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CAYLEY GRAPHS UNDER GRAPH OPERATIONS II

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Abstract. The aim of this paper is to investigate the behavior of Cayley graphs under some graph operations. It is proved that the $NEPS$, corona, hierarchical, strong, skew and converse skew products of Cayley graphs are again Cayley graphs under some conditions.

1. Introduction and preliminaries

All groups considered here are finite. For notations and definitions not defined here we refer the reader to [4, 5]. Let X and Y be two graphs. Their corona of X and Y, $X \circ Y$, is defined as the graph obtained by taking one copy of X and joining the $i-$th vertex of X to every vertex in $i-$th copy of Y. Following Petrović [7], we assume that $\Gamma_i = (X_i, U_i)$, $1 \leq i \leq n$ are finite graphs, where X_i and U_i denote the corresponding sets of vertices and of edges. Further, let β be a set of $n-$tuples $(\beta_1, \cdots, \beta_n)$ of symbols 0 and 1, which does not contain the $n-$tuple $(0, ..., 0)$. The NEPS with basis β of the graphs $G_1, \cdots, G_n$ is the graph $Z = (X, U)$, where $X = X_1 \times \cdots \times X_n$ and in which two vertices $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are adjacent if and only if there is an $n-$tuple $(\beta_1, \cdots, \beta_n)$ in β such that $x_i = y_i$ holds exactly when $\beta_i = 0$, and x_i is adjacent to y_i in Γ_i, exactly when $\beta_i = 1$. It is easy to see that Cartesian product, tensor product and strong product of graphs are special types of NEPS.

The Strong product $X \boxtimes Y$ of graphs X and Y has the vertex set $V(X \boxtimes Y) = V(X) \times V(Y)$ and $(a, x)(b, y)$ is an edge of $X \boxtimes Y$ if $a = b$ and $xy \in E(Y)$, or $ab \in E(X)$ and $x = y$, or $ab \in E(X)$ and $xy \in E(Y)$.

Let $G_i = (V_i, E_i)$ be N graphs with each vertex set V_i, $1 \leq i \leq N$, having a distinguished or root vertex, labeled 0. The hierarchical product [3] $H =$

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$G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is the graph with vertices the $N-$tuples $x_N \cdots x_3x_2x_1$, $x_i \in V_i$, and edges defined by the following adjacencies:

$$x_N \cdots x_3x_2x_1 \sim \begin{cases} x_N \cdots x_3x_2x_1 & \text{if } y_1 \sim x_1 \text{ in } G_1, \\ x_N \cdots x_3y_2x_1 & \text{if } y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\ x_N \cdots y_3x_2x_1 & \text{if } y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\ \vdots & \vdots \\ y_N \cdots x_3x_2x_1 & \text{if } y_N \sim x_N \text{ in } G_N \text{ and } x_1 = x_2 = \cdots = x_N = 0. \end{cases}$$

We encourage the interested readers to consult papers [2] for a generalization and more information on this topic.

We now define the skew product and converse skew product of two graphs G_1 and G_2 denoted by $G_1\Delta G_2$ and $G_1\nabla G_2$, respectively. These graph products are having $V(G_1) \times V(G_2)$ as their vertex set. The edge sets of these graphs are:

$$\begin{aligned} E(G_1\Delta G_2) &= \{(u_1,u_2)(v_1,v_2) \mid [u_1 = v_1 \textit{ and } u_2v_2 \in E(G_2)] \textit{ or } [u_1v_1 \in E(G_1) \textit{ and } u_2v_2 \in E(G_2)]\}, \\ E(G_1\nabla G_2) &= \{(u_1,u_2)(v_1,v_2) \mid [u_2 = v_2 \textit{ and } u_1v_1 \in E(G_1)] \textit{ or } [u_1v_1 \in E(G_1) \textit{ and } u_2v_2 \in E(G_2)]\}, \end{aligned}$$

see [8] for details.

Suppose G_1 and G_2 are two arbitrary graphs. The cluster $G_1\{G_2\}$, is obtained by taking one copy of G_1 and $|G_1|$ copies of a rooted graph G_2, and by identifying the root of the $i-$th copy of G_2 with the $i-$th vertex of G_1, $1 \le i \le |G_1|$. The cluster of graphs was introduced by Yeh and Gutman [9] in the context of distance in graphs.

