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Sign changes of error terms related to arithmetical functions

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Résumé. Soit $H(x) = \sum_{n\leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$. Motivé par une conjecture de Erdös, Lau a développé une nouvelle méthode et il a démontré que $\#\{n \leq T : H(n)H(n+1) < 0\} \gg T$. Nous considérons des fonctions arithmétiques $f(n) = \sum_{d|n} \frac{b_d}{d}$ dont l'addition peut être exprimée comme $\sum_{n\leq x} f(n) = \alpha x + P(\log(x)) + E(x)$. Ici $P(x)$ est un polynôme, $E(x) = -\sum_{n\leq y(x)} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + o(1)$ avec $\psi(x) = x - \lfloor x\rfloor - 1/2$. Nous généralisons la méthode de Lau et démontrons des résultats sur le nombre de changements de signe pour ces termes d'erreur.

Abstract. Let $H(x) = \sum_{n\leq x} \frac{\phi(n)}{n} - \frac{6}{\pi^2}x$. Motivated by a conjecture of Erdös, Lau developed a new method and proved that $\#\{n \leq T : H(n)H(n+1) < 0\} \gg T$. We consider arithmetical functions $f(n) = \sum_{d|n} \frac{b_d}{d}$ whose summation can be expressed as $\sum_{n\leq x} f(n) = \alpha x + P(\log(x)) + E(x)$, where $P(x)$ is a polynomial, $E(x) = -\sum_{n\leq y(x)} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + o(1)$ and $\psi(x) = x - \lfloor x\rfloor - 1/2$. We generalize Lau's method and prove results about the number of sign changes for these error terms.

1. Introduction

We say that an arithmetical function $f(x)$ has a *sign change on integers* at $x = n$, if $f(n)f(n+1) < 0$. The number of sign changes on integers of $f(x)$ on the interval $[1,T]$ is defined as

$$N_f(T) = \#\{n \leq T, n \text{ integer} : f(n)f(n+1) < 0\}.$$

We also define $z_f(T) = \#\{n \leq T, n \text{ integer} : f(n) = 0\}$. Throughout this work, $\psi(x) = x - \lfloor x\rfloor - 1/2$ and $f(n)$ will be an arithmetical function such

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that

$$f(n)=\sum_{d|n}\frac{b_d}{d}\ \text{ for some sequence of real numbers } b_n.$$

The motivation for our work was a paper by Y.-K. Lau [5], where he proves that the error term, $H(x)$, given by

$$\sum_{n\le x}\frac{\phi(n)}{n}=\frac{6}{\pi^2}x+H(x)$$

has a positive proportion of sign changes on integers solving a conjecture stated by P. Erdös in 1967.

An important tool that Lau used to prove his theorem, was that the error term $H(x)$ can be expressed as

$$H(x)=-\sum_{n\le\frac{x}{\log^5 x}}\frac{\mu(n)}{n}\psi\left(\frac{x}{n}\right)+O\left(\frac{1}{\log^{20}x}\right),\quad\text{(S. Chowla [3])}\tag{1}$$

We generalize Lau's result in the following way

Theorem 1.1. *Suppose $H(x)$ is a function that can be expressed as*

$$H(x)=-\sum_{n\le y(x)}\frac{b_n}{n}\psi\left(\frac{x}{n}\right)+O\left(\frac{1}{k(x)}\right),\tag{2}$$

where each b_n is a real number and

(i) *$y(x)$ increasing, $x^{\frac{1}{4}}\ll y(x)\ll\frac{x}{(\log x)^{5+\frac{D}{2}}}$, for some $D>0$, and*

$$\sum_{n\le x}b_n^4\ll x\log^D x;\tag{3}$$

(ii) *$k(x)$ is an increasing function, satisfying $\lim_{x\to\infty}k(x)=\infty$.*

(iii) *$H(x)=H(\lfloor x\rfloor)-\alpha\{x\}+\theta(x)$, where $\alpha\neq 0$ and $\theta(x)=o(1)$.*

Let $\prec\in\{<,=,\le\}$. If $\#\{1\le n\le T:\alpha H(n)\prec 0\}\gg T$ then there exists a positive constant c_0 and c_0T disjoint subintervals of $[1,T]$, with each of them having at least two integers, m and n, such that $\alpha H(m)>0$ and $\alpha H(n)\prec 0$. In particular,

(1) *$\#\{n\le T:\alpha H(n)>0\}\gg T$;*

(2) *if $\#\{n\le T:\alpha H(n)<0\}\gg T$, then $N_H(T)\gg T$ or $z_H(T)\gg T$.*

We consider arithmetical functions $f(n)$ for which, the error term of the summation function satisfies the conditions of Theorem 1.1. A first class is described in the following result

Theorem 1.2. *Let $f(n)$ be an arithmetical function and suppose the sequence b_n satisfies condition (3) and*

$$\sum_{n\le x} b_n = Bx + O\left(\frac{x}{\log^A x}\right) \tag{4}$$

for some B real, $D>0$ and $A>6+\frac{D}{2}$, respectively. Let $\alpha=\sum_{n=1}^{\infty}\frac{b_n}{n^2}$,

$$\gamma_b=\lim_{x\to\infty}\Big(\sum_{n\le x}\frac{b_n}{n}-B\log x\Big) \text{ and } H(x)=\sum_{n\le x} f(n)-\alpha x+\frac{B\log 2\pi x}{2}+\frac{\gamma_b}{2}.$$

If $\alpha\neq 0$, then Theorem 1.1 is valid for the error term $H(x)$. Moreover, if $f(n)$ is a rational function, then, except when $\alpha=0$, or $B=0$ and α is rational, we have

$$N_H(T)\gg T \quad \text{if and only if} \quad \#\{n\le T:\alpha H(n)<0\}\gg T.$$

Notice that this class of arithmetical functions is closed for addition, i.e., if $f(n)$ and $g(n)$ are members of the class then also is $(f+g)(n)$. In the case considered by Lau, it was known that $H(x)$ has a positive proportion of negative values (Y.-F. S. Pétermann [6]), so the second part of Theorem 1.2 generalizes Lau's result. Another example is $f(n)=\frac{n}{\phi(n)}$.

Using a result of U. Balakrishnan and Y.-F. S. Pétermann [2] we are able to apply Theorem 1.1 to more general arithmetical functions:

Theorem 1.3. *Let $f(n)$ be an arithmetical function and suppose the sequence b_n satisfies condition (3) and*

$$\sum_{n=1}^{\infty}\frac{b_n}{n^s}=\zeta^{\beta}(s)g(s) \tag{5}$$

for some β real, $D>0$, and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma>1-\lambda$, for some $\lambda>0$. Let $\alpha=\zeta^{\beta}(2)g(2)$ and

$$H(x)=\begin{cases}\sum_{n\le x} f(n)-\alpha x, & \text{if } \beta<0,\\ \sum_{n\le x} f(n)-\alpha x-\sum_{j=0}^{\lfloor\beta\rfloor}B_j(\log x)^{\beta-j} & \text{if } \beta>0,\end{cases}$$

where the constants B_j are well defined. If $\alpha\neq 0$, then Theorem 1.1 is valid for the error term $H(x)$.

Theorem 1.3 is valid for the following examples, where $r\neq 0$ is real:

$$\left(\frac{\phi(n)}{n}\right)^r, \qquad \left(\frac{\sigma(n)}{n}\right)^r, \qquad \left(\frac{\phi(n)}{\sigma(n)}\right)^r.$$

2. Main Lemma

The main tool used by Y.-K. Lau was his Main Lemma, where he proved that if $H(x)=\sum_{n\le x}\frac{\phi(n)}{n}-\frac{6}{\pi^2}x$ then

$$\int_T^{2T}\left(\int_t^{t+h}H(u)\,\mathrm{d}u\right)^2\mathrm{d}t\ll Th,$$

for sufficiently large T and any $1\le h\ll\log^4 T$. Lau's argument depends essentially on the formula (1). In this section, we obtain a generalization of Lau's Main Lemma.

Main Lemma. *Suppose $H(x)$ is a function that can be expressed as (2) and satisfies conditions (i) and (ii) of Theorem 1.1. Then, for all large T and $h\le\min\big(\log T,k^2(T)\big)$, we have*

$$\int_T^{2T}\left(\int_t^{t+h}H(u)\,\mathrm{d}u\right)^2\mathrm{d}t\ll Th^{\frac{3}{2}}. \tag{6}$$

For any positive integer N, define

$$H_N(x)=-\sum_{d\le N}\frac{b_d}{d}\psi\left(\frac{x}{d}\right). \tag{7}$$

The Main Lemma will follow from the next result.

