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The Weak Paley-Wiener Property for Group Extensions

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Abstract. The paper studies weak Paley-Wiener properties for group extensions by use of Mackey's theory. The main theorem establishes sufficient conditions on the dual action to ensure that the group has the weak Paley-Wiener property. The theorem applies to yield the weak Paley-Wiener property for large classes of simply connected, connected solvable Lie groups (including exponential Lie groups), but also criteria for non-unimodular groups or motion groups.
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0. Introduction

The *weak Paley-Wiener property (wPW)* can be formulated as follows: Let G be a second countable, type I locally compact group, and define

$$\mathrm{L}^\infty_c(G) = \{f : G \to \mathbb{C} : f \text{ measurable, bounded and compactly supported}\}.$$

Then G has wPW if

$$\forall f \in \mathrm{L}^\infty_c(G)\ \forall \Gamma \subset \widehat{G} : \left(\widehat{f}|_\Gamma = 0 \text{ and } \nu_G(\Gamma) > 0 \Rightarrow f = 0 \right) \tag{1}$$

where $\widehat{f}$ denotes the operator-valued Fourier transform on $\mathrm{L}^1(G)$, and the measure ν_G on $\widehat{G}$ is the Plancherel measure of G.

Remark 0.1. In principle the weak Paley-Wiener property may be formulated for other spaces than $L^\infty_c(G)$, in particular for the larger space $L^2_c(G)$ of compactly supported L^2-functions, or for the smaller space $C_c(G)$. It turns out however that wPW for $C_c(G)$ implies wPW for the larger space, by the following convolution argument: Let $f \in L^2_c(G)$ be such that $\widehat{f}$ vanishes on a set A of positive Plancherel measure. Then for all $g \in \mathrm{L}^2_c(G)$ we have $f * g \in C_c(G)$, and $\widehat{f * g}$ vanishes on A by the convolution theorem. Hence wPW on $C_c(G)$ implies $f * g = 0$. Since this holds for all $g \in \mathrm{L}^2_c(G)$, $f = 0$ follows.

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The same argument establishes for a Lie group G that wPW for $C_c^\infty(G)$ implies wPW for compactly supported distributions.

For our purposes $\mathrm{L}_c^\infty(G)$ seems the most convenient space to deal with. Imposing additional smoothness assumptions on f does not really help shorten the proofs, as we need to lift functions onto quotients by use of a cross-section. Since usually no smooth cross-section is available, this procedure destroys any smoothness.

The statement originated from real Fourier analysis. The fact that $\mathbb{R}$ has wPW follows easily from the observation that the Fourier transform of a compactly supported function is analytic. An even simpler argument works for $\mathbb{Z}$; Fourier transforms of elements of $\mathrm{L}_c^\infty(\mathbb{Z})$ are trigonometric polynomials. On the other hand, the circle group $\mathbb{T}$ does not have wPW, for obvious reasons. wPW has been proved for various (unimodular, type I) groups, in particular connected, simply connected nilpotent Lie groups [17, 18, 9, 15, 1]; with generalisations to completey solvable groups [10].

The property can be viewed as an uncertainty principle: If f is compactly supported, $\widehat{f}$ is spread over all of $\widehat{G}$. It is interesting to compare wPW to the so-called **qualitative uncertainty property (Q.U.P.)** studied in [11, 7, 1], stating for all $f \in \mathrm{L}^1(G)$ that

$$\mu_G(\mathrm{supp}(f)) + \nu_G(\mathrm{supp}(\widehat{f})) < \infty \Rightarrow f = 0, \tag{2}$$

where μ_G is left Haar measure, ν_G is Plancherel measure and supp denotes the measure-theoretic support (unique up to a null set). By contrast to wPW, this property is obviously not invariant under choice of equivalent Plancherel measures, but rather rests on a canonical choice (which is available for unimodular groups).

Throughout this paper G denotes a type I locally compact group and N a closed normal subgroup of type I, which is in addition regularly embedded. All groups are assumed to be second countable. The aim is to give sufficient criteria that G has wPW, in terms of analogous properties of N and the little fixed groups.

The paper is structured as follows: We first give a review of the Mackey machine and its uses for the computation of Plancherel measure via techniques due to Kleppner and Lipsman. We then present explicit formulas for the induced representations arising in the construction of $\widehat{G}$, and for the associated representations of $\mathrm{L}^1(G)$, which act via certain operator-valued integral kernels. These formulas will allow to prove the main result of this paper, Theorem 1.2, essentially by repeated application of Fubini's theorem. In the final section we apply Theorem 1.2 to prove wPW for a large class of simply connected, connected solvable Lie groups (Theorem 3.8), thereby considerably generalising the previously published results for nilpotent and completely solvable groups given in [17, 18, 9, 15, 1, 10]. Further consequences are criteria for nonunimodular groups (Corollary 3.2), and a characterisation of wPW for motion groups (Theorem 3.3).

1. Plancherel measure of group extensions

We follow the exposition in [14]. For further details and notation not explained here, the reader is referred to this monograph. Throughout the paper we assume that G is a type I group, and that $N \lhd G$ is a regularly embedded, type I normal subgroup. Left Haar measure on a locally compact group is denoted by $|\cdot|$, and integration against left Haar measure by $\int_G \cdot \, dx$. Given a multiplier ω on G, $\widehat{G}^\omega$ denotes the (equivalence classes of) irreducible unitary ω-representations; ω is omitted when it is trivial. $\overline{\omega}$ is the multiplier obtained by complex conjugation. A multiplier ω is called type I if all ω-representations generate type I von Neumann algebras.

Since we are also dealing with nontrivial Mackey-obstructions, we need to recall a multiplier version of the Plancherel theorem. Given a multiplier ω on G, we denote by $\lambda_{G,\omega}$ the ω-representation of G, acting on $\mathrm{L}^2(G)$ via

$$\lambda_{G,\omega}(x)f(y) = \omega(y^{-1}, x) f(x^{-1}y).$$

If ω is type I, there exists a **Plancherel measure** (unique up to equivalence) $\nu_{G,\omega}$ on $\widehat{G}^\omega$ decomposing $\lambda_{G,\omega}$,

$$\lambda_{G,\omega} \simeq \int_{\widehat{G}^\omega}^{\oplus} \dim(\pi) \cdot \pi \, d\nu_{G,\omega}(\pi)$$

The associated Fourier transform is given by

$$\mathcal{F}_\omega : \mathrm{L}^1(G) \ni f \mapsto (\rho(f))_{\rho \in \widehat{G}^\omega}.$$

ω is omitted whenever it is trivial.

For completeness, we mention the construction of the Plancherel transform associated to the Plancherel measure. In the unimodular case, the measure ν_G can be chosen so that for $f \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ the operator field $\mathcal{F}_\omega(f)$ is in fact a field of Hilbert-Schmidt operators satisfying

$$\int_{\widehat{G}} \|\mathcal{F}_\omega(f)(\sigma)\|_{HS}^2 d\nu_G(\sigma) = \|f\|_2^2,$$

and the Fourier transform extends by density to a unitary equivalence between $\mathrm{L}^2(G)$ and the direct integral of Hilbert-Schmidt spaces. The nonunimodular setting requires right multiplication of $\mathcal{F}_\omega(f)$ with a field of unbounded, densely defined selfadjoint operators with densely defined inverse. In particular, it does not matter whether we formulate wPW with reference to operator-valued Fourier- or Plancherel transform, though the first one is obviously simpler.

