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Counting discriminants of number fields

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Dedicated to Michael Pohst for his 60th birthday

Résumé. Pour tout groupe de permutations transitif sur n lettres G avec $n \leq 4$ nous donnons sans démonstration des résultats, des conjectures et des calculs numériques sur le nombre de discriminants de corps de nombres L de degré n sur $\mathbb{Q}$ tels que le groupe de Galois de la clôture galoisienne de L soit isomorphe à G.

Abstract. For each transitive permutation group G on n letters with $n \leq 4$, we give without proof results, conjectures, and numerical computations on discriminants of number fields L of degree n over $\mathbb{Q}$ such that the Galois group of the Galois closure of L is isomorphic to G.

1. Introduction

The aim of this paper is to regroup results and conjectures on discriminant counts of number fields of degree less than or equal to 4, from a theoretical, practical, and numerical point of view. Proofs are given elsewhere, see the bibliography. We only consider absolute number fields.

If G is a permutation group on n letters, we write

$$\Phi_n(G,s) = \sum_{L/\mathbb{Q}} \frac{1}{|d(L)|^s} \quad \text{and} \quad N_n(G,X) = \sum_{L/\mathbb{Q},\ |d(L)|\leq X} 1\ ,$$

where in both cases the summation is over isomorphism classes of number fields L of degree n over $\mathbb{Q}$ such that the Galois group of the Galois closure of L is isomorphic to G and $d(L)$ denotes the absolute discriminant of L. When we specify the signature (r_1, r_2), we will instead write $\Phi_{r_1,r_2}(G,s)$ and $N_{r_1,r_2}(G,X)$.

We denote by C_n the cyclic group of order n, by S_n the symmetric group on n letters, by A_n the alternating group on n letters, and by D_n the dihedral group with $2n$ elements.

We note that certain authors, in particular Datskovsky, Wright and Yukie (see [20], [32], [34]) count number fields in a fixed algebraic closure of $\mathbb{Q}$. This is the same as $N_n(G,X)$ when G is of cardinality equal to n, i.e., when

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the extensions L are Galois. Otherwise, in the range of our study ($n \leq 4$), their count is equal to $m(G)N_n(G,X)$ with $m(S_3) = 3$, $m(D_4) = 2$, and $m(A_4) = m(S_4) = 4$.

For each group G, we give the results in the following form. Whenever possible, we first give expressions for $\Phi_n(G,s)$ and $\Phi_{r_1,r_2}(G,s)$ which are as explicit as possible. Then we give asymptotic formulas for $N_n(G,X)$ and $N_{r_1,r_2}(G,X)$ which are usually directly deduced from the formula for $\Phi_n(G,s)$ and $\Phi_{r_1,r_2}(G,s)$, in the form $N_n(G,X) = P_n(G,X) + R_n(G,X)$ and $N_{r_1,r_2}(G,X) = P_{r_1,r_2}(G,X) + R_{r_1,r_2}(G,X)$, where the $P(G,X)$ are main terms, and the quantities $R(G,X)$ (which denote any one of $R_n(G,X)$ or $R_{r_1,r_2}(G,X)$) are error terms. We then give conjectural estimates of the form $R(G,X) = \widetilde{O}(X^\alpha)$ for some exponent α, where we use the convenient "soft O" notation: $f(X) = \widetilde{O}(X^\alpha)$ means that $f(X) = O(X^{\alpha+\varepsilon})$ for any $\varepsilon > 0$. (note that this does not necessarily mean $f(X) = O(X^\alpha \log(X)^\beta)$ for some β). In most cases, a suitable value for α can be rigorously obtained by complex integration methods, but we have not made any attempt in this direction, citing existing references when possible.

Note that those among the explicit constants occurring in the main terms which occur as products or sums over primes are all given numerically to at least 30 decimal digits. This is computed using a now rather standard method which can be found for example in [10].

Finally, we give tables of $N_{r_1,r_2}(G,10^k)$ for all possible signatures (r_1,r_2) and increasing values of k, as well as a comment on the comparison between this data with the most refined result or conjecture on the asymptotic behavior. To save space, we do not give $N(G,10^k)$ which is of course trivially obtained by summing over all possible signatures. The upper bound chosen for k depends on the time and space necessary to compute the data: usually a few weeks of CPU time and 1GB of RAM.

We have noticed that in most of the tables that we give, the error term (which we do not indicate explicitly) changes sign and is rather small, indicating both that there is no systematic bias, in other words no additional main term, and that the conjectured exponent in the error term is close to the correct value. Whenever there seems to be such a systematic bias, a least squares method has been used to find a conjectured additional main term, and these terms have been used in the tables. When appropriate, this is indicated in the corresponding sections.

It should be stressed that although we only give the *number* of suitable fields, the same methods can also be used to compute explicitly a defining equation for these number fields, but the storage problem makes this impractical for more than a few million fields. See [14] and [15] for details.

General references. Outside from specific references which will be given in each section, the following papers give general results and/or conjectures. The paper of Wright [32] gives a general formula for $N_n(G,X)$ for abelian groups G (and even for general abelian extensions of number fields). The exponents of X and $\log X$ are easily computable, however the multiplicative constant is only given as an adelic integral which is *in principle* computable, but in practice very difficult to compute. In fact, for general base fields these constants have been computed only in very few cases, and by quite different methods, in particular by the authors (see [16], [17], and [18]).

The thesis and paper of Mäki [25] and [26] give $\Phi_n(G,X)$ and estimates for $N_n(G,X)$ with error terms (easily deduced from $\Phi_n(G,X)$ by contour integration) again in the case of abelian groups G, but only for absolute extensions, i.e., when the base field is $\mathbb{Q}$, as we do here. This nicely complements the results of Wright, but is limited to the base field $\mathbb{Q}$. She does not give results with signatures, although they could probably be obtained using her methods.