Lemma 2.1. *Assume the conditions of the Main Lemma and take $D>0$ satisfying condition (i). Let $E=4+\frac{D}{2}$, then*

(a) *For any $\delta>0$, large T, any $Y\ll T$ and $N\le y(T)$, we have*

$$\int_T^{T+Y}\left(H(u)-H_N(u)\right)^2\,\mathrm{d}u\ll\frac{Y}{N^{1-\delta}}+\frac{Y}{k^2(T)}+y(T+Y)\left(\log T\right)^E;$$

(b) *For all large T, $N\le y(T)$ and $1\le h\le\min\big(\log T,k^2(T)\big)$, we have*

$$\int_T^{2T}\left(\int_t^{t+h}H_N(u)\,\mathrm{d}u\right)^2\mathrm{d}t\ll Th^{\frac{3}{2}}+N^3(\log N)^E.$$

Now we prove the Main Lemma:

Proof. Take $N=T^{\frac{1}{4}}$ and $\delta>0$ small. Cauchy's inequality gives us

$$\left(\int_t^{t+h}H(u)\,\mathrm{d}u\right)^2\le 2\left(\int_t^{t+h}H_N(u)\,\mathrm{d}u\right)^2 + 2\left(\int_t^{t+h}\left(H(u)-H_N(u)\right)\mathrm{d}u\right)^2.$$

Since $N = T^{\frac{1}{4}}$ then, for sufficiently large T, $N^3 \log^E N \ll T$. So, using part (b) of Lemma 2.1 we have

$$\int_T^{2T} \left(\int_t^{t+h} H_N(u)\,\mathrm{d}u \right)^2 \mathrm{d}t \ll Th^{\frac{3}{2}} + N^3 \log^E N \ll Th^{\frac{3}{2}}.$$

Using Cauchy's inequality and interchanging the integrals,

$$\begin{aligned}
&\int_T^{2T} \left(\int_t^{t+h} (H(u) - H_N(u))\,\mathrm{d}u \right)^2 \mathrm{d}t \\
&\quad \le h \int_T^{2T} \left(\int_t^{t+h} (H(u) - H_N(u))^2\,\mathrm{d}u \right) \mathrm{d}t \\
&\quad \le h \int_T^{2T+h} \left(\int_{\max(u-h,T)}^{\min(u,2T)} (H(u) - H_N(u))^2\,\mathrm{d}t \right) \mathrm{d}u \\
&\quad \le h^2 \int_T^{2T+h} (H(u) - H_N(u))^2\,\mathrm{d}u \\
&\quad \ll h^2 \left(\frac{T+h}{N^{1-\delta}} + \frac{T+h}{k^2(T)} + y(2T+h)\,(\log T)^E \right) \\
&\quad \ll T + Th \ll Th^{\frac{3}{2}}
\end{aligned}$$

since $y(2T+h) \ll \frac{T}{(\log T)^{E+1}}$ and $h \le \min(\log T, k^2(T))$. Hence

$$\int_T^{2T} \left(\int_t^{t+h} H(u)\,\mathrm{d}u \right)^2 \mathrm{d}t \ll Th^{\frac{3}{2}}.$$

□

3. Step I

In this section, we will prove part (a) of Lemma 2.1. Using expression (2) and Cauchy's inequality, we obtain

$$\int_T^{T+Y} (H(u) - H_N(u))^2 \mathrm{d}u \le 2 \int_T^{T+Y} \left(\sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi\left(\frac{u}{m}\right) \right)^2 \mathrm{d}u + O\left(\frac{Y}{k^2(T)} \right).$$

Let $\eta(T, m, n) = \max\left(T, y^{-1}(m), y^{-1}(n)\right)$, then

$$2 \int_T^{T+Y} \left(\sum_{m=N+1}^{y(u)} \frac{b_m}{m} \psi\left(\frac{u}{m}\right) \right)^2 \mathrm{d}u = 2 \sum_{m,n=N+1}^{y(T+Y)} \frac{b_m b_n}{mn} \int_{\eta(T,m,n)}^{T+Y} \psi\left(\frac{u}{m}\right) \psi\left(\frac{u}{n}\right) \mathrm{d}u.$$

The Fourier series of $\psi(u) = u - \lfloor u \rfloor - \frac{1}{2}$, when u is not an integer, is given by

$$\psi(u) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi k u)}{k}, \tag{8}$$

so we obtain

$$\frac{2}{\pi^2}\sum_{m,n=N+1}^{y(T+Y)}\frac{b_m b_n}{mn}\sum_{k,l=1}^{\infty}\frac{1}{kl}\int_{\eta(T,m,n)}^{T+Y}\sin\left(2\pi\frac{ku}{m}\right)\sin\left(2\pi\frac{lu}{n}\right)\,\mathrm{d}u.$$

Now, the integral above is equal to

$$\frac{1}{2}\int_{\eta(T,m,n)}^{T+Y}\cos\left(2\pi u\left(\frac{k}{m}+\frac{l}{n}\right)\right)-\cos\left(2\pi u\left(\frac{k}{m}-\frac{l}{n}\right)\right)\,\mathrm{d}u.$$

For the first term we get

$$\int_{\eta(T,m,n)}^{T+Y}\cos\left(2\pi u\left(\frac{k}{m}+\frac{l}{n}\right)\right)\,\mathrm{d}u \ll \frac{1}{\left(\frac{k}{m}+\frac{l}{n}\right)}.$$

If $\frac{k}{m}=\frac{l}{n}$, then

$$\int_{\eta(T,m,n)}^{T+Y}\cos\left(2\pi u\left(\frac{k}{m}-\frac{l}{n}\right)\right)\,\mathrm{d}u \le Y,$$

otherwise

$$\int_{\eta(T,m,n)}^{T+Y}\cos\left(2\pi u\left(\frac{k}{m}-\frac{l}{n}\right)\right)\,\mathrm{d}u \ll \frac{1}{\left|\frac{k}{m}-\frac{l}{n}\right|}.$$

Part (a) of Lemma 2.1 will now follow from the next three lemmas.

Lemma 3.1. *Let $E = 4 + \frac{D}{2}$ as in Lemma 2.1. Then*

$$\sum_{m,n\le X}|b_m b_n|\sum_{\substack{k,l=1\\ kn\ne lm}}^{\infty}\frac{1}{kl\,|\,kn-lm|} \ll X\,(\log X)^{E}.$$

Lemma 3.2. *If $D>0$ satisfies condition (3), then*

$$\sum_{m,n\le X}|b_m b_n|\sum_{k,l=1}^{\infty}\frac{1}{kl\,(\,kn+lm)} \ll X\,(\log X)^{1+\frac{D}{2}}.$$

Lemma 3.3. *For any $\delta>0$,*

$$\sum_{N<m,n\le X}\frac{|b_m b_n|}{mn}\sum_{\substack{k,l=1\\ kn=lm}}^{\infty}\frac{1}{kl} \ll \frac{1}{N^{1-\delta}}.$$

In order to finish the proof of part (a) of Lemma 2.1 we just need to take $X = y(T+Y)$ in the previous lemmas. Hence

$$\int_T^{T+Y} (H(u) - H_N(u))^2 \, \mathrm{d}u \ll \frac{Y}{N^{1-\delta}} + y(T+Y)(\log T)^E + \frac{Y}{k^2(T)}.$$

□

Before we prove the three lemmas above, we need the following technical result

Lemma 3.4. *Let b_n be a sequence satisfying condition (3). Then*

$$\sum_{n\le N} b_n^2 \ll N \log^{\frac{D}{2}} N, \quad \sum_{n\le N} |b_n| \ll N \log^{\frac{D}{4}} N, \quad \sum_{n\le N} \frac{b_n^2}{n} \ll (\log N)^{1+\frac{D}{2}},$$

$$\sum_{n\le N} \frac{|b_n|}{n} \ll (\log N)^{1+\frac{D}{4}}, \quad \sum_{n>N} \frac{b_n^4}{n^2}\tau(n) \ll \frac{1}{N^{1-\delta}}, \quad \textit{for any } \delta > 0.$$

Proof. Follows from Cauchy's inequality, partial summations and the fact that, for any $\epsilon > 0$, $\tau(n) = O(n^\epsilon)$. □

