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Complete solutions of a family of cubic Thue equations

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Résumé. Dans cet article, nous utilisons la méthode de Baker, basée sur les formes linéaires en logarithmes, pour résoudre une famille d'équations de Thue liée à une famille de corps de nombres de degré 3. Nous obtenons toutes les solutions de l'équation de Thue

$$\Phi_n(x,y) = x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y - (n^3 - 2)n^2xy^2 - y^3 = \pm 1,$$

pour $n \geq 0$.

Abstract. In this paper, we use Baker's method, based on linear forms of logarithms, to solve a family of Thue equations associated with a family of number fields of degree 3. We obtain all solutions to the Thue equation

$$\Phi_n(x,y) = x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y - (n^3 - 2)n^2xy^2 - y^3 = \pm 1,$$

for $n \geq 0$.

1. Introduction

A Diophantine equation of the form

$$F(x,y) = k,$$

is called *a Thue equation*, where $F \in \mathbb{Z}[X,Y]$ is an irreducible binary form of degree $d \geq 3$ and k is a non-zero rational integer, the unknown x and y being rational integers. The name is given in honour of A. Thue [9] who proved that it has only finitely many solutions. Upper bounds for the solutions have been given using Baker's theory on linear forms in logarithms

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of algebraic numbers (see [1]). So the goal of this paper is to solve the following Thue equation

$$\begin{aligned}\Phi_n(x,y) &= x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y \\ &\quad - (n^3-2)n^2xy^2 - y^3 = \pm 1. \qquad (1.1)\end{aligned}$$

using Baker's method. Since E. Thomas and M. Mignotte ([8], [6]) have solved the first parameterized family of Thue equations of positive discriminant, several families of parametrized Thue equations have been studied. The success of the method is undeniable as it is widely used. In [4], C. Heuberger, A. Togbé, and V. Ziegler applied the method to solve the first family of Thue equations of degree 8. A list of families of Thue equations studied recently can be obtained at http://www.opt.math.tu-graz.ac.at/~cheub/thue.html. In this list, the most frequent methods used are Baker's method and the hypergeometric method. In 2004, we have applied Baker's method to completely solve a family of Thue equations related with a family of cubic number fields defined by O. Lecacheux and L. C. Washington (see [10]). We will use exactly the same method to obtain the main result of this paper that is the following:

Theorem 1.1. *For $n \geq 0$, the family of parametrized Thue equations*

$$\begin{aligned}\Phi_n(x,y) &= x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2y \\ &\quad - (n^3-2)n^2xy^2 - y^3 = \pm 1. \qquad (1.2)\end{aligned}$$

has only the integral solutions

$$\pm\{(1,0),\ (0,1)\}, \qquad (1.3)$$

except for $n = 0, 1$ where we have:

$$\begin{cases} \pm\{(0,1),\ (1,0),\ (1,-1),\ (1,2),\ (2,-3),\ (3,-1)\} & \text{if } n = 0, \\ \pm\{(0,1),\ (1,0),\ (1,-1)\} & \text{if } n = 1. \end{cases} \qquad (1.4)$$

Recently, (see [5], pages 100–103), Y. Kishi studied the following family

$$\begin{aligned}\phi_n(x) &= x^3 + (n^8 + 2n^6 - 3n^5 + 3n^4 - 4n^3 + 5n^2 - 3n + 3)x^2 \\ &\quad - (n^3-2)n^2x - 1. \qquad (1.5)\end{aligned}$$

This defines one of the two new families of cubic number fields. Therefore solving the related Thue equation is of great interest. There are three real roots $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ of $\phi_n(x)$. For a solution (x, y) of (1.1), we have the norm equation

$$\Phi_n(x,y) = \prod_{j=1}^{3}\left(x - \theta^{(j)}y\right) = N_{\mathbb{Q}(\theta^{(1)})/\mathbb{Q}}\left(x - \theta^{(j)}y\right) = \pm 1. \qquad (1.6)$$

This means that $x-\theta^{(j)}y$ is a unit in the order $\mathcal{O} := \mathbb{Z}[\theta^{(1)}, \theta^{(2)}]$, associated with ϕ_n. Easily, one can check that:

- the couples in (1.3) are solutions to (1.1);
- $\Phi_n(-x,-y) = -\Phi_n(x,y)$; hence if (x,y) is a solution to (1.1), so is $(-x,-y)$. Without loss of generality, we will consider only the solutions (x,y) to (1.1) with y positive.