The Cartesian product $G_1\Box G_2$ is graph with $V(G_1)\times V(G_2)$ as vertex set such that (a, x) and (b, y) are adjacent if and only if $a = b$ and $xy \in E_2$, or $ab \in E_1$ and $x = y$.

Baik et al. [1, Lemma 2.6], proved that if $G = G_1 \times G_2$ are the direct product of two finite groups G_1, G_2 and S_1 and S_2 are subsets of G_1 and G_2, respectively, then by choosing S to be the disjoint union of S_1 and S_2, we have $Cay(G, S) \cong Cay(G_1, S_1)\Box Cay(G_2, S_2)$. The aim of this paper is to extend this result to corona, hierarchical product, skew product, converse skew product and NEPS of graphs. We refer to [1, 6] for more study on the main problem of this paper.

2. Main results

The investigation of graph operations under graph invariants is a well-known problem in metric graph theory. Here, we are interested to algebraic invariants. We begin by considering corona of two graphs.

Proposition 2.1. *Suppose $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$ and $\Gamma = Cay(G, S)$, where G is a group of order $|G_1|.|G_2| + |G_1|$ and $S \subseteq G$. Then $\Gamma = \Gamma_1 \circ \Gamma_2$ if and only if Γ_1 is empty, Γ_2 is a complete graph and Γ is a disconnected graph with G_1 components that each of them are regular of degree $|G_2|$.*

Proof. We first assume that $\Gamma = \Gamma_1 \circ \Gamma_2$. Choose vertices $u, v \in V(\Gamma)$ such that $u \in V(\Gamma_1)$ and v is vertex of the $i-$th copy of Γ_2. Then $deg_{\Gamma_1}(u) + |V(\Gamma_2)| = deg_{\Gamma_2}(v) + 1$. Since $|V(\Gamma_2)| \ge deg_{\Gamma_2}(v) + 1$, $deg_{\Gamma_1}(u) = 0$. This implies that Γ_1 is empty, Γ_2 is complete and $\Gamma_1\circ\Gamma_2$ is a disconnected graph with G_1 components that each of them are regular of degree $|G_2|$, as desired. The converse is obvious. □

Proposition 2.2. *Suppose G is a group, G_1 and G_2 are subgroups of G, $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$, $\Gamma = Cay(G, S_1)$ and $G = G_2 \cdot G_1 = \{xy \mid x \in G_2 \ \& \ y \in G_1\}$. Then $\Gamma = \Gamma_2 \sqcap \Gamma_1$ if and only if Γ_2 is empty.*

Proof. We first assume that $\Gamma = \Gamma_2 \sqcap \Gamma_1$. Choose vertices $x_2 0, x_2 x_1 \in \Gamma$. Since $deg_{\Gamma_2 \sqcap \Gamma_1}(x_2 x_1) = deg_{\Gamma_2 \sqcap \Gamma_1}(x_2 0)$, $deg_{\Gamma_1}(x_1) = deg_{\Gamma_2}(x_2) + deg_{\Gamma_1}(0)$. This shows that $deg_{\Gamma_2}(x_2) = 0$, which implies that Γ_2 is empty. The converse is a direct consequence of the definition. □

Corollary 2.3. *Let $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$, …, $\Gamma_n = Cay(G_n, S_n)$ and $\Gamma = Cay(G, S_1)$, where $G = G_n G_{n-1} \cdots G_1$. Then $\Gamma = \Gamma_n \sqcap \Gamma_{n-1} \sqcap \cdots \sqcap \Gamma_1$ if and only if Γ_2, …, Γ_n are empty graphs.*

Proposition 2.4. *Suppose $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$, r is a root vertex of Γ_2, $G = G_1 \times G_2$, $S = \{(e, x) \mid x \in S_2\}$ and $\Gamma = Cay(G, S)$. Then $\Gamma_1\{\Gamma_2\} \cong \Gamma$ if and only if Γ_1 is empty.*

Proof. We first assume that $\Gamma_1\{\Gamma_2\} \cong \Gamma$. Then $\Gamma_1\{\Gamma_2\}$ is a $|S|-$regular graph. Suppose $u \in G_1$ and $v \in G_2$. Then $deg_{\Gamma_1}(u) + deg_{\Gamma_2}(r) = deg_{\Gamma_2}(v)$. Since Γ_2 is regular, $deg_{\Gamma_1}(u) = 0$. So, Γ_1 is empty. The converse is a direct consequence of the definition. □

