ ,\ )$ be a G-invariant inner product on V. For each i, we choose a vector v_i with $(v_i, v_i) = 1$ such that v_i generates an irreducible K_i-subspace isomorphic to ρ_i. (Note that, having chosen v_1, we cannot choose all v_i to be equal to v_1 because we do not (yet) know that v_1 generates an irreducible K_i-space.) We also fix a Haar measure $d\dot{g}$ on G/Z. Of course, this restricts to a Haar measure $d\dot{k}$ on the open subgroup K_i/Z. Then

$$\int_{G/Z} |(\pi(g)v_i, v_i)|^2 d\dot{g} \geq \int_{K_i/Z} |(\pi(k)v_i, v_i)|^2 d\dot{k}.$$

Let $d(\pi)$ denote the formal degree of (π, V) and write $|K_i/Z|$ for the measure of K_i/Z (both with respect to $d\dot{g}$). Then, since $(v_i, v_i) = 1$, (1) and the succeeding sentence imply that

$$\frac{1}{d(\pi)} \geq \frac{|K_i/Z|}{\dim \rho_i}.$$

Thus the sequence $(\frac{|K_i/Z|}{\dim \rho_i})$ is bounded above.

Consider the restriction $\rho_{i+1}|_{K_i}$. Clifford theory clearly applies in this setting. It then follows from (3) that $\rho_{i+1}|_{K_i}$ is multiplicity free. Further, if we put $\widetilde{K_i} = \{y \in K_{i+1} : {}^y\rho_i \cong \rho_i\}$, then

$$\rho_{i+1}|_{K_i} \cong \bigoplus_{y \in K_{i+1}/\widetilde{K_i}} {}^y\rho_i.$$

Therefore

$$\dim \rho_{i+1} = [K_{i+1} : \widetilde{K_i}] \dim \rho_i.$$

This divides

$$[K_{i+1} : K_i] \dim \rho_i = [K_{i+1}/Z : K_i/Z] \dim \rho_i = \frac{|K_{i+1}/Z|}{|K_i/Z|} \dim \rho_i,$$

and so there are positive integers N_i such that

$$\frac{|K_i/Z|}{\dim \rho_i} N_i = \frac{|K_{i+1}/Z|}{\dim \rho_{i+1}}, \quad \forall i. \tag{4}$$

Since the sequence $(\frac{|K_i/Z|}{\dim \rho_i})$ is bounded above, we deduce that it must eventually be constant. We again reindex, if necessary, so that it is actually constant. Then

$$\dim \rho_i = \frac{|K_i/Z|}{|K_{i-1}/Z|} \dim \rho_{i-1} = [K_i : K_{i-1}] \dim \rho_{i-1}, \quad \forall i \geq 2.$$

Hence

$$\dim \rho_i = [K_i : K_1] \dim \rho_1, \quad \forall i \geq 1,$$

and thus

$$\rho_i \cong \operatorname{ind}_{K_1}^{K_i} \rho_1, \quad \forall i \geq 1.$$

It follows that $\operatorname{ind}_{K_1}^{G} \rho_1$ is irreducible, since this is an increasing union of the irreducible K_i-subspaces ρ_i. Since $\pi|_{K_1}$ contains ρ_1, there is a non-zero G-map from $\operatorname{ind}_{K_1}^{G} \rho_1$ to π. As both representations are irreducible, this map is an isomorphism. ■

Remark 2.2. Equation (4) is key to the above proof. We used Clifford theory to deduce it from the multiplicity-one statement (3). With a slightly more involved use of Clifford theory and some related notions, one can directly establish (4) and so avoid the multiplicity considerations of the beginning of the proof.

Corollary 2.3. *Let (π, V) be an irreducible smooth discrete series representation of G. Then (π, V) is supercuspidal in the sense that all of its matrix coefficients have compact support modulo Z.*

Proof. As a matrix coefficient of ρ extends (by zero) to a matrix coefficient of π, it is clear that (π, V) has at least one non-zero matrix coefficient, say $f_{w,w^\vee}$, that has compact support modulo Z. Now consider the space of all vectors v in V such that $f_{v,w^\vee}$ has compact support modulo Z. This is a non-zero G-subspace, as it contains w. It therefore equals V.

Since (π, V) is irreducible and admissible, its smooth dual $(\pi^\vee, V^\vee)$ is again irreducible. For each $v \in V$, we consider the space of vectors $v^\vee \in V^\vee$ such that $f_{v,v^\vee}$ has compact support modulo Z. By the preceding paragraph, this space contains $w^\vee$ and so is non-zero, and hence equals $V^\vee$. This completes the proof. ∎

Remark 2.4. The proof shows that if an irreducible admissible representation has one matrix coefficient that has compact support (or is square-integrable) modulo the center, then all coefficients have this property. In the absence of admissibility, this implication no longer holds (for a general locally profinite group). For example, the group consisting of the matrices in $GL_2(F)$ with second row $(0\ 1)$ has trivial center and admits an irreducible smooth representation such that some, but not all, matrix coefficients have compact support.

Suppose now that the group K in the statement of Theorem 2.1 is supersolvable. Then the irreducible smooth representation ρ of K is necessarily monomial, that is, is induced from a linear character of a closed (equivalently, open) subgroup. (This follows from the proof of the corresponding fact for irreducible representations of a supersolvable finite group. See, for example, [6] §8.5.) Now if G is the group of F-points of a unipotent algebraic group defined over F, then G is nilpotent as an abstract group, whence its subgroup K is also nilpotent, and so, *a fortiori*, supersolvable. We can therefore slightly refine Theorem 2.1 in this case and, in particular, recover the main result of [3].

Corollary 2.5. *Let G be the group of F-points of a unipotent algebraic group defined over F. Let (π, V) be an irreducible smooth discrete series representation of G. Then there is an open compact modulo center subgroup H of G and a linear character λ of H such that $\pi \cong ind_H^G \lambda$.*

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