The structure of our proof is as follows. In Section 2, we will determine some asymptotic expressions of $\phi_n(x)$ and prove that $\{\theta^{(1)}, \theta^{(2)}\}$ is almost a fundamental system of units of the number field $\mathbb{K}_n$ related with $\phi_n(x)$. In Section 3, we will study approximation properties of solutions to (1.1) and determine an upper bound for $\log y$. This is an important step for the proof of Theorem 1.1. We use some upper and lower bounds on linear forms in logarithms of algebraic numbers to prove that this equation has only the trivial solutions for large n in Section 4. Solutions for the other values of n are discussed in Section 5 using heavy computational verifications and Kash [3]. Most of the computations involve manipulations with asymptotic approximations done using Maple.

2. Associated Number Field

As the roots $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$ of $\phi_n(x)$ are not exact, we compute some asymptotic expressions of $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$:

$$\begin{aligned}\theta^{(1)} &= -n^8 - 2n^6 + 3n^5 - 3n^4 + 4n^3 - 5n^2 + 3n - 3\\ &\quad - \frac{1}{n^3} + \frac{2}{n^5} - \frac{1}{n^6} - \frac{1}{n^7} + \frac{4+\delta_1}{n^8},\\ \theta^{(2)} &= \frac{1}{n^3} - \frac{1}{n^5} + \frac{1}{n^6} + \frac{-2+\delta_2}{n^8},\\ \theta^{(3)} &= -\frac{1}{n^5} + \frac{1}{n^7} + \frac{-2+\delta_3}{n^8},\end{aligned} \tag{2.1}$$

where $|\delta_k| < 0.1$ for $n \geq 22$ and $k = 1,2,3$. We will use the following result that is a modification of Lemma 2.3 in [10]:

Lemma 2.1. *Let $a_1, a_2, a_3, n \in \mathbb{R}$ with $n \geq 29$, $|a_1| \leq 1, |a_2| \leq 2, |a_3| \leq 3$. Then*

$$\log\left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3}\right) = \frac{a_1}{n} + \frac{a_2 - a_1^2/2}{n^2} + \frac{a_1^3/3 - a_1a_2 + a_3 + \bar{\delta}}{n^3}$$

for some $\bar{\delta} \in \mathbb{R}$ with $|\bar{\delta}| < 0.1$.

One can check this result as it is proved exactly as Lemma 2.3 in [10]. Therefore we use (2.1) to determine some asympotic expressions for $\log\left|\theta^{(i)}\right|$

and $\log\left|\theta^{(i)}-\theta^{(k)}\right|$:

$$\begin{aligned}\log|\theta^{(1)}| &= 8\log(n)+\frac{2}{n^2}-\frac{3}{n^3}+\frac{\delta_4}{n^4},\\ \log|\theta^{(2)}| &= -3\log(n)-\frac{1}{n^2}+\frac{\delta_5}{n^3},\\ \log|\theta^{(3)}| &= -5\log(n)-\frac{1}{n^2}+\frac{2}{n^3}+\frac{\delta_6}{n^4},\end{aligned}\tag{2.2}$$

where $|\delta_4|<3/2$, $|\delta_5|<3/2$, $|\delta_6|<1$ for $n\geq 29$ and

$$\begin{aligned}\log|\theta^{(1)}-\theta^{(2)}| &= 8\log(n)+\frac{2}{n^2}-\frac{3}{n^3}+\frac{\delta_7}{n^4},\\ \log|\theta^{(2)}-\theta^{(2)}| &= 8\log(n)+\frac{2}{n^2}-\frac{3}{n^3}+\frac{\delta_8}{n^4},\\ \log|\theta^{(2)}-\theta^{(3)}| &= -3\log(n)+\frac{1}{n^3}+\frac{\delta_9}{n^4},\end{aligned}\tag{2.3}$$

where $|\delta_k|<5/2$ for $k=7,8,9$ and $n\geq 29$.

As we are using Baker's method, we need a system of units for $\mathbb{K}_n$ that can be fundamental or almost fundamental. So let us prove that $\{\theta^{(1)},\,\theta^{(2)}\}$ is an almost fundamental system of units of the number field $\mathbb{K}_n$ related with $\phi_n(x)$.

Lemma 2.2. *Let us consider* $\mathcal{O}=\mathbb{Z}[\theta^{(1)},\,\theta^{(2)}]$ *the order associated with* ϕ_n*, and* $\langle -1,\theta^{(1)},\,\theta^{(2)}\rangle$ *a subgroup of the unit group. We have*

$$I := [\mathcal{O}^{\times} : \langle -1,\theta^{(1)},\,\theta^{(2)}\rangle] \;\leq\; 2,\tag{2.4}$$

for $n\geq 29$.