Remark. If $H(x)$ can be expressed in the form (2), then

$$|H(x)| \le \sum_{n\le y(x)} \frac{|b_n|}{n} + O\left(\frac{1}{k(x)}\right) \ll (\log x)^{1+\frac{D}{4}}. \tag{9}$$

Proof of Lemma 3.2 : Since the arithmetical mean is greater or equal to the geometrical mean, we have

$$\begin{aligned}\sum_{m,n\le X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl(kn+lm)} &\le 2 \sum_{m,n\le X} |b_m b_n| \sum_{k,l=1}^{\infty} \frac{1}{kl\sqrt{knlm}} \\ &\ll \Big(\sum_{m\le X} \frac{|b_m|}{\sqrt{m}} \Big)^2 \\ &\ll \Big(\sum_{m\le X} 1 \Big) \Big(\sum_{M\le X} \frac{b_M^2}{M} \Big) \\ &\ll X (\log X)^{1+\frac{D}{2}}.\end{aligned}$$

□

Proof of Lemma 3.3 : For the second sum, take $d = (m,n)$, $m = d\alpha$ and $n = d\beta$. Since $kn = lm$, then $\alpha | k$ and $\beta | l$. Taking $k = \alpha\gamma$, we also have $l = \beta\gamma$. As

$$\sum_{\substack{k,l=1 \\ kn=lm}}^{\infty} \frac{1}{kl} = \frac{1}{\alpha\beta} \sum_{\gamma=1}^{\infty} \frac{1}{\gamma^2} = \frac{\pi^2}{6} \frac{(m,n)^2}{mn}. \tag{10}$$

Then,

$$\sum_{N<m,n\le X}\frac{|b_mb_n|}{mn}\sum_{\substack{k,l=1\\kn=lm}}^{\infty}\frac{1}{kl}=\frac{\pi^2}{6}\sum_{N<m,n\le X}\frac{|b_mb_n|(m,n)^2}{m^2n^2}$$

$$\le\frac{\pi^2}{6}\sum_{d\le X}\Bigg(d\sum_{\substack{N<m\le X\\ d|m}}\frac{|b_m|}{m^2}\Bigg)^2.$$

The next step is to estimate the inner sum using Hölder inequality

$$\Bigg(\sum_{\substack{N<m\le X\\ d|m}}\frac{|b_m|}{m^2}\Bigg)^2\le\Bigg(\sum_{\substack{N<m\le X\\ d|m}}\frac{b_m^4}{m^2}\Bigg)^{\frac12}\Bigg(\sum_{\substack{N<M\le X\\ d|M}}\frac{1}{M^2}\Bigg)^{\frac32}.$$

Set $M=\beta d$, then

$$\Bigg(\sum_{\substack{N<M\le X\\ d|M}}\frac{1}{M^2}\Bigg)^{\frac32}=\frac{1}{d^3}\Bigg(\sum_{\frac{N}{d}<\beta\le\frac{X}{d}}\frac{1}{\beta^2}\Bigg)^{\frac32}\ll\frac{1}{d^3}\Bigg(\min\left\{1,\frac{d^3}{N^3}\right\}\Bigg)^{\frac12}. \tag{11}$$

To complete the proof of Lemma 3.3 we use Cauchy's inequality and Lemma 3.4. For any $\delta>0$,

$$\sum_{N<m,n\le X}\frac{|b_mb_n|}{mn}\sum_{\substack{k,l=1\\kn=lm}}^{\infty}\frac{1}{kl}\ll\sum_{d\le X}\Bigg(\frac{1}{d}\Big(\min\left\{1,\frac{d^3}{N^3}\right\}\Big)^{\frac12}\Big(\sum_{\substack{N<m\le X\\ d|m}}\frac{b_m^4}{m^2}\Big)^{\frac12}\Bigg),$$

therefore

$$\Bigg(\sum_{N<m,n\le X}\frac{|b_mb_n|}{mn}\sum_{\substack{k,l=1\\kn=lm}}^{\infty}\frac{1}{kl}\Bigg)^2\ll\sum_{d\le X}\frac{1}{d^2}\min\left\{1,\frac{d^3}{N^3}\right\}\sum_{D\le X}\Bigg(\sum_{\substack{N<m\le X\\ D|m}}\frac{b_m^4}{m^2}\Bigg)$$

$$\ll\Big(\frac{1}{N^3}\sum_{d\le N}d+\sum_{d>N}\frac{1}{d^2}\Big)\sum_{N<m\le X}\Big(\frac{b_m^4}{m^2}\sum_{D|m}1\Big)$$

$$\ll\frac{1}{N}\sum_{N<m\le X}\frac{b_m^4}{m^2}\tau(m)\ll\frac{1}{N^{2-\delta}}.$$

□

Proof of Lemma 3.1 : This lemma is a generalization of *Hilfssatz* 6 in [9] of A. Walfisz. Notice first that

$$\sum_{m,n\le X}\Big(|b_mb_n|\sum_{\substack{k,l=1\\kn\ne lm}}^{\infty}\frac{1}{kl\,|\,kn-lm|}\Big)\le 2\sum_{m\le n\le X}\Big(|b_mb_n|\sum_{\substack{k,l=1\\kn\ne lm}}^{\infty}\frac{1}{kl\,|\,kn-lm|}\Big).$$

Like in [9] we begin by separating the interior sum into four terms:

$$\sum_{\substack{k,l=1\\ kn\neq lm}}^{\infty} \frac{1}{kl\,|\,kn-lm|} = \sum_{\substack{k,l=1\\ lm\leq \frac{kn}{2}}}^{\infty} \left(\frac{1}{kl\,|\,kn-lm|}\right) + \sum_{\substack{k,l=1\\ \frac{kn}{2}<lm<kn}}^{\infty} \left(\frac{1}{kl\,|\,kn-lm|}\right)$$
$$+ \sum_{\substack{k,l=1\\ kn<lm<2\,kn}}^{\infty} \left(\frac{1}{kl\,|\,kn-lm|}\right) + \sum_{\substack{k,l=1\\ lm\geq 2\,kn}}^{\infty} \left(\frac{1}{kl\,|\,kn-lm|}\right).$$

For the first term, we use Lemma 3.4,

$$\begin{aligned}
\sum_{m\leq n\leq X} |b_m b_n| \sum_{\substack{k,l=1\\ lm\leq \frac{kn}{2}}}^{\infty} \frac{1}{kl\,|\,kn-lm|} &\leq 2 \sum_{m\leq n\leq X} |b_m b_n| \sum_{\substack{k,l=1\\ lm\leq \frac{kn}{2}}}^{\infty} \frac{1}{k^2 ln} \\
&= 2 \sum_{m\leq n\leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l\leq \frac{kn}{2m}} \frac{1}{l} \\
&\ll \sum_{m\leq n\leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \left(\frac{\log k}{k^2} + \frac{\log X}{k^2}\right) \\
&\ll \log X \sum_{n\leq X} \frac{|b_n|}{n} \sum_{m\leq n} |b_m| \ll X\,(\log X)^{1+\frac{D}{2}}.
\end{aligned}$$

From the forth inequality of Lemma 3.4, we get

$$\sum_{m\leq n\leq X} |b_m b_n| \sum_{\substack{k,l=1\\ kn\leq \frac{lm}{2}}}^{\infty} \frac{1}{kl\,|\,kn-lm|} \ll X\,(\log X)^{1+\frac{D}{2}}\,.$$

The estimation of the third term is more complicated and we have to use a different approach. In this case, $\frac{1}{l} < \frac{2m}{kn}$, so that

$$\begin{aligned}
&\sum_{m\leq n\leq X} |b_m b_n| \sum_{\substack{k,l=1\\ \frac{kn}{2}<lm<kn}}^{\infty} \frac{1}{kl\,|\,kn-lm|} \\
&\qquad < 2 \sum_{m\leq n\leq X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\frac{kn}{2m}<l<\frac{kn}{m}} \frac{m}{kn-lm}
\end{aligned}$$

$$< 2 \sum_{m\le n\le X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l\le \frac{kn}{m}-1} \left(\frac{1}{\frac{kn}{m}-l} \right)$$

$$+ 2 \sum_{m\le n\le X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\frac{kn}{m}-1<l<\frac{kn}{m}} \frac{m}{kn-lm}.$$