The papers of Malle [27] and [28] give very general and quite precise conjectures on $N_n(G,X)$ for arbitrary transitive subgroups G of S_n, up to an unknown multiplicative constant, as well as results and heuristics supporting these conjectures. Although the conjectures must be corrected as stated (see [24] for a counter-example), the general form is believed to be correct.

Finally, the ICM talk [12] can be considered as a summary of the present paper.

2. Degree 2 fields with $\mathbf{G \simeq C_2}$

2.1. Dirichlet series and asymptotic formulas. The results are elementary.

Dirichlet series:

$$\begin{aligned}\Phi_2(C_2,s) &= \left(1+\frac{1}{2^{2s}}+\frac{2}{2^{3s}}\right)\prod_{p\equiv 1 \pmod 2}\left(1+\frac{1}{p^s}\right)-1\\ &= \left(1-\frac{1}{2^{s}}+\frac{2}{2^{2s}}\right)\prod_{p}\left(1+\frac{1}{p^s}\right)-1\\ &= \left(1-\frac{1}{2^{s}}+\frac{2}{2^{2s}}\right)\frac{\zeta(s)}{\zeta(2s)}-1\end{aligned}$$

$$\Phi_{2,0}(C_2,s) = \frac{1}{2}\Phi_2(C_2,s)+\frac{1}{2}\left(1-\frac{1}{2^{2s}}\right)\prod_{p\equiv 1 \pmod 2}\left(1+\frac{(-1)^{(p-1)/2}}{p^s}\right)-\frac{1}{2}$$

$$\Phi_{0,1}(C_2,s) = \Phi_2(C_2,s)-\Phi_{2,0}(C_2,s)\ .$$

Asymptotic formulas:

$$N_2(C_2,X) = c(C_2)\,X + R_2(C_2,X)$$

$$N_{2,0}(C_2,X) = \frac{c(C_2)}{2}\,X + R_{2,0}(C_2,X)$$

$$N_{0,1}(C_2,X) = \frac{c(C_2)}{2}\,X + R_{0,1}(C_2,X)\ ,$$

with

$$c(C_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

and

$$R(C_2,X) = O(X^{1/2}\exp(-c\,(\log X)^{3/5}(\log\log X)^{-1/5}))$$

for some positive constant c, and under the Riemann Hypothesis

$$R(C_2,X) = \widetilde{O}(X^{8/25})$$

(see for example [31], Notes du Chapitre I.3). It is conjectured, and this is strongly confirmed by the data, that $R(C_2,X) = \widetilde{O}(X^{1/4})$, hence to compare with the data we use $\alpha = 1/4$.

2.2. Tables. These tables have been computed using the methods explained in [8].

X	$N_{2,0}(C_2,X)$	$N_{0,1}(C_2,X)$
10^1	2	4
10^2	30	31
10^3	302	305
10^4	3043	3043
10^5	30394	30392
10^6	303957	303968
10^7	3039653	3039632
10^8	30396324	30396385
10^9	303963559	303963510
10^{10}	3039635379	3039635443
10^{11}	30396355148	30396355052
10^{12}	303963551039	303963550712
10^{13}	3039635509103	3039635509360
10^{14}	30396355093462	30396355092880
10^{15}	303963550926173	303963550926479

X	$N_{2,0}(C_2, X)$	$N_{0,1}(C_2, X)$
10^{16}	3039635509271389	3039635509273025
10^{17}	30396355092697223	30396355092696593
10^{18}	303963550927008744	303963550927017751
10^{19}	3039635509270131961	3039635509270110507
10^{20}	30396355092701313737	30396355092701291066
10^{21}	303963550927013401272	303963550927013312649
10^{22}	3039635509270133130535	3039635509270133092175
10^{23}	30396355092701331456323	30396355092701331531457
10^{24}	303963550927013314010676	303963550927013314554179
10^{25}	3039635509270133143448215	3039635509270133143069580

The relative error between the actual data and the predictions varies between -0.57% and 0.57%.

3. Degree 3 fields with $\mathbf{G \simeq C_3}$

3.1. Dirichlet series and asymptotic formulas. The results are due to Cohn [19], and can easily be obtained from the much older characterization of cyclic cubic fields due to Hasse [23], see for example [6], Section 6.4.2.

Dirichlet series:

$$\Phi_3(C_3, s) = \frac{1}{2}\left(1 + \frac{2}{3^{4s}}\right) \prod_{p \equiv 1 \pmod 6} \left(1 + \frac{2}{p^{2s}}\right) - \frac{1}{2}$$

$$\Phi_{3,0}(C_3, s) = \Phi_3(C_3, s)$$

$$\Phi_{1,1}(C_3, s) = 0\ .$$

Asymptotic formulas:

$$N_3(C_3, X) = c(C_3)\, X^{1/2} + R_3(C_3, X)$$

$$N_{3,0}(C_3, X) = N_3(C_3, X)$$

$$N_{1,1}(C_3, X) = 0\ ,$$

with

$$c(C_3) = \frac{11\sqrt{3}}{36\pi} \prod_{p \equiv 1 \pmod 6} \left(1 - \frac{2}{p(p+1)}\right)$$

$$= 0.1585282583961420602835078203575\ldots$$

and $R_3(C_3, X) = \widetilde{O}(X^{1/3})$.

It is reasonable to conjecture that we should have $R_3(C_3, X) = \widetilde{O}(X^{1/6})$, hence to compare with the numerical data we use $\alpha = 1/6$.