Proposition 2.5. *Let $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$, $\overline{S_1} = \Big\{(e, y) \mid y \in S_2\Big\}$, $\overline{S_2} = \Big\{(t, z) \mid t \in S_1, z \in S_2\Big\}$ and $\Gamma = Cay(G, S)$, where $G = G_1 \times G_2$.If $S = \overline{S_1} \cup \overline{S_2}$ then $\Gamma = \Gamma_1 \vartriangle \Gamma_2$.*

Proof. By the definition, $V(\Gamma) = V(\Gamma_1 \vartriangle \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$, as desired. On the other hand,

$$
\begin{aligned}
E(\Gamma) &= \Big\{(a,b)(c,d) \,\Big|\, (a,b)(c^{-1}, d^{-1}) \in S\Big\} \\
&= \Big\{(a,b)(c,d) \,\Big|\, (ac^{-1}, bd^{-1}) \in S\Big\} \\
&= \Big\{(a,b)(c,d) \,\Big|\, (ac^{-1}, bd^{-1}) \in \overline{S_1} \vee (ac^{-1}, bd^{-1}) \in \overline{S_2}\Big\} \\
&= \Big\{(a,b)(c,d) \,\Big|\, (ac^{-1} = e \wedge bd^{-1} \in S_2) \vee (ac^{-1} \in S_1 \wedge bd^{-1} \in S_2)\Big\} \\
&= \Big\{(a,b)(c,d) \,\Big|\, (a = c \wedge bd \in E(\Gamma_2)) \vee (ac \in E(\Gamma_1) \wedge bd \in E(\Gamma_2))\Big\} \\
&= E(\Gamma_1 \vartriangle \Gamma_2)
\end{aligned}
$$

Therefore, $\Gamma = \Gamma_1 \vartriangle \Gamma_2$, which completes our argument. □

Notice that the set S in Proposition 2.5, is not necessarily unique. To do this, we find another set S' such that $Cay(G, S) \cong Cay(G, S') \cong \Gamma_1 \vartriangle \Gamma_2$. To do this, it is enough to consider the Cayley graphs

$$
\begin{gathered}
Cay\Big(V_4 \times \mathbb{Z}_2, \{(a,0), (c,1), (b,0)\}\Big), \\
Cay\Big(V_4, \{a, b\}\Big) \vartriangle Cay\Big(\mathbb{Z}_2, \{1\}\Big),
\end{gathered}
$$

and notice that they are isomorphic graphs and

$$\{(a,1),(b,1),(e,1)\} \neq \{(a,0),(c,1),(b,0)\}.$$

Corollary 2.6. *Let* $\Gamma_1 = Cay(G_1, S_1)$, $\Gamma_2 = Cay(G_2, S_2)$, $\dots$, $\Gamma_n = Cay(G_n, S_n)$ *and* $\Gamma = Cay(G, S)$, *where* $G = G_1 \times G_2 \times \cdots \times G_n$. *Define:*

$$\begin{aligned} \overline{S_i} &= \big\{(e,e,\cdots,e,x_i,x_{i+1},\dots,x_n) \mid x_j \in S_j;\ i \le j \le n\big\},\ 1 \le i \le n, \\ S &= \overline{S_1} \cup \overline{S_2} \cup \cdots \cup \overline{S_n}. \end{aligned}$$

Then

$$\Gamma = \Big(\cdots\big((\Gamma_1 \triangle \Gamma_2) \triangle \Gamma_3\big) \triangle \cdots\Big) \triangle \Gamma_n.$$

Proof. By the definition, $V(\Gamma) = V\bigg(\Big(\cdots\big((\Gamma_1 \triangle \Gamma_2) \triangle \Gamma_3\big) \triangle \cdots\Big) \triangle \Gamma_n\bigg) = V(\Gamma_1) \times \cdots \times V(\Gamma_n)$. On the other hand,

$$\begin{aligned} E(\Gamma) &= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; (a_1b_1^{-1}, a_2b_2^{-1},\dots,a_nb_n^{-1}) \in S\Big\} \\ &= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; (a_1b_1^{-1}, a_2b_2^{-1},\dots,a_nb_n^{-1}) \in \overline{S_1} \\ &\quad \vee (a_1b_1^{-1}, a_2b_2^{-1},\dots,a_nb_n^{-1}) \in \overline{S_2} \vee \cdots \vee (a_1b_1^{-1}, a_2b_2^{-1},\dots,a_nb_n^{-1}) \in \overline{S_n}\Big\} \end{aligned}$$