Now, taking $L = \left[\frac{kn}{m} - l\right]$,

$$\sum_{m\le n\le X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{l\le \frac{kn}{m}-1} \left(\frac{1}{\frac{kn}{m}-l} \right) \le \sum_{m\le n\le X} |b_m| \frac{|b_n|}{n} \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{L\le \frac{kn}{m}-1} \frac{1}{L}$$

$$\ll X (\log X)^{1+\frac{D}{2}},$$

as in the first term. If there exists an integer l with $\frac{kn}{m} - 1 < l < \frac{kn}{m}$, then $m \nmid kn$. In this case, $kn - lm = m\left\{\frac{kn}{m}\right\}$ and $m < n$. So, we have to estimate

$$\sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k=1 \\ m\nmid kn}}^{\infty} \frac{1}{k^2\left\{\frac{kn}{m}\right\}}. \tag{12}$$

Notice that the fractional part of $\frac{kn}{m}$ is at least $\frac{1}{m}$. So, when $k \ge m$,

$$\sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \sum_{k=m}^{\infty} \frac{m}{k^2} \ll \sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \ll X (\log X)^{\frac{D}{2}}.$$

We are left with the estimation of

$$\sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k<m \\ m\nmid kn}} \frac{1}{k^2\left\{\frac{kn}{m}\right\}}.$$

Since $m \nmid kn$, given k and n, we can take $a_{k,n}$, such that $1 \le a_{k,n} < m$ and $a_{k,n} \equiv kn \mod m$. Then,

$$\sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \sum_{\substack{k<m \\ m\nmid kn}} \frac{1}{k^2\left\{\frac{kn}{m}\right\}} \le \sum_{m<n\le X} |b_m| \frac{|b_n|}{n} \sum_{k<m} \frac{m}{k^2 a_{k,n}}$$

$$\le \sum_{a,k\le X} \frac{1}{ak^2} \sum_{\max(a,k)<m\le X} m|b_m| \sum_{\substack{m<n\le X \\ kn\equiv a \\ \bmod m}} \frac{|b_n|}{n}.$$

We need to estimate the inner sums. In order to do that, we partition the interval $[1, X]$ in intervals of the form $[M, 2M)$ and apply Cauchy's inequality. Take $1 \leq P \leq Q \leq X$, then,

$$\sum_{P \leq m < 2P} m|b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} \frac{|b_n|}{n} \ll \frac{P}{Q} \sum_{P \leq m < 2P} |b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} |b_n|.$$

Next, we apply Cauchy's inequality twice, first to the first sum on the right and afterwards to the second sum:

$$\begin{aligned}
&\left(\sum_{P \leq m < 2P} \left(|b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} |b_n| \right) \right)^2 \\
&\leq \sum_{P \leq M < 2P} b_M^2 \sum_{P \leq m < 2P} \left(\sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} |b_n| \right)^2 \\
&\ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(\sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} b_n^2 \sum_{\substack{Q \leq N < 2Q \\ kN \equiv a \mod m}} 1 \right) \\
&\ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(\left(1 + \frac{Q}{\frac{m}{(k,m)}} \right) \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} b_n^2 \right).
\end{aligned}$$

Since $m \leq 2P \leq 2Q$, we have $\frac{Q}{m} \geq \frac{1}{2}$. Using also $(k,m) \leq k$, we obtain

$$1 + \frac{Q}{\frac{m}{(k,m)}} \leq 1 + \frac{Qk}{m} \leq \frac{Q}{m}(k+2) \leq 3\frac{Qk}{m}.$$

Therefore,

$$\begin{aligned}
\left(\sum_{P \leq m < 2P} \left(|b_m| \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} |b_n| \right) \right)^2 &\ll P \log^{\frac{D}{2}} P \sum_{P \leq m < 2P} \left(3\frac{Qk}{m} \sum_{\substack{Q \leq n < 2Q \\ kn \equiv a \mod m}} b_n^2 \right) \\
&\ll P \frac{3kQ}{P} \log^{\frac{D}{2}} P \sum_{Q \leq n < 2Q} \left(b_n^2 \sum_{\substack{P \leq m < 2P \\ m | kn - a}} 1 \right) \\
&\ll kQ \left(\log P \right)^{\frac{D}{2}} \sum_{Q \leq n < 2Q} b_n^2 \ \tau(kn - a).
\end{aligned}$$

By a theorem of S. Ramanujan [7], $\sum_{n\le X}\tau^2(n)\sim X\log^3 X$ and by another application of Cauchy inequality and condition (3), we get

$$\begin{aligned}\Big(\sum_{Q\le n<2Q} b_n^2\ \tau(kn-a)\Big)^2 &\le \Big(\sum_{Q\le n<2Q} b_n^4\Big)\Big(\sum_{Q\le n<2Q}\tau^2(kn-a)\Big)\\ &\ll Q\log^D Q\sum_{kQ-a\le N<2kQ-a}\tau^2(N)\\ &\ll kQ^2\log^{D+3}X.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{P\le m<2P}\Big(m|b_m|\sum_{\substack{Q\le n<2Q\\ kn\equiv a \mod m}}\frac{|b_n|}{n}\Big) &\ll \frac{P}{Q}\Big(kQ(\log P)^{\frac{D}{2}}\sum_{Q\le n<2Q} b_n^2\,\tau(kn-a)\Big)^{\frac12}\\ &\ll \frac{P}{Q}\Big(kQ(\log X)^{\frac{D}{2}}(kQ^2\log^{D+3}X)^{\frac12}\Big)^{\frac12}\\ &\ll Pk^{\frac34}(\log X)^{1+\frac{D}{2}}.\end{aligned}$$

The number of pairs of intervals of the form $([P,2P),[Q,2Q))$ to be considered is at most $\ll\log^2 X$, hence

$$\begin{aligned}\sum_{a,k\le X}\frac{1}{ak^2}\sum_{a<m\le X} m|b_m|\sum_{\substack{m<n\le X\\ kn\equiv a \mod m}}\frac{|b_n|}{n} &\ll \sum_{a,k\le X}\frac{k^{\frac34}}{ak^2}\sum_{P,Q} P\,(\log X)^{1+\frac{D}{2}}\\ &\ll X\,(\log X)^{4+\frac{D}{2}}.\end{aligned}$$

The fourth term is treated as the third, hence

$$\sum_{m\le n\le X}|b_m b_n|\sum_{\substack{k,l=1\\ kn<lm<2\,kn}}^{\infty}\frac{1}{kl\,|\,kn-lm|}\ll X\,(\log X)^{4+\frac{D}{2}}.$$

This completes the proof of Lemma 3.1. □

4. Step II

In this section we prove part (b) of Lemma 2.1. From equation (8), we get

$$\int_a^b\psi(u)\,\mathrm{d}u=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\Big(\frac{\cos(2\pi kb)}{k^2}-\frac{\cos(2\pi ka)}{k^2}\Big).$$

Using the definition of H_N stated in (7), we obtain

$$\int_t^{t+h} H_N(u)\,\mathrm{d}u = -\sum_{m\le N}\frac{b_m}{m}\int_t^{t+h}\psi(\frac{u}{m})\,\mathrm{d}u$$
$$= -\frac{1}{2\pi^2}\sum_{m\le N} b_m \sum_{k=1}^{\infty}\frac{\cos\left(2\pi\frac{k(t+h)}{m}\right)-\cos\left(2\pi\frac{kt}{m}\right)}{k^2}.$$

As usual, let us write $e(t)$ for $e^{2\pi it}$, then

$$\int_t^{t+h} H_N(u)\mathrm{d}u = \frac{1}{4\pi^2}\sum_{m\le N} b_m\sum_{k=1}^{\infty}\frac{\left(e\left(\frac{kh}{m}\right)-1\right)e\left(\frac{kt}{m}\right)\left(e\left(-k\frac{(2t+h)}{m}\right)-1\right)}{k^2}.$$

Therefore,

$$16\pi^4\int_T^{2T}\left|\int_t^{t+h} H_N(u)\mathrm{d}u\right|^2\mathrm{d}t$$
$$=\int_T^{2T}\left|\sum_{m\le N} b_m\sum_{k=1}^{\infty}\frac{\left(e\left(\frac{kh}{m}\right)-1\right)e\left(\frac{kt}{m}\right)\left(e\left(-k\frac{(2t+h)}{m}\right)-1\right)}{k^2}\right|^2\mathrm{d}t$$
$$=\sum_{m,n\le N} b_m b_n\sum_{k,l=1}^{\infty}\frac{\left(e\left(\frac{kh}{m}\right)-1\right)\left(e\left(-\frac{lh}{n}\right)-1\right)}{(kl)^2}$$
$$\times\int_T^{2T} e\left(\frac{kt}{m}\right)e\left(-\frac{lt}{n}\right)\left(e\left(-k\ \frac{(2t+h)}{m}\right)-1\right)\left(e\left(l\ \frac{(2t+h)}{n}\right)-1\right)\mathrm{d}t.$$