Proof. We would like to determine an upper bound for the index of $\langle -1,\theta^{(1)},\,\theta^{(2)}\rangle$ in the unit group $\mathcal{O}^{\times}$ of $\mathbb{K}_n$ by estimating the regulators of the two groups. The discriminant $D_{\mathcal{O}}=D(\phi_n)$ is given by

$$\begin{aligned}D(\phi_n) = {}&(n^2-n+1)^2(n^3+n-1)^2\\ &\times(n^4+2n^3+4n^2+3n+3)^2(n^4-n^3+n^2-3n+3)^2.\end{aligned}$$

From Theorem 1 of [2], the regulator R of a totally real cubic field $\mathbb{K}$ of discriminant D satisfies:

$$R\geq\frac{1}{16}\log^2(D/4).$$

Let R be the regulator of $\langle -1,\theta^{(1)},\,\theta^{(2)}\rangle$. Then we have

$$R=\begin{vmatrix}\log\left|\theta^{(1)}\right| & \log\left|\theta^{(2)}\right|\\ \log\left|\theta^{(2)}\right| & \log\left|\theta^{(3)}\right|\end{vmatrix}.$$

Applying (2.2), we obtain

$$49\log^2(n)+\frac{22}{n^2}\log(n)<R<49\log^2(n)+\left(\frac{24}{n^2}-\frac{36}{n^3}\right)\log(n)+\frac{3}{n^4}. \tag{2.5}$$

In fact, using asymptotic expressions, one can get:

$$R=(49+\delta_{10})\log^2(n), \tag{2.6}$$

where $|\delta_{10}|<0.009$, for $n\geq 29$. So $R>0$ and $\theta^{(1)}$, $\theta^{(2)}$ are independent units. From [7], page 361, we find a bound for the index

$$I=[\mathcal{O}^\times:\langle -1,\,\theta^{(1)},\,\theta^{(2)}\rangle]$$

by

$$I=\frac{R}{R_{\mathcal{O}}}<\frac{49\log^2(n)+\left(\frac{24}{n^2}-\frac{36}{n^3}\right)\log(n)+\frac{3}{n^4}}{\frac{1}{16}\log^2\left(D(\phi_n)/4\right)}\leq 2, \tag{2.7}$$

for $n\geq 29$. Hence $I=\{1,\,2\}$. □

Therefore, in the next sections we will use $I=2$.

3. Approximation Properties of Solutions

Let $(x,y)\in\mathbb{Z}^2$ be a solution to (1.1). We define $\beta:=x-\theta y$, with $\theta:=\theta^{(1)}$. We define the type j of a solution (x,y) of (1.1) such that

$$\left|\beta^{(j)}\right|:=\min_{i=1,2,3}\left\{\left|\beta^{(i)}\right|\right\}. \tag{3.1}$$

So we have seen with (1.6) that each $\beta^{(i)}$ is a unit in $\mathbb{K}_n$. For the proof of Theorem 1.1, we need the expressions of the $\beta^{(i)}$. The following lemma will be very useful to obtain their asymptotic expressions:

Lemma 3.1. *Let $n\geq 29$ and (x,y) be a solution to (1.1) of type j such that $y\geq 2$. Then*

$$\left|\beta^{(j)}\right|\leq c_j\frac{1}{y^2},\qquad \text{where } c_j=\begin{cases}\frac{4}{n^{16}} & \text{if } j=1,\\ \frac{4}{n^5} & \text{if } j=2,3,\end{cases} \tag{3.2}$$

$$\log\left|\beta^{(i)}\right|=\log(y)+\log\left|\theta^{(i)}-\theta^{(j)}\right|+\frac{1/2+\delta_{11}}{n^2},\quad i\neq j,\ \ |\delta_{11}|<0.1. \tag{3.3}$$

Proof. For $i\neq j$, we have

$$y\left|\theta^{(i)}-\theta^{(j)}\right|\leq 2\left|\beta^{(i)}\right|,$$

then

$$\left|\beta^{(j)}\right|=\frac{1}{\prod_{i\neq j}\left|\beta^{(i)}\right|}\leq\frac{4}{y^2}\cdot\frac{1}{\prod_{i\neq j}\left|\theta^{(i)}-\theta^{(j)}\right|}. \tag{3.4}$$