Mackey's theory rests on the dual action of G on $\widehat{N}$. We assume that N is regularly embedded, i.e., the orbit space $\widehat{N}/G$ is countably separated. For $\gamma \in \widehat{N}$ let G_γ denote the fixed group under the dual action. Now Mackey's theory provides $\widehat{G}$ as the disjoint union

$$\bigcup_{G.\gamma \in \widehat{N}/G} \{\mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \rho : \rho \in \widehat{G_\gamma/N}^{\overline{\omega_\gamma}}\}.$$

Here ω_γ denotes the multiplier associated to γ, and γ' denotes a fixed choice of an ω_γ-representation of G_γ on $\mathcal{H}_\gamma$ extending γ. Finally, $\rho \in \widehat{G_\gamma/N}^{\overline{\omega_\gamma}}$ is identified with its "lift" to G_γ. In the following, we use the notation

$$\pi_{G.\gamma,\rho} = \mathrm{Ind}_{G_\gamma}^G (\gamma' \otimes \rho)$$

Note that under our assumptions, it follows from a theorem due to Mackey that all $(G_\gamma/N, \overline{\omega_\gamma})$ are type I (see e.g. [14, Chapter III, Theorem 4]). Thus we have description of $\widehat{G}$ as a "fibred set", with base space $\widehat{N}/G$ and fibre

$$\{\mathrm{Ind}_{G_\gamma}^G \gamma' \otimes \rho : \rho \in \widehat{G_\gamma/N}^{\overline{\omega_\gamma}}\} \simeq \widehat{G_\gamma/N}^{\overline{\omega_\gamma}}$$

associated to $G.\gamma$. Moreover, there exist Plancherel measures on $\widehat{N}$ as well as on $\widehat{G_\gamma/N}^{\overline{\omega_\gamma}}$. Now ν_G is obtained by taking the projective Plancherel measures in the fibres, and "glueing" them together using a pseudo-image $\overline{\nu}$ of ν_N on $\widehat{N}$, i.e. the quotient measure of a finite measure equivalent to ν_N. In formulas, up to equivalence ν_G is given according to [13, I, 10.2] by

$$d\nu_G(\pi_{G.\gamma,\rho}) = d\nu_{\widehat{G_\gamma/N}^{\overline{\omega_\gamma}}}(\rho) d\overline{\nu}(G.\gamma). \tag{3}$$

But $\overline{\nu}$ also enters in a measure decomposition of ν_N: There exists quasi-invariant measures $\mu_{G.\gamma}$ on the orbits $G.\gamma \in \widehat{N}/G$ such that

$$d\nu_N(\pi) = d\mu_{G.\gamma}(\pi) d\overline{\nu}(G.\gamma). \tag{4}$$

In view of the "fibrewise" description of $\widehat{G}$ it is useful to generalise the wPW notion to multipliers:

Definition 1.1. Let G be a locally compact group and ω a type I multiplier on G. Then G has ω-wPW, if for every nonzero $f \in \mathrm{L}_c^\infty(G)$, $\mathcal{F}_\omega(f)$ does not vanish on a set of positive ω-Plancherel measure.

Now we have collected enough terminology to formulate the main result of this paper.

Theorem 1.2. *Let G be type I, $N \lhd G$ a type I regularly embedded normal subgroup, with the additional property that for almost $\gamma \in \widehat{N}$, G/G_γ carries an invariant measure.*

(a) Assume that, for almost all $\gamma \in \widehat{G}$, G_γ/N has $\overline{\omega_\gamma}$-wPW. Moreover assume that the following condition holds:

$$\left.\begin{array}{c} \forall \varphi \in \mathrm{L}_c^\infty(G) \\ \forall\ G\text{ - } invariant\ \Gamma \subset \widehat{N} \end{array}\right\} \left(\widehat{f}|_\Gamma = 0 \text{ and } \nu_N(\Gamma) > 0 \Rightarrow f = 0\right). \tag{5}$$

Then G has wPW.

(b) Conversely, if G has wPW, then (5) holds.

The existence of invariant measures on G/G_γ is ensured if G/N is abelian or compact. The fibrewise description of Plancherel measure that we used above can be employed to convenient label the different assumptions: Condition (5) clearly constitutes a kind of "base space wPW", whereas the requirement that G_γ/N has $\overline{\omega_\gamma}$-wPW may be called "fibre wPW". Note that base space wPW holds in particular when N has wPW.

2. Proof of Theorem 1.2

The proof uses ideas and techniques very similar to those in [13]. It rests on the interplay of several measure decompositions: Of Haar measure on G along shifts of subgroups on the group side, and of the Plancherel measures ν_G and ν_N according to (3) and (4).

For explicit calculations it is convenient to realize induced representations $\mathrm{Ind}_H^G\sigma$ on $\mathrm{L}^2(G/H;\mathcal{H}_\sigma)$. Then the integrated representation acts on this space via operator-valued kernels. The proof of Theorem 1.2 uses explicit formulas for these kernels, and their relationship to Fourier transforms of restrictions of φ to cosets mod N. In the following, the restriction of a map f to a subset Y of its domain is denoted by $f|_Y$. $f|_Y = 0$ is to be understood in the sense of vanishing almost everywhere (with respect to a measure that is clear from the context).

First let us recall a few basic results concerning cross-sections, quasi-invariant measures and measure decompositions. If $H < G$, then a cross-section $\alpha : G/H \to G$ is a measurable mapping fulfilling $\alpha(\xi)H = \xi$, for all $\xi = gH \in G/H$. A cross-section $G/H \to G$ is called **regular** if images of compact subsets are relatively compact. All cross-sections in this paper are assumed to be regular, which is justified by the following lemma.

Lemma 2.1. *If G is second countable and $H < G$ closed, there exists a regular cross-section.*

Proof. Denote by $q : G \to G/H$ the quotient map. By [16, Lemma 1.1] there exists a Borel set C of representatives mod H, such that in addition for all $K \subset G$ compact the set $C \cap q^{-1}(q(K))$ is relatively compact.

The associated cross-section is constructed by observing that $q|_C : C \to G/H$ is bijective. Then $q|_C$ is a measurable bijection between standard Borel spaces, hence the inverse map α is also Borel, and it is a cross-section. Moreover, given any compact $K \subset G/H$, there exists a compact $K_0 \subset K$ with $q(K_0) = K$ [8, Lemma 2.46]. Then $\alpha(K) = C \cap q^{-1}(q(K_0))$ is relatively compact. ■

Given a cross-section, we may parametrise G by the map

$$H \times G/H \ni (h, \xi) \mapsto h\alpha(\xi)^{-1}. \tag{6}$$

This particular choice of parametrisation seems a bit peculiar, since it refers to right cosets rather than left ones. Its benefit will become apparent in the proof of Lemma 2.4.

Let us first take a closer look at the form that left Haar measure on G takes in the parametrisation (6). We assume that G/H carries an invariant measure, denoted in the following by $d\xi$.