3.2. Table. This table has been computed using the methods explained in [8].

X	$N_3(C_3,X)$	X	$N_3(C_3,X)$	X	$N_3(C_3,X)$
10^1	0	10^{14}	1585249	10^{27}	5013103697105
10^2	2	10^{15}	5013206	10^{28}	15852825840369
10^3	5	10^{16}	15852618	10^{29}	50131036986701
10^4	16	10^{17}	50131008	10^{30}	158528258396671
10^5	51	10^{18}	158528150	10^{31}	501310370020343
10^6	159	10^{19}	501309943	10^{32}	1585282583932681
10^7	501	10^{20}	1585282684	10^{33}	5013103700345884
10^8	1592	10^{21}	5013103291	10^{34}	15852825839615504
10^9	5008	10^{22}	15852826251	10^{35}	50131037003076114
10^{10}	15851	10^{23}	50131036382	10^{36}	158528258396205064
10^{11}	50152	10^{24}	158528255967	10^{37}	501310370031289126
10^{12}	158542	10^{25}	501310368157		
10^{13}	501306	10^{26}	1585282578080		

The relative error between the actual data and the predictions varies between -0.68% and 0.33%.

4. Degree 3 fields with $\mathbf{G \simeq S_3 \simeq D_3}$

4.1. Dirichlet series and asymptotic formulas. The main terms in the asymptotic formulas are due to Davenport and Heilbronn [21], [22]. The other terms are conjectural and can be attributed to Datskovsky–Wright [20] and Roberts [30].

Dirichlet series:

In this case, the Dirichlet series do not seem to have any nice form.

Asymptotic formulas:

$$N_3(S_3,X) = c(S_3)\,X + c'(S_3)\,X^{5/6} - \frac{c(C_3)}{3}X^{1/2} + R_3(S_3,X)$$

$$N_{3,0}(S_3,X) = \frac{c(S_3)}{4}\,X + \frac{c'(S_3)}{\sqrt{3}+1}\,X^{5/6} - \frac{c(C_3)}{3}X^{1/2} + R_{3,0}(S_3,X)$$

$$N_{1,1}(S_3,X) = \frac{3}{4}c(S_3)\,X + \frac{\sqrt{3}}{\sqrt{3}+1}c'(S_3)\,X^{5/6} + R_{1,1}(S_3,X)\ ,$$

with

$$c(S_3) = \frac{1}{3\zeta(3)} = 0.27730245752690248956104209294\ldots$$

$$c'(S_3) = \frac{4(\sqrt{3}+1)}{5\Gamma(2/3)^3}\frac{\zeta(1/3)}{\zeta(5/3)}$$
$$= -0.40348363666394679863364025671534\ldots$$

and

$$R(S_3, X) = \widetilde{O}(X^{19/20})\ .$$

This remainder term is due to Belabas–Bhargava–Pomerance [4], and evidently, in these estimates the remainder term is of larger order than the additional main term. The reason that we have given this additional term is that much more is conjectured to be true. From heuristics of Roberts and Wright (see [33] and [30]), it is believed that $R(S_3, X)$ is negligible compared to the additional main term, in other words that $R(S_3, X) = o(X^{1/2})$. Thus, to compare with the numerical data we use these additional main terms and choose $\alpha = 1/2$, although the tables would seem to indicate that even $\alpha = 5/12$ could be possible.

4.2. Tables. These tables have been computed by Belabas in [2] using his methods, based on the Davenport–Heilbronn theory, and also explained in detail in [7], Chapter 8. It should not be too difficult to extend them to $X = 10^{13}$, say, using the improved methods given in [3].

X	$N_{3,0}(S_3, X)$	$N_{1,1}(S_3, X)$
10^1	0	0
10^2	0	7
10^3	22	127
10^4	366	1520
10^5	4753	17041
10^6	54441	182417
10^7	592421	1905514
10^8	6246698	19609185
10^9	64654353	199884780
10^{10}	661432230	2024660098
10^{11}	6715773873	20422230540

The relative error between the actual data and the predictions varies between -0.2% and 0.04%.

5. Degree 4 fields with $G \simeq C_4$

5.1. Dirichlet series and asymptotic formulas. The results are not difficult. The paper which is cited in this context is [1], which unfortunately contains several misprints. These are corrected in the papers of Mäki [25] and ours, in particular here.

Dirichlet series:

$$\Phi_4(C_4,s)=\frac{\zeta(2s)}{2\zeta(4s)}\left(\left(1-\frac{1}{2^{2s}}+\frac{2}{2^{4s}}+\frac{4}{2^{11s}+2^{9s}}\right)\prod_{\substack{p\equiv 1\\ \pmod 4}}\left(1+\frac{2}{p^{3s}+p^s}\right)-\left(1-\frac{1}{2^{2s}}+\frac{2}{2^{4s}}\right)\right)$$

$$\Phi_{4,0}(C_4,s)=\frac{1}{2}\Phi_4(C_4,s)+\frac{L(2s,\left(\frac{-4}{\cdot}\right))}{4\zeta(4s)}\left(\prod_{\substack{p\equiv 1\\ \pmod 4}}\left(1+\frac{2(-1)^{(p-1)/4}}{p^{3s}+p^s}\right)-1\right)$$

$$\Phi_{2,1}(C_4,s)=0$$

$$\Phi_{0,2}(C_4,s)=\Phi_4(C_4,s)-\Phi_{4,0}(C_4,s)\ .$$

Asymptotic formulas:

$$N_4(C_4,X)=c(C_4)\,X^{1/2}+c'(C_4)\,X^{1/3}+R_4(C_4,X)$$

$$N_{4,0}(C_4,X)=\frac{c(C_4)}{2}\,X^{1/2}+\frac{c'(C_4)}{2}\,X^{1/3}+R_{4,0}(C_4,X)$$

$$N_{2,1}(C_4,X)=0$$

$$N_{0,2}(C_4,X)=\frac{c(C_4)}{2}\,X^{1/2}+\frac{c'(C_4)}{2}\,X^{1/3}+R_{0,2}(C_4,X)\ ,$$

with

$$c(C_4)=\frac{3}{\pi^2}\left(\left(1+\frac{\sqrt{2}}{24}\right)\prod_{p\equiv 1 \pmod 4}\left(1+\frac{2}{p^{3/2}+p^{1/2}}\right)-1\right)$$

$$=0.12205267325139676092260805289 65\ldots$$

$$c'(C_4)=\frac{3+2^{-1/3}+2^{-2/3}}{1+2^{-2/3}}\frac{\zeta(2/3)}{4\pi\zeta(4/3)}\prod_{p\equiv 1 \pmod 4}\left(1+\frac{2}{p+p^{1/3}}\right)\left(\frac{1-1/p}{1+1/p}\right)$$

$$=-0.11567519939427878830185483678\ldots$$

Although easy to obtain by contour integration, we have not found the additional $X^{1/3}$ main term in the literature.

It is reasonable to conjecture that we should have $R(C_4,X)=\widetilde{O}(X^{1/6})$, hence to compare with the numerical data we use $\alpha=1/6$, although the tables seem to indicate that even $\alpha=1/8$ could be possible.

5.2. Tables. These tables have been computed using the methods explained in [8].

X	$N_{4,0}(C_4, X)$	$N_{0,2}(C_4, X)$
10^{1}	0	0
10^{2}	0	0
10^{3}	0	1
10^{4}	6	4
10^{5}	15	17
10^{6}	59	54
10^{7}	182	181
10^{8}	586	582
10^{9}	1867	1865
10^{10}	5966	5964
10^{11}	19017	19028
10^{12}	60456	60469
10^{13}	191736	191764
10^{14}	607589	607609
10^{15}	1924160	1924059
10^{16}	6090130	6090110
10^{17}	19271385	19271321
10^{18}	60968525	60968399
10^{19}	192857593	192857870
10^{20}	609994937	609994964
10^{21}	1929244391	1929243674
10^{22}	6101387381	6101387860
10^{23}	19295537531	19295537010
10^{24}	61020552533	61020552938
10^{25}	192969762398	192969758223
10^{26}	610236520653	610236519548
10^{27}	1929764373961	1929764373161
10^{28}	6102509058257	6102509054460
10^{29}	19297953643936	19297953644691
10^{30}	61025758244048	61025758248309
10^{31}	192980974911603	192980974923193
10^{32}	610260681684841	610260681669563

The relative error between the actual data and the predictions varies between -0.64% and 0.45%, and seems clearly to tend to 0 as $k \to \infty$.

6. Degree 4 fields with $G \simeq V_4 = C_2 \times C_2$

6.1. Dirichlet series and asymptotic formulas. The results are not difficult. Once again the paper which is cited in this context is [1], which contains several misprints which are corrected in the papers of Mäki [25] and in ours.

Dirichlet series:

$$\Phi_4(V_4,s) = \frac{1}{6}\left(1+\frac{3}{2^{4s}}+\frac{6}{2^{6s}}+\frac{6}{2^{8s}}\right)\prod_{\substack{p\equiv 1\\ \pmod 2}}\left(1+\frac{3}{p^{2s}}\right) - \frac{1}{2}\Phi_2(C_2,2s)-\frac{1}{6}$$

$$\Phi_{4,0}(V_4,s) = \frac{1}{4}\Phi_4(V_4,s)-\frac{1}{2}\Phi_{2,0}(C_2,2s)+\frac{1}{8}\Phi_2(C_2,2s)-\frac{1}{8} + \frac{1}{8}\left(1-\frac{1}{2^{4s}}+\frac{2}{2^{6s}}-\frac{2}{2^{8s}}\right)\prod_{\substack{p\equiv 1\\ \pmod 2}}\left(1+\frac{1+2(-1)^{(p-1)/2}}{p^{2s}}\right)$$

$$\Phi_{2,1}(V_4,s) = 0$$

$$\Phi_{0,2}(V_4,s) = \Phi_4(V_4,s)-\Phi_{4,0}(V_4,s)\ .$$

Asymptotic formulas:

$$N_4(V_4,X) = (c(V_4)\log^2 X + c'(V_4)\log X + c''(V_4))\,X^{1/2} + R_4(V_4,X)$$

$$N_{4,0}(V_4,X) = \left(\frac{c(V_4)}{4}\log^2 X + \frac{c'(V_4)}{4}\log X + \frac{c'''(V_4)}{4}\right)X^{1/2} + R_{4,0}(V_4,X)$$

$$N_{2,1}(V_4,X) = 0$$

$$N_{0,2}(V_4,X) = \left(\frac{3}{4}c(V_4)\log^2 X + \frac{3}{4}c'(V_4)\log X + \left(c''(V_4)-\frac{c'''(V_4)}{4}\right)\right)X^{1/2} + R_{0,2}(V_4,X),\quad \text{with}$$

$$c(V_4) = \frac{23}{960}\prod_p\left(\left(1+\frac{3}{p}\right)\left(1-\frac{1}{p}\right)^3\right)$$

$$c'(V_4) = 12c(V_4)\left(\gamma-\frac{1}{3}+\frac{9\log 2}{23}+4\sum_{p\ge 3}\frac{\log p}{(p-1)(p+3)}\right)$$