Since

$$\prod_{i\neq j}\left|\theta^{(i)}-\theta^{(1)}\right|\geq n^{16},\qquad \text{for } n\geq 29 \text{ and } j=1,$$

$$\prod_{i\neq j}\left|\theta^{(i)}-\theta^{(j)}\right|\geq n^{5},\qquad \text{for } n\geq 29 \text{ and } j=2,\ 3,$$

so we obtain (3.2). Therefore, for $n \geq 29$, we have $|\theta^{(j)} - x/y| < 1/(2y^2)$, hence x/y is a convergent to $\theta^{(j)}$. Moreover, we know that

$$\frac{|\beta^{(i)}|}{y\left|\theta^{(i)}-\theta^{(j)}\right|}=\left|1+\frac{\beta^{(j)}}{y(\theta^{(i)}-\theta^{(j)})}\right|,$$

then taking the log of the previous expression and using (2.1) and (3.2) we have

$$\log\left|\beta^{(i)}\right|=\log y+\log\left|\theta^{(i)}-\theta^{(j)}\right|+\frac{1/2+\delta_{11}}{n^2}, \tag{3.5}$$

with $i \neq j$, $|\delta_{11}| < 0.1$ for $n \geq 29$. This completes the proof. □

Now we will use the almost fundamental system of units and the asymptotic expressions (2.2), (2.3), (3.2), and (3.3) to determine an upper bound for $\log y$. Obtaining this bound is very crutial for the proof of Theorem 1.1.

Lemma 3.2. *Let (x, y) be a solution to (1.1) with $y \geq 2$ and $n \geq 29$. Then*

$$\log y \geq \frac{7}{12}\log(n)[7n^2\log(n)-16]. \tag{3.6}$$

Proof. If (x, y) is a solution to (1.1), then β is a unit in $\mathbb{Z}[\theta]$. By Lemma 2.2, there are integers u_1, u_2 such that

$$\beta^I=\pm\left(\theta^{(1)}\right)^{u_1}\left(\theta^{(2)}\right)^{u_2}. \tag{3.7}$$

So using the conjugates of β and taking the absolute values, we have

$$\begin{cases}|(\beta^{(1)})^I|=|\theta^{(1)}|^{u_1}|\theta^{(2)}|^{u_2},\\ |(\beta^{(2)})^I|=|\theta^{(2)}|^{u_1}|\theta^{(3)}|^{u_2},\\ |(\beta^{(3)})^I|=|\theta^{(3)}|^{u_1}|\theta^{(1)}|^{u_2};\end{cases} \tag{3.8}$$

therefore we obtain

$$\begin{cases}\log|\beta^{(1)}|=\frac{u_1}{I}\log|\theta^{(1)}|+\frac{u_2}{I}\log|\theta^{(2)}|,\\ \log|\beta^{(2)}|=\frac{u_1}{I}\log|\theta^{(2)}|+\frac{u_2}{I}\log|\theta^{(3)}|,\\ \log|\beta^{(3)}|=\frac{u_1}{I}\log|\theta^{(3)}|+\frac{u_2}{I}\log|\theta^{(1)}|.\end{cases} \tag{3.9}$$

For each j, from (3.9), we consider the subsystem not containing $\beta^{(j)}$ that we solve to determine u_1 and u_2 using Cramer's method. Then we use the asymptotic expressions (2.2), (2.3), (3.2), and (3.3) to obtain

$$\frac{u_1}{I} = \begin{cases} \left(\frac{13}{49\log(n)} - \frac{165}{2401n^2\log(n)^2} + \frac{\delta_{12}}{n^3}\right)\log(y) + \frac{104}{49} \\ \qquad + \frac{937}{4802n^2\log(n)} + \frac{\delta_{13}}{n^3} \text{ if } j=1, \\ \left(\frac{11}{49\log(n)} - \frac{117}{2401n^2\log(n)^2} + \frac{\delta_{14}}{n^3}\right)\log(y) + \frac{55}{49} \\ \qquad + \frac{741}{4802n^2\log(n)} + \frac{\delta_{15}}{n^3} \text{ if } j=2, \\ \left(-\frac{2}{49\log(n)} + \frac{48}{2401n^2\log(n)^2} + \frac{\delta_{16}}{n^3}\right)\log(y) - 1 \\ \qquad + \frac{2}{49n^2\log(n)} + \frac{\delta_{17}}{n^3} \text{ if } j=3, \end{cases} \tag{3.10}$$