Lemma 2.2. *For all $f \in \mathrm{L}^\infty_c(G)$,*

$$\int_G f(x)dx = \int_{G/H}\int_H f(h\alpha(\xi)^{-1})\Delta_G(\alpha(\xi))^{-1}dhd\xi.$$

Proof.

$$\begin{aligned}\int_G f(x)dx &= \int_G f(x^{-1})\Delta_G(x^{-1})dx \\ &= \int_{G/H}\int_H f(h^{-1}x^{-1})\Delta_G(h^{-1}x^{-1})dhd(xH) \\ &= \int_{G/H}\int_H f(h^{-1}\alpha(\xi)^{-1})\Delta_G(h^{-1}\alpha(\xi)^{-1})dxd\xi \\ &= \int_{G/H}\int_H f(h\alpha(\xi)^{-1})\Delta_G(h\alpha(\xi)^{-1})\Delta_H(h^{-1})dhd\xi \\ &= \int_{G/H}\int_H f(h\alpha(\xi)^{-1})\Delta_G(\alpha(\xi))^{-1}dhd\xi,\end{aligned}$$

where we used Weil's integral formula and $\Delta_H = \Delta_G|_H$. ∎

Next we note a few technical details concerning the behaviour of restrictions of an L^∞_c-function to shifts of subgroups. For this purpose one further piece of notation is necessary: The left- and right translation operators on G, denoted by R_x, L_y, act via

$$(L_y R_x f)(g) = f(ygx).$$

Lemma 2.3. *Let $\varphi \in \mathrm{L}^\infty_c(G)$ and $H < G$; let $\alpha : G/H \to G$ be a cross-section mapping compact sets to relatively compact sets. Consider the mapping*

$$C : G/H \times G/H \to \mathbb{R}, (\xi,\xi') \mapsto \| \left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\|_1$$

where $\|\cdot\|_1$ denotes the L^1-norm on H.

(a) Given a compact set $K \subset G/H$, the set

$$\{\xi' \in G/H : C(\xi,\xi') \neq 0 \textit{ for some } \xi \in K\}$$

is relatively compact.

(b) C is bounded on compact subsets of $G/H \times G/H$.

Proof. For part (a) note that $\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H$ is not identically zero iff $H \cap \alpha(\xi)^{-1}(\mathrm{supp}(\varphi))\alpha(\xi') \neq \emptyset$. Solving for $\alpha(\xi')$ we obtain the necessary condition

$$\alpha(\xi') \in (\mathrm{supp}(\varphi))^{-1}\alpha(K)H$$

or, equivalently

$$\xi' \in \left((\mathrm{supp}(\varphi))^{-1}\alpha(K)H\right)/H.$$

By assumption on α, $\alpha(K)$ is relatively compact, hence (a) is shown.

For part (b), it is enough to obtain an upper estimate for the Haar measure of $\mathrm{supp}(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H)$, and to consider compact sets of the form $K_1 \times K_2$. Here we see that

$$\begin{aligned}\mathrm{supp}(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H) &\subset H \cap \alpha(\xi)^{-1}\mathrm{supp}(\varphi)\alpha(\xi') \\ &\subset \alpha(K_1^{-1})\mathrm{supp}(\varphi)\alpha(K_2)\end{aligned}$$

and the latter set is relatively compact. ∎

Now we can compute the action of the induced representation.

Lemma 2.4. *Let G be a locally compact group, $H < G$ closed and σ a representation of H acting on a separable Hilbert space $\mathcal{H}_\sigma$. Let $\alpha : G/H \to G$ be a Borel cross-section and assume that there exists an invariant measure on G/H. We realise $\pi = \mathrm{Ind}_H^G \sigma$ on the corresponding vector-valued space $\mathrm{L}^2(G/H; \mathcal{H}_\sigma)$ using α. Then π acts via*

$$\pi(x)f(\xi) = \sigma\left(\alpha(\xi)^{-1}x\alpha(x^{-1}\xi)\right) f(x^{-1}\xi). \tag{7}$$

For $\varphi \in \mathrm{L}_c^\infty(G)$, $\pi(\varphi)$ acts via

$$\left[\pi(\varphi)f\right](\xi) = \int_{G/H} \Phi(\xi,\xi')f(\xi')d\xi' \tag{8}$$

where the right hand side converges in the weak sense for all $\xi \in G/H$, and Φ is an operator-valued integral kernel given by

$$\Phi(\xi,\xi') = \sigma\left(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right) \cdot \Delta_G(\alpha(\xi'))^{-1}. \tag{9}$$

Moreover, we have the equivalence

$$\pi(\varphi) = 0 \Leftrightarrow \Phi(\xi,\xi') = 0 (a.e.) \tag{10}$$

Proof. Formula (7) is well-known. For weak convergence of the right-hand side of (8) let $\eta \in \mathcal{H}_\sigma$, and compute

$$\begin{aligned}
&\int_{G/H} \left|\left\langle \sigma\left(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right)\Delta_G(\alpha(\xi'))^{-1}f(\xi'),\eta\right\rangle\right| d\xi' \leq \\
&\leq \int_{G/H} \left\|\sigma\left(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right)\right\|_\infty \Delta_G(\alpha(\xi'))^{-1} \,\|f(\xi')\|\, \|\eta\|\, d\xi' \\
&\leq \int_{G/H} \left\|\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right\|_1 \Delta_G(\alpha(\xi'))^{-1} \,\|f(\xi')\| d\xi' \,\|\eta\|.
\end{aligned}$$

By Lemma 2.3 (a) the map

$$\xi' \mapsto \left\|\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right\|_1 \Delta_G(\alpha(\xi'))^{-1}$$

is compactly supported, and also bounded, by Lemma 2.3 (b) and boundedness of $\Delta_G^{-1} \circ \alpha$ on the support. Hence the map is square-integrable, and an application of the Cauchy-Schwarz inequality finishes the proof of weak convergence.

For the integrated transform, let $f, g \in \mathrm{L}^2(G/H; \mathcal{H}_\sigma)$. Then, by (7), the weak definition of $\pi(\varphi)$ yields

$$\begin{aligned}
&\langle \pi(\varphi)f, g\rangle = \\
&= \int_{G/H}\int_G \varphi(x)\left\langle \sigma\left(\alpha(\xi)^{-1}x\alpha(x^{-1}\xi)\right) f(x^{-1}\xi), g(\xi)\right\rangle dx d\xi \\
&= \int_{G/H}\int_G \varphi(x)\left\langle \sigma\left(\alpha(\xi)^{-1}x\alpha(x^{-1}\xi)\right) f(x^{-1}\alpha(\xi)H), g(\xi)\right\rangle dx d\xi \\
&= \int_{G/H}\int_G \varphi(\alpha(\xi)x)\left\langle \sigma\left(x\alpha((\alpha(\xi)x)^{-1}\xi)\right) f(x^{-1}H), g(\xi)\right\rangle dx d\xi \qquad (11) \\
&= \int_{G/H}\int_{G/H}\int_H \varphi(\alpha(\xi)h\alpha(\xi')^{-1})\Delta_G(\alpha(\xi'))^{-1} \\
&\qquad \langle \sigma(h)f(\xi'), g(\xi)\rangle \, dh d\xi' d\xi. \qquad (12)
\end{aligned}$$