After multiplying the terms inside the integral above, we obtain the following four terms that we will estimate below:

$$\int_T^{2T} e\left(\frac{kt}{m}-\frac{lt}{n}\right)\,\mathrm{d}t + e\left(\frac{lh}{n}-\frac{kh}{m}\right)\int_T^{2T} e\left(\frac{lt}{n}-\frac{kt}{m}\right)\,\mathrm{d}t$$
$$-\,e\left(-\frac{kh}{m}\right)\int_T^{2T} e\left(-\frac{lt}{n}-\frac{kt}{m}\right)\,\mathrm{d}t - e\left(\frac{lh}{n}\right)\int_T^{2T} e\left(\frac{lt}{n}+\frac{kt}{m}\right)\,\mathrm{d}t.$$

Notice that, $\left|\int_T^{2T} e^{2\pi irt}\,\mathrm{d}t\right| \le \frac{1}{\pi|r|}$, for any $r \ne 0$. We begin with the last term and use $|e(t)-1| \le 2$ and Lemma 3.2. Then

$$\begin{aligned}
&\left|\sum_{m,n\le N} b_m b_n \sum_{k,l=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right)-1\right)\left(e\left(-\frac{lh}{n}\right)-1\right)e\left(\frac{lh}{n}\right)}{(kl)^2}\int_T^{2T} e\left(\frac{lt}{n}+\frac{kt}{m}\right)\mathrm{d}t\right| \\
&\quad\le \frac{4}{\pi}\sum_{m,n\le N}|b_m b_n|\sum_{k,l=1}^{\infty}\frac{1}{(kl)^2\left(\frac{l}{n}+\frac{k}{m}\right)} \\
&\quad\ll \sum_{m,n\le N}|b_m b_n|\sum_{k,l=1}^{\infty}\left(\frac{m}{k}\right)\left(\frac{n}{l}\right)\frac{1}{kl(lm+kn)} \ll N^3(\log N)^{1+\frac{D}{2}}\ .
\end{aligned}$$

The third term is treated similarly to obtain

$$\begin{aligned}
&\left|\sum_{m,n\le N} b_m b_n \sum_{k,l=1}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right)-1\right)\left(e\left(-\frac{lh}{n}\right)-1\right)e\left(-\frac{kh}{m}\right)}{(kl)^2}\int_T^{2T} e\left(-\frac{lt}{n}-\frac{kt}{m}\right)\mathrm{d}t\right| \\
&\qquad\ll N^3(\log N)^{1+\frac{D}{2}}\ .
\end{aligned}$$

Now, if $kn = lm$, then

$$\int_T^{2T} e\left(\frac{kt}{m}-\frac{lt}{n}\right)\mathrm{d}t + e\left(\frac{lh}{n}-\frac{kh}{m}\right)\int_T^{2T} e\left(\frac{lt}{n}-\frac{kt}{m}\right)\mathrm{d}t = 2T.$$

If $kn \ne lm$ then,

$$\int_T^{2T} e\left(\frac{kt}{m}-\frac{lt}{n}\right)\mathrm{d}t + e\left(\frac{lh}{n}-\frac{kh}{m}\right)\int_T^{2T} e\left(\frac{lt}{n}-\frac{kt}{m}\right)\mathrm{d}t \ll \frac{1}{\left|\frac{k}{m}-\frac{l}{n}\right|}.$$

Let us study first the case when $kn \ne lm$,

$$\begin{aligned}
&\left|\sum_{m,n\le N} b_m b_n \sum_{\substack{k,l=1\\ kn\ne lm}}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right)-1\right)\left(e\left(-\frac{lh}{n}\right)-1\right)}{(kl)^2}\int_T^{2T} e\left(\frac{kt}{m}-\frac{lt}{n}\right)\mathrm{d}t\right| \\
&\ll \sum_{m,n\le N}|b_m b_n|\sum_{\substack{k,l=1\\ kn\ne lm}}^{\infty}\frac{1}{(kl)^2\left|\frac{k}{m}-\frac{l}{n}\right|} \\
&\ll \sum_{m,n\le N}|b_m b_n|\sum_{\substack{k,l=1\\ kn\ne lm}}^{\infty}\left(\frac{m}{k}\right)\left(\frac{n}{l}\right)\frac{1}{kl\,|kn-lm|} \ll N^3\log^E N,
\end{aligned}$$

by Lemma 3.1. Similarly,

$$\left|\sum_{m,n\le N} b_m b_n \sum_{\substack{k,l=1\\ kn\neq lm}}^{\infty} \frac{\left(e\left(\frac{kh}{m}\right)-1\right)\left(e\left(-\frac{lh}{n}\right)-1\right)e\left(\frac{lh}{n}-\frac{kh}{m}\right)}{(kl)^2}\int_T^{2T} e\left(\frac{lt}{n}-\frac{kt}{m}\right)\mathrm{d}t\right| \ll N^3\log^E N.$$

If $kn = ml$, we will use $|e(t)-1| \ll \min(1,|t|)$ instead. The expression obtained has some similarities with Lemma 3.3. We are going to use the same argument to prove:

$$\sum_{m,n\le N} |b_m b_n| \sum_{\substack{k,l=1\\ kn=lm}}^{\infty} \frac{1}{(kl)^2}\min\left(1,\frac{kh}{m}\right)\min\left(1,\frac{lh}{n}\right) \ll h^{\frac{3}{2}}. \tag{13}$$

As in Lemma 3.3, take $d=(m,n)$, $\alpha=\frac{m}{d}$, $\beta=\frac{n}{d}$ and $\gamma=\frac{k}{\alpha}$. So $l=\beta\gamma$ and then

$$\sum_{\substack{k,l=1\\ kn=lm}}^{\infty} \frac{1}{(kl)^2}\min\left(1,\frac{kh}{m}\right)\min\left(1,\frac{lh}{n}\right) = \frac{1}{\alpha^2\beta^2}\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\left(\min\left(1,\frac{h\gamma}{d}\right)\right)^2.$$

If $d\le h$, we obtain $\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\left(\min\left(1,\frac{h\gamma}{d}\right)\right)^2=\frac{\pi^4}{90}$, and if $h<d\le N$,

$$\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\left(\min\left(1,\frac{h\gamma}{d}\right)\right)^2 = \left(\frac{h}{d}\right)^2\sum_{\gamma\le\frac{d}{h}}\frac{1}{\gamma^2}+\sum_{\gamma>\frac{d}{h}}\frac{1}{\gamma^4} \ll \left(\frac{h}{d}\right)^2+\left(\frac{h}{d}\right)^3 \ll \left(\frac{h}{d}\right)^2.$$

Therefore,

$$\begin{aligned}
&\sum_{m,n\le N} |b_m b_n| \sum_{\substack{k,l=1\\ kn=lm}}^{\infty} \frac{1}{(kl)^2}\min\left(1,\frac{kh}{m}\right)\min\left(1,\frac{lh}{n}\right)\\
&= \sum_{m,n\le N}|b_m b_n|\frac{(m,n)^4}{m^2n^2}\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\left(\min\left(1,\frac{h\gamma}{(m,n)}\right)\right)^2\\
&\le \sum_{d\le N} d^4 \sum_{\substack{m,n\le N\\ d=(m,n)}}\frac{|b_m b_n|}{m^2n^2}\sum_{\gamma=1}^{\infty}\frac{1}{\gamma^4}\left[\min\left(1,\frac{h\gamma}{d}\right)\right]^2\\
&\ll \sum_{d\le h}\left(d^2\sum_{\substack{m\le N\\ d\mid m}}\frac{|b_m|}{m^2}\right)^2 + h^2\sum_{h<d\le N}\left(d\sum_{\substack{m\le N\\ d\mid m}}\frac{|b_m|}{m^2}\right)^2.
\end{aligned}$$

Since $d|m$ we have $m > h$ and so, $h^2 \sum_{h<d\le N} \Big(d \sum_{\substack{m\le N\\ d|m}} \frac{|b_m|}{m^2} \Big)^2 \ll h^{1+\delta}$. To estimate the first term we begin with Hölder inequality:

$$\sum_{d\le h} d^4 \Big(\sum_{\substack{m\le N\\ d|m}} \frac{|b_m|}{m^2} \Big)^2 \le \sum_{d\le h} d^4 \Big(\sum_{\substack{m\le N\\ d|m}} \frac{|b_m|^4}{m^2} \Big)^{\frac{1}{2}} \Big(\sum_{\substack{M\le N\\ d|M}} \frac{1}{M^2} \Big)^{\frac{3}{2}}.$$