$$\frac{u_2}{I} = \begin{cases} \left(\frac{2}{49\log(n)} - \frac{48}{2401n^2\log(n)^2} + \frac{\delta_{18}}{n^3}\right)\log(y) + \frac{16}{49} \\ \qquad - \frac{139}{2401n^2\log(n)} + \frac{\delta_{19}}{n^3} \text{ if } j=1, \\ \left(\frac{13}{49\log(n)} - \frac{165}{2401n^2\log(n)^2} + \frac{\delta_{20}}{n^3}\right)\log(y) + \frac{16}{49} \\ \qquad + \frac{1045}{4802n^2\log(n)} + \frac{\delta_{21}}{n^3} \text{ if } j=2, \\ \left(\frac{11}{49\log(n)} - \frac{117}{2401n^2\log(n)^2} + \frac{\delta_{22}}{n^3}\right)\log(y) \\ \qquad + \frac{27}{98n^2\log(n)} + \frac{\delta_{23}}{n^3} \text{ if } j=3, \end{cases} \tag{3.11}$$

where $|\delta_k| < 0.1$ for $12 \le k \le 23$ and $n \ge 29$. Then we get

$$\frac{v_j}{I} = \begin{cases} \left(\frac{6}{49n^2\log^2 n} + \frac{\delta_{24}}{n^3}\right)\log y + \frac{8}{7n^2\log n} + \frac{\delta_{25}}{n^3} & \text{if } j=1, \\ \left(\frac{6}{49n^2\log^2 n} + \frac{\delta_{26}}{n^3}\right)\log y - \frac{19}{49n^2\log n} + \frac{\delta_{27}}{n^3} & \text{if } j=2, \\ \left(\frac{6}{49n^2\log^2 n} + \frac{\delta_{28}}{n^3}\right)\log y + \frac{1}{n^2\log n} + \frac{\delta_{29}}{n^3} & \text{if } j=3. \end{cases} \tag{3.12}$$

where $|\delta_k| < 0.1$ for $24 \le k \le 29$. In fact, for each j, v_j is a linear combination of u_k defined by:

$$\frac{v_j}{I} := \begin{cases} 2\frac{u_1}{I} - 13\frac{u_2}{I} & \text{if } j=1, \\ 13\frac{u_1}{I} - 11\frac{u_2}{I} - 11 & \text{if } j=2, \\ 11\frac{u_1}{I} + 2\frac{u_2}{I} + 11 & \text{if } j=3. \end{cases} \tag{3.13}$$

We need to specify that a generator σ of the Galois group G of $\mathbb{K}_n$ is defined by

$$\sigma(x) = \frac{(n^3-1)x - 1}{(n^6+n^4-2n^3+n^2-n+1)x + n}. \tag{3.14}$$

As $y \ge 2$ and v_j is an integer, we have $v_j \ge 1$. Therefore, (3.12) helps to obtain (3.6). □

4. Large Solutions

Suppose that $(x, y) \in \mathbb{Z}^2$ is a non trivial solution of type j. We choose indices (i, k) depending on j:

$$(i,k) = \begin{cases} (2,3) & \text{if } j = 1, \\ (3,1) & \text{if } j = 2, \\ (1,2) & \text{if } j = 3. \end{cases}$$

We use the following Siegel identity

$$\frac{\beta^{(k)}(\theta^{(j)} - \theta^{(i)})}{\beta^{(i)}(\theta^{(j)} - \theta^{(k)})} - 1 = \frac{\beta^{(j)}(\theta^{(k)} - \theta^{(i)})}{\beta^{(i)}(\theta^{(j)} - \theta^{(k)})}.$$

We put

$$\lambda_j = \frac{\theta^{(j)} - \theta^{(i)}}{\theta^{(j)} - \theta^{(k)}}, \qquad \tau_j = \frac{\beta^{(j)}}{\beta^{(i)}} \left(\frac{\theta^{(k)} - \theta^{(i)}}{\theta^{(j)} - \theta^{(k)}} \right)$$

and we obtain the following linear form in logarithms

$$\Lambda_j = \frac{u_1}{I} \log \left| \frac{\theta^{(k)}}{\theta^{(i)}} \right| + \frac{u_2}{I} \log \left| \frac{\theta^{(j)}}{\theta^{(k)}} \right| + \log|\lambda_j| = \log|1 + \tau_j|. \tag{4.1}$$

Lemma 4.1. *We have* $\Lambda_j \neq 0$.