Here (11) was obtained by a left translation in the integration variable x. (12) used Lemma 2.2, as well as the calculations

$$\alpha(\xi')h^{-1}H = \xi'$$

and

$$\begin{aligned} h\alpha(\xi')^{-1}\alpha((\alpha(\xi)h\alpha(\xi')^{-1})^{-1}\xi) &= h\alpha(\xi')^{-1}\alpha(\alpha(\xi')h^{-1}\alpha(\xi)^{-1}\xi) \\ &= h\alpha(\xi')^{-1}\alpha(\alpha(\xi')h^{-1}H) \\ &= h\alpha(\xi')^{-1}\alpha(\alpha(\xi')H) \\ &= h. \end{aligned}$$

Using the definition of weak integrals, we may continue from (12) to obtain

$$\begin{aligned} &\langle \pi(\varphi)f, g\rangle = \\ &= \int_{G/H}\int_{G/H} \langle \Delta_G(\alpha(\xi'))^{-1}\sigma\left(\left(L_{\alpha(\xi)}R_{\alpha(\xi')^{-1}}\varphi\right)|_H\right) f(\xi'), g(\xi)\rangle d\xi' d\xi \\ &= \int_{G/H}\left\langle \int_{G/H} \Phi(\xi,\xi')f(\xi')d\xi', g(\xi)\right\rangle d\xi, \end{aligned}$$

hence the pointwise definition (8) indeed coincides with $\pi(\varphi)$.

Now the direction "$\Leftarrow$" of (10) is immediate. For the other direction assume that Φ does not vanish almost everywhere. Pick an ONB $(\eta_i)_{i\in I}$ of $\mathcal{H}_\sigma$; since $\mathcal{H}_\sigma$ is separable, I is countable. Since

$$\Phi(\xi,\xi') = 0 \Leftrightarrow \forall i,j \in I : \langle \Phi(\xi,\xi')\eta_i, \eta_j\rangle = 0$$

there exists a pair (i,j) and a set $A \subset G/H \times G/H$ of positive measure such that

$$\langle \Phi(\xi,\xi')\eta_i, \eta_j\rangle \neq 0$$

for all $(\xi,\xi') \in A$. By passing to a smaller set we may assume that in addition $A \subset G/H \times K$ for a compact $K \subset G/H$ (observing that G/H is σ-compact).

Now define the auxiliary operator $T : \mathrm{L}^2(X) \to \mathrm{L}^2(X)$ by

$$(Tf)(\xi) = \chi_K(\xi)\langle \sigma(\varphi)(f\cdot\eta_i)(\xi), \eta_j\rangle.$$

Then T is an integral operator with kernel

$$(\xi,\xi') \mapsto \chi_K(\xi)\langle \Phi(\xi,\xi')\eta_i, \eta_j\rangle,$$

which by construction is nonzero. By Lemma 2.3 this kernel is bounded and compactly supported, hence in $\mathrm{L}^2(G/H \times G/H)$. But for this space the map from kernel to integral operator is a unitary operator onto the space of Hilbert-Schmidt operators on $\mathrm{L}^2(G/H)$; in particular the map is one-to-one. Hence $T \neq 0$, which implies $\sigma(\varphi) \neq 0$. ∎

Proof of Theorem 1.2. Let $\varphi \in \mathrm{L}^\infty_c(G)$ be given with $\pi(\varphi) = 0$ for π in a set of positive Plancherel measure. Then, by (3), there exists a G-invariant subset $\Gamma \subset \widehat{N}$ of positive Plancherel measure and subsets $B_{G.\gamma} \subset \widehat{G_\gamma/N}^{\overline{\omega_\gamma}}$ $(\gamma \in \Gamma)$ with

$$\nu_{G_\gamma/N,\overline{\omega_\gamma}}(B_{G.\gamma}) > 0 \text{ and } \pi_{G.\gamma,\sigma}(\varphi) = 0, \text{ for all } \sigma \in B_{G.\gamma}.$$

Now fix $\gamma \in \Gamma$. Our aim is to relate the equation $\pi_{G.\gamma,\sigma}(\varphi) = 0$ for σ in a set of positive projective Plancherel measure in $\widehat{G_\gamma/N}^{\overline{\omega_\gamma}}$ to certain Fourier transforms, using the integral kernel calculus. For this purpose we use Borel cross-sections $\alpha : G/G_\gamma \to G$ and $\vartheta : G_\gamma/N \to G_\gamma$. In the following calculations, all quotients carry invariant measures. We also need the continuous homomorphism $\delta : G \to (\mathbb{R}^+, \cdot)$ defined by picking $B \subset N$ of positive finite measure and letting $\delta(x) = \frac{|B|}{|xBx^{-1}|}$.

By Lemma 2.4, $\pi_{G.\gamma,\sigma}(\varphi)$ has the operator-valued kernel

$$
\begin{aligned}
&\Phi(\xi,\xi') = \\
&= \int_{G_\gamma} (\gamma'(y)\otimes\sigma(y))\,\varphi(\alpha(\xi)y\alpha(\xi')^{-1})dy\ \Delta_G(\alpha(\xi'))^{-1} \\
&= \int_{G_\gamma/N}\int_N \left(\gamma'\left(n\vartheta(h)^{-1}\right)\otimes\sigma(h)^{-1}\right)\varphi(\alpha(\xi)n\vartheta(h)^{-1}\alpha(\xi')^{-1}) \\
&\qquad\qquad \Delta_{G_\gamma}(\vartheta(h))^{-1}dndh\ \Delta_G(\alpha(\xi'))^{-1} \\
&= \int_{G_\gamma/N}\left(\int_N \gamma(n)\varphi(\alpha(\xi)n\vartheta(h)^{-1}\alpha(\xi')^{-1})dn\circ\gamma'(\vartheta(h)^{-1})\Delta_{G_\gamma}(\vartheta(h))^{-1}\right) \\
&\qquad\qquad \otimes\sigma(h^{-1})dh\ \Delta_G(\alpha(\xi'))^{-1} \\
&= \int_{G_\gamma/N} F_{\xi,\xi'}(h)\otimes\sigma(h^{-1})dh\ \Delta_G(\alpha(\xi'))^{-1},
\end{aligned}
$$

where

$$
\begin{aligned}
&F_{\xi,\xi'}(h) = \\
&= \int_N \gamma(n)\varphi(\alpha(\xi)n\vartheta(h)^{-1}\alpha(\xi')^{-1})dn\circ\gamma'(\vartheta(h)^{-1})\Delta_{G_\gamma}(\vartheta(h))^{-1} \\
&= \delta(\alpha(\xi))\int_N \gamma\left(\alpha(\xi)^{-1}n\alpha(\xi)\right)\ \varphi(n\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1})dn\circ\gamma'(\vartheta(h)^{-1}) \\
&\qquad\qquad \cdot\Delta_{G_\gamma}(\vartheta(h))^{-1} \\
&= \delta(\alpha(\xi))\left[(\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right)\right]\circ\gamma'(\vartheta(h)^{-1})\Delta_{G_\gamma}(\vartheta(h))^{-1}. \quad (13)
\end{aligned}
$$