$$c''(V_4) = \frac{c'(V_4)^2}{4c(V_4)}-\frac{3}{\pi^2} + 24c(V_4)\left(\frac{1}{6}-\gamma_1-\frac{\gamma^2}{2}-\frac{340}{529}\log^2 2-4\sum_{p\ge 3}\frac{p(p+1)\log^2 p}{(p-1)^2(p+3)^2}\right)$$

$$c'''(V_4) = c''(V_4) - \frac{3}{\pi^2} + \frac{7}{8\pi^2} \prod_{p\equiv 1 \pmod 4} \frac{(1+3/p)(1-1/p)}{(1+1/p)^2}\ ,$$

where γ is Euler's constant and

$$\gamma_1 = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log k}{k} - \frac{\log^2 n}{2n} \right) = -0.0728158454836767248605863758\ldots$$

Numerically, we have

$$\begin{aligned} c(V_4) &= \ \ 0.0027524302227554813966383118376\ldots \\ c'(V_4) &= \ \ 0.05137957621042353770883347445\ldots \\ c''(V_4) &= -0.2148583422482281175118362061\ldots \\ c'''(V_4) &= -0.4438647800546969108664219885\ldots \end{aligned}$$

Although not difficult to compute, we have not found the additional main terms in the literature.

It is reasonable to conjecture that we should have $R(V_4, X) = \widetilde{O}(X^{1/4})$, hence to compare with the numerical data we use $\alpha = 1/4$.

6.2. Tables. These tables have been computed using the methods explained in [8].

X	$N_{4,0}(V_4, X)$	$N_{0,2}(V_4, X)$
10^1	0	0
10^2	0	0
10^3	0	8
10^4	6	41
10^5	42	201
10^6	196	818
10^7	876	3331
10^8	3603	13076
10^9	14249	50067
10^{10}	54940	187770
10^{11}	207295	694262
10^{12}	769284	2536801
10^{13}	2814497	9167570
10^{14}	10181802	32835581
10^{15}	36478693	116677591

X	$N_{4,0}(V_4, X)$	$N_{0,2}(V_4, X)$
10^{16}	129620531	411762457
10^{17}	457321963	1444383361
10^{18}	1603453447	5039360330
10^{19}	5590953378	17497040934
10^{20}	19398735478	60486267277
10^{21}	67009600870	208270612830
10^{22}	230548142363	714545063480
10^{23}	790326082314	2443396436299
10^{24}	2700275901104	8329834172525
10^{25}	9197857451663	28317754338743
10^{26}	31242564815515	96017758881843
10^{27}	105847491463943	324784293743259
10^{28}	357742322840950	1096127612328756
10^{29}	1206393568766650	3691598900680670
10^{30}	4059776186016270	12408334995379417
10^{31}	13635417115241023	41630433288940969
10^{32}	45713153519958996	139429524939542248
10^{33}	152991934395591362	466217622608203817
10^{34}	511204072681788782	1556512861826445892
10^{35}	1705526466144745140	5188997592667511054
10^{36}	5681952310883424255	17274863370464181629

The relative error between the actual data and the predictions varies between -0.73% and 0.51%, and once again seems clearly to tend to 0 as $k \to \infty$.

7. Degree 4 fields with $\mathbf{G} \simeq \mathbf{D_4}$

7.1. Dirichlet series and asymptotic formulas. The results of this section are due to the authors, see [9] and [13]. In the totally complex case (signature $(0, 2)$) we will distinguish between fields having a real quadratic subfield (using the superscript $^+$) and those having a complex quadratic subfield (using the superscript $^-$), which gives important extra information (the behavior of the "Frobenius at infinity").

Furthermore, it is convenient both in theory and in practice to introduce the set of *imprimitive* quartic number fields but now in a fixed algebraic closure of $\mathbb{Q}$, and to denote by $\Phi(I, s)$ and $N(I, X)$ the corresponding Φ and N functions, possibly with signatures. We then have

$$\begin{aligned}
\Phi_4(I,s) &= 2\Phi_4(D_4,s) + 3\Phi_4(V_4,s) + \Phi_4(C_4,s)\\
\Phi_{4,0}(I,s) &= 2\Phi_{4,0}(D_4,s) + 3\Phi_{4,0}(V_4,s) + \Phi_{4,0}(C_4,s)\\
\Phi_{2,1}(I,s) &= 2\Phi_{2,1}(D_4,s)\\
\Phi_{0,2}(I,s) &= 2\Phi_{0,2}(D_4,s) + 3\Phi_{0,2}(V_4,s) + \Phi_{0,2}(C_4,s)\\
\Phi_{0,2}^{+}(I,s) &= 2\Phi_{0,2}^{+}(D_4,s) + \Phi_{0,2}(V_4,s) + \Phi_{0,2}(C_4,s)\\
\Phi_{0,2}^{-}(I,s) &= 2\Phi_{0,2}^{-}(D_4,s) + 2\Phi_{0,2}(V_4,s)\ ,
\end{aligned}$$

and the same linear combinations for the N functions. The last two formulas come from the fact that the quadratic subfield of a C_4 field is always real and that a complex V_4 field always contains two complex and one real quadratic subfield.

Thus we will give the formulas only for I (the above combinations allowing to easily get back to D_4), but the tables only for D_4. Note the important fact that, as a consequence, the asymptotic constants for D_4 are *one half* of the ones for I.

Denote by $\mathcal{D}$ the set of all fundamental discriminants, in other words 1 and discriminants of quadratic fields. For any $d \in \mathcal{D}$, denote by $L(s,d)$ the Dirichlet L-series for the quadratic character $\left(\frac{d}{n}\right)$.