Proof. Suppose that $\Lambda_j = 0$, then from (4.1) we have $\tau_j = 0$ or $\tau_j = -2$. It is impossible that $\tau_j = 0$ because the polynomial $\phi_n(x)$ has three distinct nonzero roots. In the other side, if $\tau_j = -2$, then by the Siegel identity used the conjugate τ_{j+1} (the index is reduced mod 3) of τ_j would be equal to 1. This is also impossible in the normal closure of $\mathbb{K}_n$. □

From (4.1), we have

$$\log|\Lambda_j| = \log\log|1 + \tau_j| \leq \log|\tau_j| = \log \left| \frac{\beta^{(j)}}{\beta^{(i)}} \left(\frac{\theta^{(k)} - \theta^{(i)}}{\theta^{(j)} - \theta^{(k)}} \right) \right|.$$

So by (2.3), (3.2), and (3.3), we obtain the following upper bounds of Λ_j:

$$\log|\Lambda_j| \leq -3\log y + \log 4 + \begin{cases} -35\log n - \frac{3}{n^2} & \text{if } j = 1, \\ -2\log n - \frac{0.1}{n^2} & \text{if } j = 2, 3. \end{cases} \tag{4.2}$$

Our goal is to use Theorem 4.3 in [10] to obtain lower bounds for Λ_j. So in order to use linear forms in two logaritms, by (3.13) and (4.1), we rewrite Λ_j as

$$\text{(4.3a)}\qquad 13I\Lambda_1 = u_1 \log\left|\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{13}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{2}\right| + \log\left|\lambda_1^{13I}\left(\frac{\theta^{(3)}}{\theta^{(1)}}\right)^{v_1}\right|,$$

$$\text{(4.3b)}\qquad 13I\Lambda_2 = u_2 \log\left|\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{13}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{11}\right| + \log\left|\lambda_2^{13I}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{v_2+11I}\right|,$$

$$\text{(4.3c)}\qquad 11I\Lambda_3 = u_2 \log\left|\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{11}\left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{2}\right| + \log\left|\lambda_3^{11I}\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{v_3-11I}\right|.$$

We consider $D = 3$ and

$$\Delta_j = \begin{vmatrix} \log|\gamma_1| & \log|\gamma_2| \\ \log|\sigma(\gamma_1)| & \log|\sigma(\gamma_2)| \end{vmatrix}$$

for $j = 1, 2, 3$.

• For $j = 1$, we consider

$$\gamma_1 = \left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{13}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{2}; \quad \gamma_2 = \lambda_1^{13I}\left(\frac{\theta^{(3)}}{\theta^{(1)}}\right)^{v_1}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because $\Delta_1 > 1910\log^2 n$. After studying the conjugates of γ_1 and γ_2, we obtain

$$h(\gamma_1) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{13}\left(\frac{\theta^{(2)}}{\theta^{(3)}}\right)^{2}\right|;$$

$$h(\gamma_2) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}-\theta^{(2)}}{\theta^{(3)}-\theta^{(2)}}\right)^{13I}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{v_1}\right|.$$

• For $j = 2$, we take

$$\gamma_1 = \left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{13}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{11}; \quad \gamma_2 = \lambda_2^{13I}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{v_2+11I}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because $\Delta_2 > 1910\log^2 n$. The study of the conjugates of γ_1 and γ_2 leads to

$$h(\gamma_1) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{13}\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{11}\right|;$$

$$h(\gamma_2) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}-\theta^{(2)}}{\theta^{(3)}-\theta^{(2)}}\right)^{13I}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{v_2+11I}\right|.$$

- For $j = 3$, we take

$$\gamma_1 = \left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{11}\left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{2}; \quad \gamma_2 = \lambda_3^{11I}\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{v_3-11I}.$$

The algebraic numbers γ_1 and γ_2 are multiplicatively independent because $\Delta_3 < -69531\log^2 n$. After studying the conjugates of γ_1 and γ_2, therefore we have

$$h(\gamma_1) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{11}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{2}\right|;$$

$$h(\gamma_2) \le \frac{1}{3}\log\left|\left(\frac{\theta^{(1)}-\theta^{(2)}}{\theta^{(3)}-\theta^{(2)}}\right)^{11I}\left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{v_3+11I}\right|.$$

Thus the choice of h_1, h_2, and b' depending on j is given in Table 1 below.