Here it is important to note that for fixed (ξ,ξ'), the operator-valued function $F_{\xi,\xi'}$ has compact support: A short calculation establishes that

$$\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N \neq 0$$

only if $h \in \alpha(\xi')^{-1}(\mathrm{supp}(\varphi))^{-1}\alpha(\xi)^{-1}N =: K_0$, and K_0 is a compact subset of $G/N \supset G_\gamma/N$. Moreover, for $h \in K_0$

$$
\begin{aligned}
&\left\|\delta(\alpha(\xi))\,(\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right)\circ\gamma'(\vartheta(h)^{-1})\Delta_{G_\gamma}(\vartheta(h))^{-1}\right\|_\infty \\
&\leq\ \delta(\alpha(\xi))\left\|\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right\|_1\Delta_{G_\gamma}(\vartheta(h))^{-1} \\
&\leq\ \delta(\alpha(\xi))\|\varphi\|_\infty\left|N\cap\mathrm{supp}(\varphi)\alpha(\xi')\vartheta(K_0)\alpha(\xi)^{-1}\right|\Delta_{G_\gamma}(\vartheta(h))^{-1}.
\end{aligned}
$$

The middle term is the measure of a fixed relatively compact subset of N, by regularity of ϑ, and the last term is bounded on the compact support. Hence the map $h \mapsto \|F_{\xi,\xi'}(h)\|_\infty$ is in $\mathrm{L}^\infty_c(G_\gamma/N)$.

Now, for fixed $\gamma \in \Gamma$ and $\sigma \in B_{G.\gamma}$, relation (10) and $\Delta_G > 0$ imply that

$$\int_{G_\gamma/N} F_{\xi,\xi'}(h) \otimes \sigma(h^{-1}) dh = 0 \tag{14}$$

for almost every (ξ, ξ'), where the set of these (ξ, ξ') may depend on σ. However, an application of Fubini's Theorem provides a conull subset $C \subset G/G_\gamma \times G/G_\gamma$, such that (14) holds for all $(\xi, \xi') \in C$ and all σ in a conull subset of $B_{G.\gamma}$, possibly depending on (ξ, ξ'). Now fix $(\xi, \xi') \in C$, as well as ONB's $(\eta_i)_{i \in I} \subset \mathcal{H}_\gamma$, $(\beta_j)_{j \in J} \subset \mathcal{H}_\sigma$. It follows that

$$\begin{aligned} 0 &= \langle \Phi_{\xi,\xi'}(\eta_i \otimes \beta_j), (\eta_k \otimes \beta_\ell) \rangle \\ &= \int_{G_\gamma/N} \langle F_{\xi,\xi'}(h)\eta_i, \eta_k \rangle \langle \beta_j, \sigma(h)^{-1}\beta_\ell \rangle dh \\ &= \langle \sigma(\Psi_{i,k})\beta_j, \beta_\ell \rangle, \end{aligned}$$

where we used that dh is left Haar measure on G_γ/N, and the scalar-valued function $\Psi_{i,k}$ given by

$$\Psi_{i,k}(h) = \langle \eta_i, F_{\xi,\xi'}(h)\eta_k \rangle, \text{ satisfying } |\Psi_{i,k}(h)| \le \|F_{\xi,\xi'}(h)\|_\infty \|\eta\| \, \|\eta'\|.$$

In particular $\Psi_{i,k} \in \mathrm{L}_c^\infty(G_\gamma/N)$. Thus we are finally in a position to use the assumption that G_γ/N has $\overline{\omega_\gamma}$-wPW, yielding for all i, k that $\Psi_{i,k} = 0$ on a joint conull subset. But this clearly entails $F_{\xi,\xi'}(h) = 0$ almost everywhere.

Since obviously

$$F_{\xi,\xi'}(h) = 0 \Leftrightarrow (\alpha(\xi).\gamma) \left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi \right) |_N \right) = 0$$

our considerations so far have established for fixed $\gamma \in \Gamma$, that for almost all $(\xi, \xi', h) \in G/G_\gamma \times G/G_\gamma \times G_\gamma/N$

$$(\alpha(\xi).\gamma) \left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi \right) |_N \right) = 0 \tag{15}$$

Now choose a measurable cross-section $\Lambda : G/N \to G$. By definition of the Borel structure on $\widehat{N}$, the map

$$\widehat{N} \times G/N \ni (\gamma, s) \mapsto \left\| \gamma \left(\left(R_{\Lambda(s)^{-1}}\varphi \right) |_N \right) \right\|_\infty$$

is Borel. Moreover, for a fixed $\xi \in G/G_\gamma$, the set

$$\{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1} : h \in G_\gamma/N, \xi' \in G/G_\gamma\}$$

is a set of representatives of G/N, since N is normal. In particular, for every $s \in G/N$ there exists $(\xi', h) \in G/G_\gamma \times G_\gamma/N$ such that $N\Lambda(s)^{-1} = N\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}$. Hence $\Lambda(s)^{-1} = n\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}$, for suitable $n \in N$. Thus (15) implies that for fixed $\gamma \in \Gamma$ and almost all $\xi \in G/G_\gamma$ the set

$$\{s \in G/N : \left\| (\alpha(\xi).\gamma) \left(\left(R_{\Lambda(s)^{-1}}\varphi \right) |_N \right) \right\|_\infty = 0\}$$

has a complement of measure zero.

On the other hand, $\xi \mapsto \alpha(\xi).\gamma$ yields a bijection between G/G_γ and $G.\gamma$, and the image of the invariant measure on G/G_γ is equivalent to the measure $\mu_{G.\gamma}$ appearing in (4). Summarising, we obtain that

$$\begin{aligned} 0 &= \int_{\Gamma/N}\int_{G.\gamma}\int_{G/N} \|\gamma\left(\left(R_{\Lambda(s)^{-1}}\varphi\right)|_N\right)\|_\infty \, ds \, d\mu_{G.\gamma}(\gamma) \, d\overline{\nu}(G.\gamma) \\ &= \int_{G/N}\int_{\Gamma} \|\gamma\left(\left(R_{\Lambda(s)^{-1}}\varphi\right)|_N\right)\|_\infty d\nu_N(\gamma) ds. \end{aligned}$$

Here the second equation uses the measure decomposition (4) and Fubini's Theorem. But the latter integral implies for almost all $s \in G/N$, that $((R_{\Lambda(s)^{-1}}\varphi)|_N)^\wedge$ $=0$ on the G-invariant subset $\Gamma \subset \widehat{N}$. Since $\left(R_{\Lambda(s)^{-1}}\varphi\right)|_N \in \mathrm{L}^\infty_c(N)$, an appeal to assumption (5) yields $\left(R_{\Lambda(s)^{-1}}\varphi\right)|_N = 0$ for almost every coset s. Hence $\varphi = 0$, which finishes the proof of (a).