The third sum is $O\left(\frac{1}{d^3}\right)$ (similar to (11)). Then

$$\begin{aligned}\sum_{d\le h} d^4 \Big(\sum_{\substack{m\le N\\ d|m}} \frac{|b_m|^4}{m^2} \Big)^{\frac{1}{2}} \Big(\sum_{\substack{M\le N\\ d|M}} \frac{1}{M^2} \Big)^{\frac{3}{2}} &\ll \sum_{d\le h} d \Big(\sum_{\substack{d\le m\le N\\ d|m}} \frac{|b_m|^4}{m^2} \Big)^{\frac{1}{2}} \\ &\ll \Big(\sum_{d\le h} d^2 \Big)^{\frac{1}{2}} \Big[\sum_{D\le h} \Big(\sum_{\substack{D\le m\le N\\ D|m}} \frac{|b_m|^4}{m^2} \Big) \Big]^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \Big(\sum_{m\le N} \frac{|b_m|^4}{m^2} \sum_{\substack{D\le h\\ D|m}} 1 \Big)^{\frac{1}{2}} \\ &\ll h^{\frac{3}{2}} \Big(\sum_{m\le N} \frac{|b_m|^4}{m^2} \tau(m) \Big)^{\frac{1}{2}}.\end{aligned}$$

The last part of Lemma 3.4 implies $\sum_{m\le N} \frac{|b_m|^4}{m^2} \tau(m) = O(1)$, where the underlying constant doesn't depend on N. Therefore, we obtain inequality (13) and part (b) of Lemma 2.1, follows. □

5. A general Theorem

In this section, we prove Theorem 1.1, from which the main Theorems 1.2 and 1.3, will be deduced.

Proof of Theorem 1.1 : From the Main Lemma, we have, for all large T and $h \le \min\left(\log T, k^2(T)\right)$,

$$\int_T^{2T} \left(\int_t^{t+h} H(u)\,\mathrm{d}u \right)^2 \mathrm{d}t \ll T h^{\frac{3}{2}}.$$

Assume $\#\{n \le T : \alpha H(n) \prec 0\} \gg T$. Let $c > 0$ be a constant and T be sufficiently large, such that $\#\{n \le 2T : \alpha H(n) \prec 0\} > cT$. Divide the interval $[1, 2T]$ into subintervals of length h, where h is a sufficiently large integer satisfying $h \le \log T$. Then more than cT/h of those subintervals must have at least one integer n with $\alpha H(n) \prec 0$. Let $\mathcal{C}$ be the set of the subintervals which satisfy this property. Write $\mathcal{C} = \{J_r \,|\, 1 \le r \le R\}$, where the subintervals are indexed by their positions in the interval $[1, 2T]$ and where $R > cT/h$. Define $K_s = J_{3s-2}$, for $1 \le s \le R/3$, and let $\mathcal{D}$ be the set of these subintervals. We have $\#(\mathcal{D}) > cT/3h$. Notice that any two members of $\mathcal{D}$ are separated by a distance of at least $2h$.

Let M be the number of subintervals K in $\mathcal{D}$ for which there exists an integer n in K such that $\alpha H(n) \prec 0$ and $\alpha H(m) \leq 0$ for every integer $m \in (n, n+2h)$, and let $\mathcal{S}$ be the set of the corresponding values of n.

Lemma 5.1. *For some absolute constant c_1, we have $M \leq c_1 \frac{T}{h^{\frac{3}{2}}}$.*

Proof. Since $H(x) = H(\lfloor x \rfloor) - \alpha\{x\} + \theta(x)$, then

$$\alpha H(x) - \alpha H(\lfloor x \rfloor) = -\alpha^2\{x\} + \alpha\theta(x).$$

So, if x is sufficiently large and not an integer then

$$-\frac{5}{4}\alpha^2\{x\} < \alpha H(x) - \alpha H(\lfloor x \rfloor) < -\frac{3}{4}\alpha^2\{x\}. \tag{14}$$

Let n_1 be the smallest integer such that any non integer $x > n_1$ satisfies condition (14). If $\#\{n \in \mathcal{S} : n \geq n_1\} = 0$ then $M \leq n_1$, so, the lemma is clearly true for sufficiently large T. Otherwise,

$$\#\{n \in \mathcal{S} : n \geq n_1\} \geq M - n_1 \gg M.$$

Take $n \in \mathcal{S}$ with $n \geq n_1$ and $t \in [n, n+h]$. For any integer $m \in [t, t+h]$, $\alpha H(m) \leq 0$. Now, for any $1 \leq j < h$,

$$\int_{[t]+j}^{[t]+j+1} H(u)\,\mathrm{d}u = \int_{[t]+j}^{[t]+j+1} (H(u) - H([t]+j))\,\mathrm{d}u + \int_{[t]+j}^{[t]+j+1} H([t]+j)\,\mathrm{d}u.$$

Therefore, by (14),

$$\begin{aligned}\int_{[t]+j}^{[t]+j+1} \alpha H(u)\,\mathrm{d}u &< \int_0^1 \left(-\frac{3}{4}\alpha^2 x\right)\,\mathrm{d}x + \alpha H([t]+j)\\ &< -\frac{3}{8}\alpha^2,\end{aligned}$$

because $\alpha H([t]+j) \leq 0$. Since $[t] \geq n$, we also have

$$\int_t^{[t]+1} \alpha H(u)\,\mathrm{d}u < \int_{\{t\}}^1 \left(-\frac{3}{4}\alpha^2 x\right)\,\mathrm{d}x + \alpha H([t])\,(1 - \{t\}) < 0$$

and

$$\int_{[t]+h}^{t+h} \alpha H(u)\,\mathrm{d}u < \int_0^{\{t\}} \left(-\frac{3}{4}\alpha^2 x\right)\,\mathrm{d}x + \alpha H([t]+h)\{t\} \leq 0.$$

Hence,

$$\left|\int_t^{t+h} H(u)\,\mathrm{d}u\right| \geq \frac{3}{8}|\alpha|(h-1).$$

Take an integer $r = r(T)$ such that $2^r > (\log T)^{3+\frac{D}{2}}$. Using (6) and (9), we obtain

$$\begin{aligned}\int_0^{2T}\left(\int_t^{t+h} H(u)\,\mathrm{d}u\right)^2 \mathrm{d}t &= \int_0^{\frac{T}{2^r}}\left(\int_t^{t+h} H(u)\,\mathrm{d}u\right)^2 \mathrm{d}t \\ &\quad + \sum_{j=0}^{r}\int_{\frac{T}{2^j}}^{\frac{T}{2^{j-1}}}\left(\int_t^{t+h} H(u)\,\mathrm{d}u\right)^2 \mathrm{d}t \\ &\ll \frac{T}{2^r}h^2(\log T)^{2+\frac{D}{2}} + h^{\frac{3}{2}}\sum_{j=0}^{r}\frac{T}{2^j} \\ &\ll Th^{\frac{3}{2}},\end{aligned}$$

due to $h \le \log T$. On the other hand,

$$\begin{aligned}\int_0^{2T}\left(\int_t^{t+h} H(u)\,\mathrm{d}u\right)^2 \mathrm{d}t &\ge \sum_{n\in\mathcal{S}}\int_n^{n+h}\left(\int_t^{t+h} H(u)\,\mathrm{d}u\right)^2 \mathrm{d}t \\ &\ge \sum_{\substack{n\in\mathcal{S}\\ n\ge n_1}}\int_n^{n+h}\left(\frac{3}{8}|\alpha|(h-1)\right)^2 \mathrm{d}t \\ &\gg Mh^3.\end{aligned}$$