Case	h_1	h_2	b'
$j=1$	$49\log n + \frac{13}{n^2}$	$\left(\frac{52}{49n^2\log n} + \frac{12}{49n^4\log^2 n}\right)\log y + \frac{286}{3}\log n + \frac{572}{21n^2}$	$\frac{n^2}{240}$
$j=2$	$49\log n + \frac{13}{n^2}$	$\frac{52\log y}{49n^2\log n} + \frac{572}{3}\log n + \frac{5288}{147n^2}$	$\frac{431n^2}{232320}$
$j=3$	$49\log n + \frac{13}{n^2}$	$\left(\frac{52}{49n^2\log n} + \frac{12}{49n^4\log^2 n}\right)\log y + 176\log n + \frac{136}{3n^2}$	$\frac{11n^2}{5760}$

TABLE 1. Choice of h_1, h_2, and b' depending on j.

Thus we get

$$\log|\Lambda_1| \geq -1971.54\left(\log\left(\frac{n^2}{240}\right)+.14\right)^2\left(49\log n+\frac{13}{n^2}\right)$$

$$\times\left(\left(\frac{52}{49n^2\log n}+\frac{12}{49n^4\log^2 n}\right)\log y+\frac{286}{3}\log n+\frac{572}{21n^2}\right)-\log 13I, \tag{4.4a}$$

$$\log|\Lambda_2| \geq -1971.54\left(\log\left(\frac{431n^2}{232320}\right)+.14\right)^2\left(49\log n+\frac{13}{n^2}\right)$$

$$\times\left(\frac{52\log y}{49n^2\log n}+\frac{572}{3}\log n+\frac{5288}{147n^2}\right)-\log 13I, \tag{4.4b}$$

$$\log|\Lambda_3| \geq -1971.54\left(\log\left(\frac{11n^2}{5760}\right)+.14\right)^2\left(49\log n+\frac{13}{n^2}\right)$$

$$\times\left(\left(\frac{52}{49n^2\log n}+\frac{12}{49n^4\log^2 n}\right)\log y+176\log n+\frac{136}{3n^2}\right)-\log 11I. \tag{4.4c}$$

By combining (4.2), (4.4) and Lemma 3.2, we obtain the following result:

Lemma 4.2. *Let* $(x,y)\in\mathbb{Z}^2$ *be a solution to* (1.1) *of type* j *which is not listed in* (1.3)*. Then* $n \leq N_j$*, where*

$$N_j := \begin{cases} 11907 & \text{if } j=1,\\ 16452 & \text{if } j=2,\\ 10595 & \text{if } j=3.\end{cases} \tag{4.5}$$

5. Solutions for $0 \leq n \leq N_j$

The aim of this section is to verify that for $0 \leq n \leq N_j$ the only solutions to (1.1) are those listed in (1.3). As a first step, we use linear forms in logarithms once again in order to obtain an upper bound for $\log y$:

Lemma 5.1. *For* $29 < n \leq N_j$*, we have*

$$\log y \leq \begin{cases} 6.57\cdot 10^{21}\log n & \text{if } j=1,\\ 7.04\cdot 10^{21}\log n & \text{if } j=2,\\ 6.32\cdot 10^{21}\log n & \text{if } j=3.\end{cases} \tag{5.1}$$

Proof. We note that (3.12) and Lemma 3.2 yield

$$v_1 \leq \frac{12.02}{49n\log^2 n}\log y,\quad v_2 \leq \frac{12}{49n\log^2 n}\log y,\quad v_3 \leq \frac{12.02}{49n\log^2 n}\log y. \tag{5.2}$$

From the asymptotic expansions of u_1 and u_2 for all j, see (3.10) and (3.11), we observe that for $1 \le j \le 3$

$$U := \max\{|u_1|, |u_2|\} = \begin{cases} u_1, & \text{if } j = 1, \\ u_2, & \text{if } j = 2, 3, \end{cases} \tag{5.3}$$

then we have

$$U \le \tilde{U}_j := \frac{d_j}{\log n} \log y \ \text{ with } \ d_j = \begin{cases} 0.54, & \text{if } j = 1, 2, \\ 0.45, & \text{if } j = 3. \end{cases} \tag{5.4}$$

Applying Theorem 5.2, in [10] page 75, to Λ_j as it is defined in (4.1), estimating U by (5.4), and combining the lower bound with (4.2) result in

$$-3 \log y \ge \log |I\Lambda_j| \ge -C(3, 6) h_1 h_2 h_3 \log\left(\tilde{U}_j\right).$$

Here we take $n = 3$, $d = 6$, and

$$h_1 = h_2 = \frac{13}{3} \log n + \frac{1}{n^2}, \ h_3 = \frac{11I}{3} \log(n) + \frac{2I}{3n^2}. \tag{5.5}$$