For the proof of (b), assume that $\varphi_0 \in \mathrm{L}^\infty_c(N) \setminus \{0\}$ is a counterexample to (5), i.e. there exists a G-invariant $\widetilde{\Gamma} \subset \widehat{N}$ of positive measure such that $\widehat{\varphi_0}$ vanishes on Γ. Let $K \subset G/N$ be some compact set of positive measure, and let $\Lambda : G/N \to G$ be a cross-section. Then

$$\varphi(n\Lambda(s)^{-1}) = \varphi_0(n)\chi_K(s)$$

defines a nonzero $\varphi \in \mathrm{L}^\infty_c(G)$. Let $\Sigma = \{\pi \in \widehat{G} : \widehat{\varphi}(\pi) = 0\}$. We intend to show $\pi_{G.\gamma,\sigma} \in \Sigma$ for all $\gamma \in \widetilde{\Gamma}$ and all $\sigma \in \widehat{G.\gamma}^{\overline{\omega_\gamma}}$. By equation (13) this amounts to proving

$$0 = (\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right),$$

where α, ϑ are cross-sections associated to G_γ/N and G/G_γ. Now for $n \in N$,

$$\begin{aligned} \left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N(n) &= \varphi(n\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}) \\ &= \varphi(nn'\Lambda(s)^{-1}) \\ &= \varphi_0(nn')\chi_K(s) \end{aligned}$$

where $s = \alpha(\xi')\vartheta(h)\alpha(\xi)^{-1}N$ and $n' \in N$ is suitably chosen, and independent of n. Since $\alpha(\xi).\gamma \in \Sigma$, it follows that

$$(\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right) = \chi_K(s)\,(\alpha(\xi).\gamma)\,(\varphi_0) \circ (\alpha(\xi).\gamma)\,(n') = 0$$

Hence we can compute

$$\begin{aligned} \nu_G(\Sigma) &= \int_{\widehat{N}/G}\int_{\widehat{G_\gamma/N}^{\overline{\omega_\gamma}}} \chi_\Sigma(\pi_{G.\gamma,\sigma}) \, d\nu_{G_\gamma/N,\overline{\omega_\gamma}}(\sigma) \, d\overline{\nu}(G.\gamma) \\ &\geq \int_{\widetilde{\Gamma}/G} \nu_{G_\gamma/N,\overline{\omega_\gamma}}(\widehat{G_\gamma/N}^{\overline{\omega_\gamma}}) \, d\overline{\nu}(G.\gamma) \\ &> 0, \end{aligned}$$

therefore φ is the desired counterexample to wPW on G. ∎

3. Applications and Examples

In this section we apply Theorem 1.2 to a variety of cases, and discuss the necessity of its assumptions. Unless otherwise stated, our standing assumptions are: G is second countable, G and $N \lhd G$ are of type I, with N regularly embedded.

Corollary 3.1. *Assume that the dual action of G/N is free ν_N-almost everywhere. Then G has wPW iff condition (5) holds.*

Corollary 3.2. *Let G be nonunimodular, and $N = \mathrm{Ker}(\Delta_G)$. Then G has wPW iff (5) holds; in particular, if N has wPW.*

Proof. This is an immediate consequence of Corollary 3.1, which applies to this setting by [6, Section 5]. ■

For split compact extensions the freeness of the operation turns out to be necessary also. Since a compact group has wPW iff it is trivial, the next theorem provides a class of extensions for which the conditions of Theorem 1.2 are necessary and sufficient.

Theorem 3.3. *Assume that $G = N \rtimes K$, with K compact. Then G has wPW iff condition (5) holds and the dual action of G/N is free ν_N-almost everywhere.*

Proof. The "if"-part is Corollary 3.1. For the "only-if"-part, necessity of (5) was noted in Theorem 1.2. We denote elements of G by pairs $(n,k) \in N \times K$, and the conjugation action of K on N by $K \times N \ni (k,n) \mapsto k.n$. Define the little fixed groups as $K_\gamma = G_\gamma \cap K$; since G is a semidirect product, $G_\gamma = N \rtimes K_\gamma$.

We define

$$\widetilde{\Sigma} = \{\gamma \in \widehat{N} : K_\gamma \neq \{1\}\}.$$

Let us first show that $\widetilde{\Sigma}$ is a Borel subset of $\widehat{N}$. For this purpose consider the space X of closed subgroups of K, endowed with the compact open topology. By [2, Proposition II.2.3], the stabilizer map $\widehat{N} \to X$ is Borel. Moreover, the complement of $\widetilde{\Sigma}$ is nothing but the inverse image of the trivial subgroup under the stabiliser map, hence Borel. Thus $\widetilde{\Sigma}$ is Borel also. Assuming that $\nu_N(\widetilde{\Sigma}) > 0$, we need to construct a $\varphi \in \mathrm{L}^\infty_c(G)$ such that $\widehat{\varphi}$ vanishes on a set of positive measure.

Let $\varphi_0 \in \mathrm{L}^\infty_c(N) \setminus \{0\}$ with $\varphi_0 \geq 0$ be given, and let

$$\varphi(n,k) = \varphi_1(n) = \int_K \varphi_0(k'.n)dk'$$

which yields a nonzero element $\varphi \in \mathrm{L}^\infty_c(G)$. Next we show

$$\forall \gamma \in \widetilde{\Sigma}\ ,\ \forall \sigma \in \widehat{K_\gamma} \setminus \{1_{K_\gamma}\}\ :\ \pi_{G.\gamma,\sigma}(\varphi) = 0 \tag{16}$$

where 1_{K_γ} denotes the trivial representation of K_γ.

Since G is a semidirect product, all Mackey obstructions are trivial. In addition, we can assume that all cross-sections arising below in fact map into $K < G$, i.e. $\vartheta(h) = (e_N, \widetilde{\vartheta}(h))$ etc. Moreover, since K is compact all involved measures can be chosen invariant. In this setting the calculations from the proof of Theorem 1.2 yield that $\pi_{G.\gamma,\sigma}(\varphi)$ acts on $\mathrm{L}^2(K/K_\gamma;\mathcal{H}_\gamma\otimes\mathcal{H}_\sigma)$ via

$$\Phi(\xi,\xi') = \int_{K_\gamma} (\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right)\otimes\sigma(h^{-1})dh.$$

In order to prove that Φ vanishes, it is enough to show for all ξ,ξ' that the map

$$F_{\xi,\xi'} : h\mapsto (\alpha(\xi).\gamma)\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right).$$

is constant on K_γ. Note first that by construction of φ, and the fact that ϑ,α map into K that

$$\left(\left(R_{\alpha(\xi)\vartheta(h)^{-1}\alpha(\xi')^{-1}}\varphi\right)|_N\right)(n) = \varphi_1(n)$$

is independent of ξ,h,ξ' and invariant under the action of K. Hence we obtain

$$F_{\xi,\xi'}(h) = (\alpha(\xi).\gamma)(\varphi_1) = \gamma(\varphi_1).$$

Hence $F_{\xi,\xi'}$ is constant, and thus $\pi_{G.\gamma,\sigma}(\varphi)=0$.