Hence $M \le c_1\frac{T}{h^{\frac{3}{2}}}$ for some absolute constant c_1. □

Take $c_0 = c/6h$. If h is a suitably large integer such that $c_1 T < c_0 T h^{\frac{3}{2}}$, then there are at least $c_0 T$ intervals K in $\mathcal{D}$ such that $\alpha H(n) \prec 0$ for some integer $n \in K$ and $\alpha H(m) > 0$ for some integer m lying in $(n, n+2h)$. Now, suppose $\#\{n \le T : \alpha H(n) \le 0\} \gg T$. Take T sufficiently large and take the order relation '$\prec$' to be '$\le$'. Therefore, we have $c_0 T$ integers m in the interval $[1, 2T]$, for which $\alpha H(m)$ is positive. In this case,

$$\#\{n \le T : \alpha H(n) > 0\} \gg T.$$

If we don't have $\#\{n \le T : \alpha H(n) \le 0\} \gg T$, then

$$\#\{n \le T : \alpha H(n) > 0\} = T(1 + o(1)).$$

Hence, part 1 of Theorem 1.1 is proved. Next, we prove part 2. Take '$\prec$' to be '$<$'. Then, there exists a positive constant c_0 and $c_0 T$ disjoint subintervals of $[1, T]$, with each of them having at least two integers, m and n, such that $H(m) > 0$ and $H(n) < 0$. Therefore, in each of those intervals we have at least one l with either $H(l) = 0$ or $H(l)H(l+1) < 0$. Whence, $z_H(T) > \frac{c_0}{2}T$ or $N_H(T) > \frac{c_0}{2}T$. □

6. A class of arithmetical functions

In this section, we consider arithmetical functions $f(n)$, such that the sequence b_n satisfies conditions (3) and (4). We begin with some elementary results about this class of arithmetical functions. Using condition (4), we immediately obtain the following lemma:

Lemma 6.1. *Let b_n be a sequence of real numbers satisfying (4), for some constants B and $A > 1$, then there exist constants γ_b and α such that*

$$\sum_{n\le x} \frac{b_n}{n} = B\log x + \gamma_b + O\left(\frac{1}{\log^{A-1} x}\right), \tag{15}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^2} = \alpha, \tag{16}$$

$$\sum_{n>x} \frac{b_n}{n^2} = \frac{B}{x} + O\left(\frac{1}{x\log^{A-1} x}\right). \tag{17}$$

Next, we calculate the sum of $f(n)$ and describe the error term $H(x)$.

Lemma 6.2. *Let b_n be a sequence of real numbers as in Lemma 6.1, then*

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{d|n} \frac{b_d}{d} = \alpha x - \frac{B\log 2\pi x}{2} - \frac{\gamma_b}{2} + H(x), \tag{18}$$

where,

$$H(x) = -\sum_{n\le \frac{x}{\log^C x}} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right), \tag{19}$$

for any $0 < C < A-1$.

Proof. We have

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{d|n}\frac{b_d}{d} = \sum_{d\le x}\frac{b_d}{d}\sum_{\substack{n\le x\\ d|n}} 1 = \sum_{m\le x}\sum_{d\le \frac{x}{m}}\frac{b_d}{d}.$$

We separate the double sum above in two parts. Let $0 < C < A-1$ and $y = \log^C x$. Then

$$\sum_{m\le x}\sum_{d\le \frac{x}{m}}\frac{b_d}{d} = \sum_{n\le y}\sum_{d\le \frac{x}{n}}\frac{b_d}{d} + \sum_{d\le \frac{x}{y}}\frac{b_d}{d}\sum_{y<n\le \frac{x}{d}} 1. \tag{20}$$

In order to evaluate the first term on the right, we start with an application of formula (15):

$$\sum_{n\le y}\sum_{d\le\frac{x}{n}}\frac{b_d}{d}=\sum_{n\le\lfloor y\rfloor}\left(B\log x-B\log n+\gamma_b+O\left(\frac{1}{\log^{A-1}(\frac{x}{n})}\right)\right)$$
$$=B\lfloor y\rfloor\log x-B\sum_{n\le\lfloor y\rfloor}\log n+\gamma_b\lfloor y\rfloor+O\left(\frac{y}{\log^{A-1}(\frac{x}{y})}\right).$$

Recall that by Stirling formula, $\sum_{n\le\lfloor y\rfloor}\log n=\lfloor y\rfloor\log\lfloor y\rfloor-\lfloor y\rfloor+\frac{\log\lfloor y\rfloor}{2}+\frac{\log 2\pi}{2}+O\left(\frac{1}{y}\right)$. Hence,

$$\sum_{n\le y}\sum_{d\le\frac{x}{n}}\frac{b_d}{d}=B\lfloor y\rfloor\left(\log x-\log\lfloor y\rfloor+1\right)+\gamma_b\lfloor y\rfloor-\frac{B(\log 2\pi y)}{2}$$
$$+O\left(\frac{y}{\log^{A-1}x}\right)+O\left(\frac{1}{y}\right).$$

For the second term, we get

$$\sum_{d\le\frac{x}{y}}\frac{b_d}{d}\sum_{y<n\le\frac{x}{d}}1=\sum_{d\le\frac{x}{y}}\frac{b_d}{d}\left(\left\lfloor\frac{x}{d}\right\rfloor-\lfloor y\rfloor\right)$$
$$=x\sum_{d=1}^{\infty}\frac{b_d}{d^2}-x\sum_{d>\frac{x}{y}}\frac{b_d}{d^2}-\sum_{d\le\frac{x}{y}}\frac{b_d}{d}\,\psi\left(\frac{x}{d}\right)-\left(\frac{1}{2}+\lfloor y\rfloor\right)\sum_{d\le\frac{x}{y}}\frac{b_d}{d}$$
$$=\alpha x-By-\sum_{d\le\frac{x}{y}}\frac{b_d}{d}\,\psi\left(\frac{x}{d}\right)-\frac{\gamma_b+B\log x}{2}$$
$$-B\lfloor y\rfloor(\log x-\log y)+\frac{B\log y}{2}-\gamma_b\lfloor y\rfloor+O\left(\frac{y}{\log^{A-1}x}\right).$$

Notice also that $B\lfloor y\rfloor(\log y-\log\lfloor y\rfloor)=B\lfloor y\rfloor\left(-\log\left(1-\frac{\{y\}}{y}\right)\right)=B\{y\}+O\left(\frac{1}{y}\right)$. Joining everything together, we obtain

$$H(x)=\sum_{n\le x}f(n)-\left(\alpha x-\frac{B\log 2\pi x}{2}-\frac{\gamma_b}{2}\right)$$
$$=-\sum_{d\le\frac{x}{\log^C x}}\frac{b_d}{d}\psi\left(\frac{x}{d}\right)+O\left(\frac{1}{\log^{A-C-1}x}\right)+O\left(\frac{1}{\log^C x}\right).$$

□

Proof of Theorem 1.2 : We just have to show that $H(x)$ satisfies the conditions of Theorem 1.1. From Lemma 6.2, for any x

$$\begin{aligned} H(x) - H(\lfloor x \rfloor) &= -\alpha\{x\} - \frac{B}{2}\left(\log 2\pi\lfloor x\rfloor - \log 2\pi x\right) \\ &= -\alpha\{x\} + \frac{B}{2}\frac{\{x\}}{x} + O\left(\frac{1}{x^2}\right). \end{aligned}$$

In Lemma 6.2, we also obtained

$$H(x) = -\sum_{n \le \frac{x}{\log^C x}} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + O\left(\frac{1}{\log^C x}\right) + O\left(\frac{1}{\log^{A-C-1} x}\right),$$

for any $0 < C < A - 1$. Take $C = 5 + \frac{D}{2}$, $y(x) = \frac{x}{\log^C x}$ and $k(x) = \min\left(\log^C x, \log^{A-C-1} x\right)$. Since $A > 6 + D/2$, then $C < A - 1$ and $A - C - 1 > 0$. The first part of Theorem 1.2 now follows from Theorem 1.1.

Suppose that $f(n)$ takes only rational values. In order to prove the second part of Theorem 1.2, we use the following result of A. Baker [1].

Proposition. *Let $\alpha_1, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$ denote nonzero algebraic numbers. Then $\beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n \neq 0$.*

Using the result above, we obtain the next lemma.

Lemma 6.3. *Let $f(n)$ be a rational valued arithmetical function and suppose the sequence b_n satisfies condition (4) for some real B and $A > 1$. Let r be a real number and suppose $H(x)$ is given by (18). Then*

(1) *If $B = 0$ and α is irrational then $\#\{n \text{ integer} : H(n) = r\} \le 1$;*
(2) *If B is a nonzero algebraic number then $\#\{n \text{ integer} : H(n) = r\} \le 2$;*
(3) *If B is transcendental then there exists a constant C that depends on r and on the function $f(n)$, such that*

$$\#\{n \le T, n \text{ integer} : H(n) = r\} < (\log T)^C.$$

Proof. Suppose that $B = 0$ and α is irrational. Suppose also that there are two integers, say $M \neq N$, such that $H(M) = H(N)$. Then

$$\sum_{n \le M} f(n) - \alpha M + \frac{\gamma_b}{2} = \sum_{n \le N} f(n) - \alpha N + \frac{\gamma_b}{2}.$$

But this implies that α is rational, a contradiction.