Dirichlet series:

$$\Phi_4(I,s) = \frac{1}{2\zeta(2s)} \sum_{D\in\mathcal{D}\smallsetminus\{1\}} \frac{2^{-r_2(D)}}{|D|^{2s}L(2s,D)} F_D(s) - \Phi_2(C_2,2s)\ ,$$

where $r_2(D)=0$ if $D>0$, $r_2(D)=1$ if $D<0$, with

$$\begin{aligned}
F_D(s) = &\sum_{\substack{d\mid D,\ d\in\mathcal{D},\ \gcd(d,D/d)=1\\ d>0 \text{ if } D>0}} f_{D,d}(s)L(s,d)L(s,D/d)\\
&+ g_D(s) \sum_{\substack{d\mid D,\ d\in\mathcal{D},\ \gcd(d,D/d)=1\\ k(D)d>0 \text{ if } D>0}} L(s,k(D)d)L(s,k(D)D/d)\ ,
\end{aligned}$$

where

$$f_{D,d}(s) = \begin{cases} 1 - \dfrac{1}{2^{2s}} + \dfrac{4}{2^{4s}} & \text{if } D \equiv 5 \pmod 8\\[2ex] 1 - \dfrac{2\left(\frac{d}{2}\right)}{2^s} + \dfrac{5}{2^{2s}} - \dfrac{4\left(\frac{d}{2}\right)}{2^{3s}} + \dfrac{4}{2^{4s}} & \text{if } D \equiv 1 \pmod 8\\[2ex] 1 - \dfrac{\left(\frac{d_1}{2}\right)}{2^s} + \dfrac{2}{2^{2s}} - \dfrac{2\left(\frac{d_1}{2}\right)}{2^{3s}} + \dfrac{4}{2^{4s}} & \text{if } D \equiv 0 \pmod 4\ , \end{cases}$$

where $d_1 = d$ if $d \equiv 1 \pmod 4$, $d_1 = D/d$ if $d \equiv 0 \pmod 4$,

$$k(D) = \begin{cases} -4 & \text{if } D \not\equiv -4 \pmod{16} \\ 8 & \text{if } D \equiv -4 \pmod{16}\ , \end{cases}$$

$$g_D(s) = \begin{cases} 1 & \text{if } D \not\equiv 8 \pmod{16} \\ 1 + \dfrac{2}{2^{2s}} & \text{if } D \equiv 8 \pmod{16}\ . \end{cases}$$

The Dirichlet series for $\Phi_{r_1,r_2}(I,s)$ are of a similar nature but are too complicated to be given here (see [9]).

Asymptotic formulas:

$$\begin{aligned} N_4(I,X) &= 2c(D_4)\,X + R_4(D_4,X) \\ N_{4,0}(I,X) &= \frac{c^+(D_4)}{2}\,X + R_{4,0}(I,X) \\ N_{2,1}(I,X) &= c^+(D_4)\,X + R_{2,1}(I,X) \\ N_{0,2}(I,X) &= \frac{c^+(D_4)+2c^-(D_4)}{2}\,X + R_{0,2}(I,X) \\ N^+_{0,2}(I,X) &= \frac{c^+(D_4)}{2}\,X + R^+_{0,2}(I,X) \\ N^-_{0,2}(I,X) &= c^-(D_4)\,X + R^-_{0,2}(I,X)\ , \end{aligned}$$

with

$$c^\pm(D_4) = \frac{3}{\pi^2}\sum_{\mathrm{sign}(D)=\pm}\frac{1}{D^2}\frac{L(1,D)}{L(2,D)} \quad \text{and}$$

$$c(D_4) = c^+(D_4) + \frac{c^-(D_4)}{2}\ ,$$

where the sum is over discriminants D of quadratic fields of given sign.

Numerically, we have

$$c^+(D_4) = 0.01971137577, \quad c^-(D_4) = 0.06522927087,$$
$$c(D_4) = 0.05232601119\ ,$$

where in each case the mean deviation seems to be less than 100 in the last given digit (i.e., $\pm 10^{-9}$).

It is possible that these constants can be expressed as finite linear combinations of simple Euler products, but we have not been able to find such expressions.

It can be shown (see [13]) that $R(I,X) = \widetilde{O}(X^{3/4})$, and it is reasonable to conjecture that we should have $R(I,X) = \widetilde{O}(X^{1/2})$. However, the tables seem to show that there are additional main terms, so that

$$R(I,X) = 2(c'(D_4)\log X + c''(D_4))X^{1/2} + O(X^{\alpha})$$

for suitable constants $c'_{R_1,R_2}(D_4)$ and $c''_{R_1,R_2}(D_4)$ (depending on the signature), and some $\alpha < 1/2$ (we include an extra factor 2 above so that it disappears in the formulas for D_4).

A least squares computation gives

$$\begin{aligned} c'(D_4) &= 0.034067 & c''(D_4) &= -0.81992 \\ c'_{4,0}(D_4) &= 0.0092312 & c''_{4,0}(D_4) &= -0.26410 \\ c'_{2,1}(D_4) &= -0.0030683 & c''_{2,1}(D_4) &= -0.027401 \\ c'_{0,2}(D_4) &= 0.027904 & c''_{0,2}(D_4) &= -0.52842 \\ c'^{+}_{0,2}(D_4) &= 0.0096442 & c''^{+}_{0,2}(D_4) &= -0.13795 \\ c'^{-}_{0,2}(D_4) &= 0.018260 & c''^{-}_{0,2}(D_4) &= -0.39047 \end{aligned}$$

Here, even though we give the values with 5 digits, they are probably accurate only to within a factor of 2 or so. Nevertheless, the least square fit is very good, hence we use these values to compare with the actual data.