Hence, defining the Borel subset

$$\Sigma = \{\pi\in\widehat{G} : \widehat{\varphi}(\pi)\neq 0\},$$

we can use (3) and (16) to estimate

$$\begin{aligned}
\nu_G(\Sigma) &= \int_{\widehat{N}/G}\int_{\widehat{K_\gamma}} \chi_\Sigma(\pi_{G.\gamma,\sigma})d\nu_{K_\gamma}(\sigma)d\overline{\nu}(G.\gamma)\\
&\geq \int_{\widetilde{\Sigma}/G}\int_{\widehat{K_\gamma}} \chi_\Sigma(\pi_{G.\gamma,\sigma})d\nu_{K_\gamma}(\sigma)d\overline{\nu}(G.\gamma)\\
&\geq \int_{\widetilde{\Sigma}/G} \nu_{K_\gamma}\left(\widehat{K_\gamma}\setminus\{1_{K_\gamma}\}\right)d\overline{\nu}(G.\gamma)\\
&> 0.
\end{aligned}$$

Here the last inequality is due to the fact that the integrand is strictly positive on $\widetilde{\Sigma}/N$, and we assumed $\nu_N(\widetilde{\Sigma})>0$. ∎

Applying the theorem to motion groups yields that $G=\mathbb{R}^n\rtimes SO(n)$ has wPW iff $n\leq 2$.

Another extreme case is given by an almost everywhere trivial action of G/N on $\widehat{G}$. The following corollary also covers direct product groups.

Corollary 3.4. *Assume that G/N acts trivially ν_N-almost everywhere. If G has wPW, then N has wPW. Conversely, if both N and G/N have wPW, then so does G.*

Note that G/N need not have wPW, even if G does: Simply take $G=\mathbb{R}$ and $N=\mathbb{Z}$.

A result similar to the following is formulated for the so-called *topological Paley-Wiener condition* in [12, Theorem 2.2].

Corollary 3.5. *Suppose that G/N is abelian and compact-free. Assume in addition that either almost all Mackey obstructions vanish, or that G/N is compactly generated. Then, if condition 5 holds, G has wPW.*

Proof. Let us first deal with the case of vanishing Mackey obstructions. Recall that G/N is compact-free iff it has no nontrivial compact subgroups. For abelian groups, this is equivalent to wPW by [12, Theorem 3.2]. Moreover, if G/N is compact-free, so are all its closed subgroups; in particular, the little fixed groups also have wPW. Hence Theorem 1.2 implies wPW for G.

If G/N is compact-free and compactly generated, the structure theorem for LCA groups yields $G/N \cong \mathbb{R}^k \times \mathbb{Z}^\ell$, and the little fixed groups have a similar structure. Hence Theorem 1.2 together with the next lemma yield that G has wPW. ∎

Lemma 3.6. *Let $G = \mathbb{R}^k \times \mathbb{Z}^\ell$, and ω a type I multiplier on G. Then G has ω-wPW.*

Proof. We use the description of $\widehat{G}^\omega$ given in [3]. We may assume that ω is normalised. Then the map

$$h_\omega : G \to \widehat{G}, h_\omega(x)(y) = \omega(x,y)\overline{\omega(y,x)}$$

defines a continuous homomorphism. Denote the kernel of this homomorphism by S_ω. ω is called *totally skew* if S_ω is trivial. By [3, Theorem 3.1], we may then assume that ω is lifted from a totally skew cocycle ω_1 of G/S_ω. Moreover, $\widehat{G/S_\omega}^{\omega_1} = \{\pi\}$, and the mapping $\widehat{G} \ni \gamma \mapsto \gamma\pi \in \widehat{G}^\omega$ is continuous and onto (also by [3, Theorem 3.1]). This map is constant on $S_\omega^\perp$, giving rise to a homeomorphism between $\widehat{G}^\omega$ and $\widehat{G}/S_\omega^\perp \simeq \widehat{S_\omega}$. Moreover, the projective Plancherel measure can be chosen as the Haar measure on $\widehat{G}/S_\omega^\perp$. To see this consider the unitary action of $\widehat{G}$ on $\mathrm{L}^2(G)$ defined by pointwise multiplication, $(M_\gamma f)(x) = \gamma(x)f(x)$. Then it is straightforward to compute that on the (projective) Fourier transform side, $\widehat{G}$ acts via shifts,

$$(\gamma_1\pi)(M_{\gamma_2}f) = (\gamma_1\gamma_2\pi)(f).$$

On the other hand, since G is unimodular, there exists a choice of $\nu_{G,\omega}$ such that $\mathcal{F}_\omega$ extends to a unitary equivalence

$$\mathrm{L}^2(G) \to \int_{\widehat{G}^\omega}^\oplus \mathrm{HS}(\mathcal{H}_\rho) d\nu_{G,\omega}(\rho) \simeq \mathrm{L}^2(\widehat{G}/S_\omega^\perp, d\nu_{G,\omega}) \otimes \mathrm{HS}(\mathcal{H}_\pi).$$

Since we already know that the shifts on $\widehat{G}/S_\omega^\perp$ yield unitary operators on $\mathrm{L}^2(\widehat{G}/S_\omega^\perp, d\nu_{G,\omega})$, it follows that $\nu_{G,\omega}$ is shiftinvariant.

Now assume that $f \in \mathrm{L}^\infty_c(G)$ is such that $\rho(f) = 0$ on a set of projective Plancherel measure zero. Then the map $\widehat{G} \ni \gamma \mapsto (\gamma\pi)(f)$ also vanishes on a set of positive Plancherel measure. Pick an ONB $(\eta_i)_{i\in I}$ of $\mathcal{H}_\pi$. Then for all $i,j \in I$

$$\begin{aligned} 0 &= \langle(\gamma\pi)(f)\eta_i, \eta_j\rangle \\ &= \int_G \gamma(x)f(x)\langle\pi(x)\eta_i,\eta_j\rangle dx, \end{aligned}$$

for all γ in a set of positive measure. Hence wPW for G implies that

$$0 = f(x)\langle\pi(x)\eta_i,\eta_j\rangle$$

for all $i,j \in I$ and almost all $x \in G$. On the other hand, the fact that $\pi(x)$ is unitary implies for all $x \in G$ that $\langle\pi(x)\eta_i,\eta_j\rangle \neq 0$ for some pair (i,j). Thus we obtain $f = 0$ almost everywhere. ∎

We will next show that iterated application of Corollary 3.5 allows to establish wPW for a large class of solvable Lie groups, thus extending the results from [15, 9, 1, 10]. In the following, the term "Lie group" is shorthand for simply connected, connected Lie group. We first start with an observation that is probably folklore, and which ensures that Corollary 3.5 can be used iteratively. We include a proof since we could not obtain a reference.

Lemma 3.7. *Let G be an exponential Lie group and $N \lhd G$ a closed connected nilpotent normal subgroup. Then N is regularly embedded.*

Proof. Denote the Lie algebras of G, N by $\mathfrak{g}, \mathfrak{n}$, and by Ad_G^* and Ad_N^* the coadjoint actions of G and N respectively.

G being exponential implies that $\mathfrak{g}$ is a $\mathfrak{g}$*-module of exponential type* under the adjoint action, which means that all roots of the $\mathfrak{g}$-module $\mathfrak{g}$ have the form

$$\Psi(X) = (1 + i\alpha)\lambda(X)$$

with λ a real linear functional and $\alpha \in \mathbb{R}$ [4, Chap. I]. It follows that the submodule $\mathfrak{n}$ is also of exponential type. Passing to the dual yields that $\mathfrak{n}^*$ is a $\mathfrak{g}$- module of exponential type under the coadjoint action. Hence we obtain for the canonically induced coadjoint action Ad_G^* of G on $\mathfrak{n}^* \cong \mathfrak{g}^*/\mathfrak{n}^\perp$ that all G-orbits in $\mathfrak{n}^*$ are locally closed [4, Chap. I, Theoréme 3.8]. But then the orbit space $\mathfrak{n}^*/\mathrm{Ad}_G^*(G)$ is countably separated, by Glimm's Theorem (e.g., [4, Chap. I, Remarque 3.9]).