Next, suppose $B \neq 0$ is algebraic number and that there are $M > N > Q$ integers, satisfying $H(M) = H(N) = H(Q)$. We have

$$\sum_{n \le M} f(n) - \alpha M + \frac{B\log 2\pi M}{2} + \frac{\gamma_b}{2} = \sum_{n \le N} f(n) - \alpha N + \frac{B\log 2\pi N}{2} + \frac{\gamma_b}{2}$$

which implies

$$\alpha = \frac{B}{M-N}\log\left(\frac{M}{N}\right) + \frac{1}{M-N}\sum_{N<n\le M} f(n).$$

Consequently

$$B\log\left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}}\right) = \frac{1}{M-Q}\sum_{Q<n\le M} f(n) - \frac{1}{M-N}\sum_{N<n\le M} f(n).$$

We are going to prove that

$$\left(\frac{M}{N}\right)^{\frac{1}{M-N}} \ne \left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}. \tag{21}$$

Since B is a nonzero algebraic number and the values of $f(n)$ are rational, for any integer n, the proposition implies

$$B\log\left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}}\right) \ne \frac{1}{M-Q}\sum_{Q<n\le M} f(n) - \frac{1}{M-N}\sum_{N<n\le M} f(n),$$

and so we get a contradiction, which implies $\#\{n \text{ integer} : H(n) = r\} \le 2$, for any real number r. In fact, instead of proving (21), we are going to prove that

$$M^{N-Q}Q^{M-N} < N^{M-Q}, \tag{22}$$

for any positive integers $M > N > Q$. Clearly, this implies (21). The inequality (22) is just a particular case of the geometric mean-analytic mean inequality

$$\left(\prod_{i=1}^{n} u_i\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{i=1}^{n} u_i, \tag{23}$$

where equality happens only if $u_1 = u_2 = \cdots = u_n$. In fact, if we take $n = M-Q$, $u_i = M$ for $1 \le i \le N-Q$ and $u_i = Q$ for $N-Q < i \le M-Q$, we derive

$$\left(M^{N-Q}Q^{M-N}\right)^{\frac{1}{M-Q}} < \frac{1}{M-Q}\left((N-Q)M + (M-N)Q\right) = N.$$

Hence, we obtain (22) and part 2 of the Lemma.

Finally we prove part 3. Suppose r is a real number such that

$$\#\{n \le T : H(n) = r\} \ge 4.$$

Let $Q < N < M$ be the three smallest positive integers in the above set, then

$$0 \neq B \log \left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = \frac{1}{M-Q} \sum_{Q<n\leq M} f(n) - \frac{1}{M-N} \sum_{N<n\leq M} f(n).$$

Suppose L is such that $H(L) = r$. Then $L > N > Q$, and as in part 2:

$$0 \neq B \log \left(\frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right) = \frac{1}{L-Q} \sum_{Q<n\leq L} f(n) - \frac{1}{L-N} \sum_{N<n\leq L} f(n).$$

After we cross multiply the two expressions above, we obtain

$$\log \left(\frac{\left(\frac{M}{N}\right)^{\frac{1}{M-N}}}{\left(\frac{M}{Q}\right)^{\frac{1}{M-Q}}} \right) = r_1 \log \left(\frac{\left(\frac{L}{N}\right)^{\frac{1}{L-N}}}{\left(\frac{L}{Q}\right)^{\frac{1}{L-Q}}} \right),$$

for some rational r_1. Therefore, there are four rational numbers r_2, r_3, r_4 and r_5, such that

$$L^{r_2} = M^{r_3} N^{r_4} Q^{r_5}.$$

Now, any prime dividing L must divide MNQ. Notice that, if p is a prime, k is an integer and $p^k \leq x$ then $k \leq \frac{\log x}{\log p}$. Therefore, the number of integers smaller than x, which have all prime divisors smaller than M is smaller than $(\log x)^{\pi(M)}$. This finishes the proof. □

Except when $\alpha = 0$, or $B = 0$ and α is rational, we cannot have $z_H(T) \gg T$. Hence, we obtain the second part of Theorem 1.2. □

Example. We finish this section by proving that Theorem 1.2 is valid for the arithmetical function $\frac{n}{\phi(n)}$.

Notice that

$$\frac{n}{\phi(n)} = \prod_{p|n} \left(1 + \frac{1}{p-1}\right) = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

Let $b_n = \frac{\mu^2(n)n}{\phi(n)}$, then $f(n) = \sum_{d|n} \frac{b_d}{d}$. In [8], R. Sitaramachandrarao proved that

$$\sum_{n\leq x} \frac{\mu^2(n)n}{\phi(n)} = x + O\left(x^{\frac{1}{2}}\right),$$

so condition (4) is satisfied for any A and with $B=1$. By Merten's Theorem $\prod_{p|n}\left(1-\frac{1}{p}\right)^{-1}\leq\prod_{p\leq n}\left(1-\frac{1}{p}\right)^{-1}\sim e^{\gamma}\log n$, hence

$$\sum_{n\leq x} b_n^4=\sum_{n\leq x}\mu^2(n)\frac{n^4}{\phi^4(n)}=\sum_{n\leq x}O\left(\log^4 n\right)=O\left(x\log^4 x\right),$$

and condition (3) is satisfied for $D\geq 4$. In this case,

$$\alpha=\frac{\zeta(2)\zeta(3)}{\zeta(6)},\quad \gamma_b=\gamma+\sum_p\frac{\log p}{p(p-1)}$$

and

$$H(x)=\sum_{n\leq x}\frac{n}{\phi(n)}-\frac{\zeta(2)\zeta(3)}{\zeta(6)}x+\frac{\log x}{2}+\frac{\log 2\pi+\gamma+\sum_p\frac{\log p}{p(p-1)}}{2}.$$

Since $B=1$ we can apply Theorem 6.3, and so $z(T)\leq 2$. Therefore, if $\#\{n\leq T:\alpha H(n)<0\}\gg T$, then $N_H(T)\gg T$.

7. Second class of arithmetical functions

Given a sequence of real numbers b_n, and a complex number s, we define the Dirichlet series $B(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$. In this section, we consider arithmetical functions $f(n)$, such that the sequence b_n satisfies conditions (3) and (5) for some $D>0$, β real and a function $g(s)$ with a Dirichlet series expansion absolutely convergent for $\sigma>1-\lambda$, for some $\lambda>0$.

U. Balakrishnan and Y.-F. S. Pétermann [2] proved that:

Proposition. *Let $f(n)$ be a complex valued arithmetical function satisfying*

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\zeta(s)\zeta^{\beta}(s+1)g(s+1),$$

for a complex number β, and $g(s)$ having a Dirichlet series expansion

$$g(s)=\sum_{n=1}^{\infty}\frac{c_n}{n^s},$$

which is absolutely convergent in the half plane $\sigma>1-\lambda$ for some $\lambda>0$. Let β_0 be the real part of β. If

$$\zeta^{\beta}(s)g(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}$$

then there is a real number b, $0 < b < 1/2$, and constants B_j, such that, taking $y(x) = x\exp\left(-(\log x)^b\right)$ and $\alpha = \zeta^{\beta}(2)g(2)$,

$$\sum_{n\le x} f(n) = \begin{cases} \alpha x - \sum_{n\le y(x)} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + o(1) & \text{if } \beta_0 < 0, \\ \alpha x + \sum_{j=0}^{[\beta_0]} B_j(\log x)^{\beta-j} - \sum_{n\le y(x)} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + o(1) & \text{if } \beta_0 \ge 0, \end{cases}$$

The real version of the previous proposition allows us to prove Theorem 1.3:

Proof. Notice that, for any $c > 0$, $\log^c\lfloor x\rfloor = \log^c x - c\frac{\{x\}}{x}\log^{c-1} x + O\left(\frac{1}{x}\right)$. So, $H(x) = H(\lfloor x\rfloor) - \alpha\{x\} + o(1)$. From the previous proposition, there is an increasing function $k(x)$, with $\lim_{x\to\infty} k(x) = \infty$, such that

$$H(x) = -\sum_{n\le y(x)} \frac{b_n}{n}\psi\left(\frac{x}{n}\right) + O\left(\frac{1}{k(x)}\right),$$

where $y(x) = x\exp\left(-(\log x)^b\right)$, for some $0 < b < 1/2$. Hence, the result follows from Theorem 1.1. $\square$

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