This seems to show that the functions $\Phi(I,s)$ have a double pole at $s = 1/2$, but we do not know how to prove this or how to compute the polar parts at $s = 1/2$, although heuristically it is easy to guess why they have at least a simple pole.

Thus, to compare with the data we use these refined estimates, and we choose $\alpha = 2/5$, which seems to give reasonable results.

7.2. Tables. See [13] and [9] for the methods used to compute these tables.

X	$N_{4,0}(D_4,X)$	$N_{2,1}(D_4,X)$	$N_{0,2}(D_4,X)$
10^1	0	0	0
10^2	0	0	0
10^3	1	6	17
10^4	25	93	295
10^5	379	968	3417
10^6	4486	9772	36238
10^7	47562	98413	370424
10^8	486314	984708	3734826
10^9	4903607	9852244	37469573
10^{10}	49188349	98546786	375154025

X	$N_{4,0}(D_4,X)$	$N_{2,1}(D_4,X)$	$N_{0,2}(D_4,X)$
10^{11}	492454432	985536549	3753258277
10^{12}	4926654580	9855572218	37538880690
10^{13}	49274156836	98556488881	375411901218
10^{14}	492769145545	985567509497	3754202033198
10^{15}	4927790007755	9855683662056	37542317217650
10^{16}	49278249627160	98556864596086	375424223055946
10^{17}	492783730187748	985568739794773	3754245940051259

X	$N_{0,2}^{+}(D_4,X)$	$N_{0,2}^{-}(D_4,X)$
10^{1}	0	0
10^{2}	0	0
10^{3}	0	17
10^{4}	27	268
10^{5}	395	3022
10^{6}	4512	31726
10^{7}	47708	322716
10^{8}	486531	3248295
10^{9}	4904276	32565297
10^{10}	49190647	325963378
10^{11}	492464630	3260793647
10^{12}	4926673909	32612206781
10^{13}	49274235813	326137665405
10^{14}	492769387400	3261432645798
10^{15}	4927790822970	32614526394680
10^{16}	49278252225484	326145970830462
10^{17}	492783738112277	3261462201938982

The relative error between the actual data and the predictions varies between -0.32% and 0.76%.

8. Degree 4 fields with $\mathbf{G} \simeq \mathbf{A_4}$

8.1. Dirichlet series and asymptotic formulas. Using Kummer theory, it is possible to obtain an explicit expression for the Dirichlet series $\Phi_4(A_4, k, s)$ where the additional parameter k indicates that we fix the resolvent cubic field (see [11]), hence an asymptotic formula for $N_4(A_4, k, s)$. However, as indicated in loc. cit., it does not seem possible to sum naively on k to obtain an asymptotic estimate for $N_4(A_4, X)$. Thus we must be content with experimental data. According to general conjectures, including that of Malle, it is reasonable to conjecture that we have an asymptotic formula of the form

$$N_4(A_4, X) \sim c(A_4)\, X^{1/2} \log X$$

for some constant $c(A_4) > 0$. As for the case $G \simeq D_4$, it is possible that the constant $c(A_4)$ can be expressed as a finite linear combination of Euler products with explicit coefficients. In view of the numerical data, it is possible that we have a sharper formula of the form

$$N_4(A_4, X) = (c(A_4) \log X + c'(A_4))\, X^{1/2} + O(X^{\alpha})$$

for some $\alpha < 1/2$, perhaps for any $\alpha > 1/4$. We obtain an excellent least squares fit by using $c(A_4) = 0.018634$ and $c'(A_4) = -0.14049$. We obtain similar fits for the tables with signatures ($c_{4,0}(A_4) = 0.0049903$, $c'_{4,0}(A_4) = -0.0373357$, $c_{0,2}(A_4) = 0.0136441$, $c'_{0,2}(A_4) = -0.103157$ with evident notations). All these values should be correct to within 5%.

To compare with the numerical data we use the values obtained above with the least squares fit and we choose $\alpha = 1/4$, which gives reasonable results.

8.2. Numerical computation. We have generated A_4 extensions using Kummer theory of quadratic extensions over cyclic cubic fields, keeping only those extensions whose discriminant is less than the required bound (see [14] for details). The computations without signatures being simpler than with signatures have been pushed to $X = 10^{16}$, while those with signatures have only been pushed to $X = 10^{13}$, although it should be easy to push them further. Thus, exceptionally we also give separately the data without signature distinction.

8.3. Tables. See [14] for the methods used to compute these tables.

X	$N_4(A_4, X)$	X	$N_4(A_4, X)$
10^1	0	10^9	7699
10^2	0	10^{10}	28759
10^3	0	10^{11}	104766
10^4	4	10^{12}	374470
10^5	27	10^{13}	1319606
10^6	121	10^{14}	4602909
10^7	514	10^{15}	15915694
10^8	2010	10^{16}	54592313

X	$N_{4,0}(A_4, X)$	$N_{0,2}(A_4, X)$
10^1	0	0
10^2	0	0
10^3	0	0
10^4	0	4
10^5	4	23
10^6	31	90
10^7	129	385
10^8	527	1483
10^9	2037	5662
10^{10}	7662	21097
10^{11}	28182	76584
10^{12}	100576	273894
10^{13}	354302	965304

The relative error between the actual data and the predictions varies between -1.01% and 1.03%.