Consequently, considering that $29 \le n \le N_j$, we obtain

$$\frac{\frac{d_j}{\log n} \log y}{\log\left(\frac{d_j}{\log n} \log y\right)} \le \begin{cases} 7.143 \cdot 10^{19} & \text{if } j = 1, \\ 7.644 \cdot 10^{19} & \text{if } j = 2, \\ 5.751 \cdot 10^{19} & \text{if } j = 3. \end{cases}$$

This yields (5.1). □

We write (4.3) as

$$m_j I \Lambda_j = \log |\gamma_{j1}| + v_j \log |\gamma_{j2}| + v'_j \log |\gamma_{j3}|, \tag{5.6}$$

where the notations are defined in Table 2.

j	m_j	γ_{j1}	γ_{j2}	γ_{j3}	v'_j
1	13	λ_1^{13I}	$\frac{\theta^{(3)}}{\theta^{(1)}}$	$\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{13} \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{2}$	u_1,
2	13	$\lambda_2^{13I} \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{11I}$	$\frac{\theta^{(1)}}{\theta^{(3)}}$	$\left(\frac{\theta^{(2)}}{\theta^{(1)}}\right)^{13} \left(\frac{\theta^{(1)}}{\theta^{(3)}}\right)^{11}$	u_2,
3	11	$\lambda_3^{11I} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{11I}$	$\frac{\theta^{(2)}}{\theta^{(1)}}$	$\left(\frac{\theta^{(3)}}{\theta^{(2)}}\right)^{11} \left(\frac{\theta^{(1)}}{\theta^{(2)}}\right)^{2}$	u_2.

TABLE 2. Notations for (5.6)

We divide (5.6) by $\log|\gamma_{j3}|$, use (4.2), (3.6), and $n \geq 29$, and obtain

$$\left|\delta_{j1} + v_j\delta_{j2} + v'_j\right| < 10^{-50696}, \tag{5.7}$$

where $\delta_{ji} := \log|\gamma_{ji}| \,/\, \log|\gamma_{j3}|$ for $i = 1, 2$.

In order to apply lemma 5.3 in [10] page 77, we note that (5.1) and (5.2) imply

(5.8)

$$2 + 2|v_1| \leq \frac{1.62 \cdot 10^{21}}{n\log n}, \quad 2 + 2|v_2| \leq \frac{1.73 \cdot 10^{21}}{n\log n}, \quad 2 + 2|v_3| \leq \frac{1.56 \cdot 10^{21}}{n\log n}.$$

For all pairs (j, n) with $1 \leq j \leq 3$ and $29 < n \leq N_j$, we calculate approximations $\tilde{\delta}_{j1}$ and $\tilde{\delta}_{j2}$ such that $|\delta_{j1} - \tilde{\delta}_{j1}| < Q^{-2}$ and $|\delta_{j2} - \tilde{\delta}_{j2}| < Q^{-2}$. In fact, we start with $Q = 10^m$, if it is not successfull we try successively 10^{m+1}, 10^{m+2}, $10^{m+3}, \ldots$ until we obtain the desired results. We use a high precision for the computations. In general, we did the computations with a precision of 100 digits. For all pairs of (j, n), we compute the successive convergents of $\tilde{\delta}_{j2}$ until we find a convergent p/q of $\tilde{\delta}_{j2}$ with $q < Q$ such that

$$q\|q\tilde{\delta}_{j1}\| > \frac{1}{n\log n} \cdot \begin{cases} 1.62 \cdot 10^{21} & \text{if } j = 1, \\ 1.73 \cdot 10^{21} & \text{if } j = 2, \\ 1.56 \cdot 10^{21} & \text{if } j = 3. \end{cases}$$

Here are a few remarks about the computations. The program was developed in Maple 9 and executed on a Pentium 4 with 3.92 GHz running under Linux 7.2.

- For $j = 1, 2, 3$ (together) and $29 \leq n \leq 10595$, we ran the program, starting with $Q = 10^{26}$. It took in average 7.74 seconds for each value of n.
- For $j = 1$ and $10595 \leq n \leq 11907$, we ran the program starting with $Q = 10^{28}$. It took in average 1.83 seconds for each value of n.
- For $j = 2$ and $10595 \leq n \leq 16452$, we ran the program starting with $Q = 10^{28}$. It took in average 2.96 seconds for each value of n.

To finish the proof of Theorem 1.1 and since some of our asymptotic expansions are not valid for $0 \leq n \leq 29$, we use Kant [3] (Kash Version 2.4) to solve (1.1) for $0 \leq n \leq 29$. We exactly get the solutions listed in (1.3).

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