On the other hand, let

$$\kappa : \mathfrak{n}^*/\mathrm{Ad}_N^*(N) \to \widehat{N}$$

denote the Kirillov map, which is a homeomorphism. Then it is straightforward to check that κ intertwines the action of Ad_G^* with the dual action, thus inducing a homeomorphism of orbit spaces

$$\mathfrak{n}^*/\mathrm{Ad}_G^*(G) \to \widehat{N}/G.$$

Hence the right-hand side is countably separated, and N is regularly embedded. ■

For the formulation of the next theorem recall that the *nilradical* N of a solvable Lie group G is defined as the maximal connected nilpotent normal subgroup of G. Hence $N \lhd G$ is simply connected, with $G/N \cong \mathbb{R}^n$ [2, Chapter III]. Recall also that nilpotent, or more generally, exponential Lie groups are of type I [19]. A class R solvable Lie group is defined by the requirement that for all $x \in G$ and for all eigenvalues λ of $\mathrm{Ad}(x)$, $|x| = 1$ [2]. By contrast, exponential Lie groups are characterised by the property that no eigenvalue of any $\mathrm{Ad}(x)$ is purely imaginary [4, Théorème 2.1].

Theorem 3.8. *Let G be a solvable Lie group, and let $N \lhd G$ denote the nilradical. Assume that G is type I and that N is regularly embedded. Then G has wPW. In particular, G has wPW if it is exponential, or if it is of class R and type I.*

Proof. wPW for N is established by straightforward iterated application of Corollary 3.5 to a Jordan-Hölder series of N, observing that normal subgroups in nilpotent Lie groups are regularly embedded by Lemma 3.7. Moreover, $G/N \cong \mathbb{R}^n$, and N is type I. Hence Corollary 3.5 once again applies to yield wPW for G.

Now if G is exponential, it is type I by [19], and N is regularly embedded by Lemma 3.7. If G is of type I and class R, N is regularly embedded by [2, Chapter III, Theorem 1]. ∎

Corollary 3.9. *If G is a solvable CCR Lie group, it has wPW.*

Proof. CCR groups are of type I, and solvable CCR Lie groups are in addition of class R [2, Chapter V, Theorem 1]. Hence the previous theorem applies. ∎

Let us next give a class of group extensions that fail to have wPW, namely those where the normal subgroup is (nontrivial and) compact. For this purpose, an alternative formulation of wPW, which has the additional advantage of applying also to the non-type I setting, is observed:

Remark 3.10. If G is type I, then the following conditions are equivalent:

(i) G has wPW.

(ii) Every nonzero $f \in \mathrm{L}^\infty_c(G)$ is cyclic for the two-sided representation of G acting on $\mathrm{L}^2(G)$.

(iii) For all nonzero $f \in \mathrm{L}^\infty_c(G)$ and nonzero every two-sided invariant operator T on $\mathrm{L}^2(G)$, $Tf \neq 0$.

(ii) ⇔ (iii) is [5, I.I.4]. For $(i) \Rightarrow (iii)$ let $f \in \mathrm{L}^\infty_c(G)$ and let T denote a two-sided invariant operator. Under the Plancherel transform,

$$T \simeq \int_{\widehat{G}}^{\oplus} m(\sigma) \cdot \mathrm{Id}_{\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}_\sigma} \, d\nu_G(\sigma)$$

for a certain Borel mapping $m \in \mathrm{L}^\infty(\widehat{G})$. If $T \neq 0$, m does not vanish identically. But then (i) implies that

$$(Tf)^\wedge(\sigma) = m(\sigma)\widehat{f}(\sigma)$$

does not vanish identically, thus $Tf \neq 0$. For $(iii) \Rightarrow (i)$ assume that $\widehat{f}$ vanishes on a set $\Sigma \subset \widehat{N}$ of positive Plancherel measure. Let P denote the projection defined by

$$P \simeq \int_{\widehat{G}} \chi_\Sigma(\sigma) \cdot \mathrm{Id}_{\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}_\sigma}.$$

Then P is two-sided invariant, nontrivial, but $Pf = 0$.

Similar arguments apply to show that the following conditions are equivalent, for regularly embedded $N \lhd G$:

(i) Condition (5) holds.

(ii) Every nonzero $f \in \mathrm{L}^\infty_c(N)$ is cyclic for the von-Neumann algebra generated by the two-sided representation of N and the representation of G acting on $\mathrm{L}^2(N)$ by conjugation.

(iii) For all nonzero $f \in \mathrm{L}^\infty_c(N)$ and nonzero two-sided invariant operator T on $\mathrm{L}^2(N)$ commuting with the conjugation action of G, $Tf \neq 0$.

Proposition 3.11. *If G has wPW, it has no nontrivial compact normal subgroups.*

Proof. Assume that $K \lhd G$ is compact, and consider the subspace

$$\begin{aligned}\mathrm{L}^2_K(G) &= \{f \in \mathrm{L}^2(G) : \forall k \in K : f(xk) = f(x)\} \\ &= \{f \in \mathrm{L}^2(G) : \forall k \in K : f(kx) = f(x)\}.\end{aligned}$$

It is easy to see that $\mathrm{L}^2_K(G)$ is closed and two-sided invariant. Moreover clearly $\mathrm{L}^2_K(G) \neq \mathrm{L}^2(G)$ and $\mathrm{L}^2_K(G) \cap \mathrm{L}^\infty_c \neq \{0\}$. Hence G does not have wPW, by the previous remark. ■

Proposition 3.11 has a parallel in [12, Lemma 1.1], where it is stated for the topological Paley-Wiener condition. [12, Theorem 1.4] shows that for SIN groups, topological wPW coincides with the necessary condition derived in the previous proposition.

Example 3.12. An example where condition (5) holds, but N does not have wPW is constructed as follows: Consider $G = \mathbb{Q}_p \rtimes \mathbb{Q}_p^\times$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers, and its unit group $\mathbb{Q}_p^\times$ acts by multiplication. $\mathbb{Q}_p$ is self-dual, and the dual action of $\mathbb{Q}_p^\times$ is again by multiplication. In particular, $\widehat{\mathbb{Q}_p}$ consists of the two dual orbits $\{0\}$ (which has measure zero) and $\mathbb{Q}_p^\times$. Moreover, the action of $\mathbb{Q}_p^\times$ on the large orbit is free. Hence Kleppner and Lipsman's theorem yields that the Plancherel measure is supported on a single point, and the wPW property is an immediate consequence of the Plancherel theorem. (Of course, Theorem 1.2 also applies, with both conditions trivially fulfilled.)

On the other hand, $\mathbb{Q}_p$ has the nontrivial compact subgroup $\mathbb{Z}_p$, hence $\mathbb{Q}_p$ does not have wPW.

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