9. Degree 4 fields with $\mathbf{G \simeq S_4}$

9.1. Dirichlet series and asymptotic formulas. By using similar methods to the A_4 case but this time with Kummer theory over noncyclic cubic fields, we can also compute explicitly the Dirichlet series $\Phi_4(S_4, k, s)$, which is quite similar in form to $\Phi_4(A_4, k, s)$, where k is a fixed cubic resolvent, see once again [11], hence also obtain an asymptotic formula for $N_4(S_4, k, X)$, with evident notation. Contrary to the A_4 case, however, it seems that it is now possible to sum the contributions coming from the different cubic resolvents and obtain an asymptotic formula for $N_4(S_4, X)$. However this does not give a very useful formula, neither in theory nor for numerical computation, and in any case is completely superseded by the work of Bhargava.

Indeed, in a series of groundbreaking papers [5], Bhargava gives a wide generalization of the methods of Davenport–Heilbronn and as a consequence obtains an asymptotic formula for $N(S_4, X)$, including a simple expression for $c(S_4)$, and also with signatures.

Asymptotic formulas:

$$\begin{aligned} N_4(S_4, X) &= r_4(S_4)z(S_4)X + c'(S_4)X^{5/6} \\ &\quad + (c''(S_4)\log X + c'''(S_4))X^{3/4} + R_4(S_4, X) \\ N_{r_1,r_2}(S_4, X) &= r_{r_1,r_2}(S_4)z(S_4) + {c'}_{r_1,r_2}(S_4)X^{5/6} \\ &\quad + ({c''}_{r_1,r_2}(S_4)\log X + {c'''}_{r_1,r_2}(S_4))X^{3/4} + R_{r_1,r_2}(S_4, X) \end{aligned}$$

with

$$z(S_4) = \prod_{p \geq 2} \left(1 + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^4}\right) = 1.2166902869063309337694390868\ldots$$

and

$$r_4(S_4) = \frac{5}{24},\ r_{4,0}(S_4) = \frac{1}{48},\ r_{2,1}(S_4) = \frac{1}{8},\ r_{0,2}(S_4) = \frac{1}{16}\ .$$

The additional main terms given above have been suggested to the authors in a personal communication of Yukie.

Bhargava proves only that $R(S_4, X) = o(X)$, but conjecturally we should have $R(S_4, X) = O(X^\alpha)$ for some $\alpha < 3/4$, perhaps for any $\alpha > 1/2$. We will use $\alpha = 1/2$ for comparisons with the actual data.

As in the D_4 and A_4 cases, we can try using a least squares method to find the additional main terms. However, note first that, using our Kummer-theoretic method, it is very costly to compute $N(S_4, 10^7)$ since it involves in particular computing the class group and units of 2.5 million (exactly $N_3(S_3, 10^7)$) cubic fields (see below, however). Second, note that in the range $X \leq 10^7$ the functions X, $X^{5/6}$, $X^{3/4} \log X$, and $X^{3/4}$ are quite close to one another (for instance the function $X^{3/4} \log X$ which is asymptotically negligible with respect to $X^{5/6}$ is still more than 4 times *larger* for $X = 10^7$), hence it will be almost impossible to distinguish between their coefficients using a least squares method. Nonetheless we have done so and found

$$\begin{aligned}
c'(S_4) &= -2.17561, & c''(S_4) &= 0.08417, & c'''(S_4) &= 1.91916\\
c'_{4,0}(S_4) &= -0.42792, & c''_{4,0}(S_4) &= 0.034743, & c''_{4,0}(S_4) &= 0.33335\\
c'_{2,1}(S_4) &= -1.28495, & c''_{2,1}(S_4) &= 0.051021, & c''_{2,1}(S_4) &= 1.11068\\
c'_{0,2}(S_4) &= -0.46274, & c''_{0,2}(S_4) &= -0.001590, & c''_{0,2}(S_4) &= 0.47514\ .
\end{aligned}$$

These values are only given to indicate how the tables have been computed, but are certainly very far from the correct ones.

9.2. Numerical computation. As for A_4 extensions, we use Kummer theory of quadratic extensions, this time over noncyclic cubic fields and we keep only those extensions whose discriminant is less than the required bound. See [14] for details. The reason that we cannot easily go above 10^7 is that we need to compute units and class groups for all (noncyclic) cubic fields of discriminant up to that bound, and this is very time-consuming. It is possible that in the same way that Belabas adapted the methods of Davenport–Heilbronn to compute rapidly tables of S_3-cubic fields by enumerating reduced cubic forms, one can adapt the method of Bhargava to efficiently compute S_4-quartic fields by enumerating reduced pairs of ternary quadratic forms, thus enabling computations to much larger discriminant bounds.

9.3. Tables. It should be emphasized that contrary to the other Galois groups, our numerical predictions should here be only considered as guesses.

X	$N_{4,0}(S_4, X)$	$N_{2,1}(S_4, X)$	$N_{0,2}(S_4, X)$
10^1	0	0	0
10^2	0	0	0
10^3	0	10	8
10^4	13	351	206
10^5	449	5916	3374
10^6	8301	80899	44122
10^7	120622	989587	525099

The relative error between the actual data and the predictions varies between -0.18% and 0.28%.

For instance, our estimates give $P_4(S_4, 10^8) = 18719128$, $P_{4,0}(S_4, 10^8) = 1521877$, $P_{2,1}(S_4, 10^8) = 11294945$, $P_{0,2}(S_4, 10^8) = 5902307$. It would be interesting to see how close to the truth are these estimates.

Note added: we have just learnt that in a large computation using Hunter's method instead of Kummer theory, G. Malle has computed

$$N_{4,0}(S_4, 10^8) = 1529634 \quad \text{and} \quad N_{4,0}(S_4, 10^9) = 17895702\ ,$$

so our above prediction for 10^8 was within 0.5% of the correct value.

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