$$\begin{aligned} &= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; (a_1b_1^{-1} \in S_1, a_2b_2^{-1} \in S_2,\dots,a_nb_n^{-1} \in S_n) \\ &\quad \vee (a_1b_1^{-1} = e, a_2b_2^{-1} \in S_2,\dots,a_nb_n^{-1} \in S_n) \\ &\quad \vee (a_1b_1^{-1} = e, a_2b_2^{-1} = e, a_3b_3^{-1} \in S_3,\dots,a_nb_n^{-1} \in S_n) \\ &\quad \vee \cdots \vee (a_1b_1^{-1} = e,\dots,a_{n-1}b_{n-1}^{-1} = e, a_nb_n^{-1} \in S_n)\Big\} \\ &= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; \bigg(\Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big) \\ &\vee \big(a_1 = b_1 \wedge a_2b_2 \in E(\Gamma_2)\big)\Big) \wedge \big(a_3b_3 \in E(\Gamma_3) \wedge \cdots \wedge a_nb_n \in E(\Gamma_n)\big)\bigg) \\ &\vee \big(a_1 = b_1 \wedge a_2 = b_2 \wedge a_3b_3 \in E(\Gamma_3) \wedge \cdots \wedge a_nb_n \in E(\Gamma_n)\big) \\ &\vee \cdots \vee \big(a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_{n-1} = b_{n-1} \wedge a_nb_n \in E(\Gamma_n)\big)\Big\} \\ &= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; \bigg(\Big(\big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangle \Gamma_2) \\ &\wedge a_3b_3 \in E(\Gamma_3)\big) \vee \big(a_1 = b_1 \wedge a_2 = b_2 \wedge a_3b_3 \in E(\Gamma_3)\big)\Big) \\ &\wedge \big(a_4 = b_4 \wedge \cdots \wedge a_{n-1} = b_{n-1} \wedge a_nb_n \in E(\Gamma_n)\big)\bigg) \\ &\vee \cdots \vee \big(a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_{n-1} = b_{n-1} \wedge a_nb_n \in E(\Gamma_n)\big)\Big\}. \end{aligned}$$

So

$$
\begin{aligned}
&E(\Gamma)\\
&= \Big\{(a_1,a_2,\ldots,a_n)(b_1,b_2,\ldots,b_n)\;\Big|\;\Big(\Big(\big((a_1,a_2,a_3)(b_1,b_2,b_3)\in E((\Gamma_1\triangle\Gamma_2)\triangle\Gamma_3)\\
&\wedge\;\; a_4b_4\in E(\Gamma_4)\big)\vee\big(a_1=b_1\wedge a_2=b_2\wedge a_3=b_3\wedge a_4b_4\in E(\Gamma_4)\big)\Big)\\
&\wedge\;\; \big(a_5b_5\in E(\Gamma_5)\wedge\cdots\wedge a_nb_n\in E(\Gamma_n)\big)\Big)\\
&\vee\;\; \cdots\vee\big(a_1=b_1\wedge a_2=b_2\wedge\cdots\wedge a_{n-1}=b_{n-1}\wedge a_nb_n\in E(\Gamma_n)\big)\Big\}\\
&= \;\cdots.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E(\Gamma) &= \Big\{(a_1,\ldots,a_n)(b_1,\ldots,b_n)\;\Big|\;\Big((a_1,\ldots,a_{n-1})(b_1,\ldots,b_{n-1})\\
&\in E\Big(\Big(\cdots\Big(\big(\Gamma_1\triangle\Gamma_2\big)\triangle\Gamma_3\Big)\triangle\cdots\triangle\Gamma_{n-1}\Big)\Big)\wedge a_nb_n\in E(\Gamma_n)\Big)\\
&\vee\;\; \Big(\big(a_1=b_1\wedge a_2=b_2\wedge\cdots\wedge a_{n-1}=b_{n-1}\wedge a_nb_n\in E(\Gamma_n)\Big)\Big\}\\
&= E\Big(\Big(\cdots\Big(\big(\Gamma_1\triangle\Gamma_2\big)\triangle\Gamma_3\Big)\triangle\cdots\triangle\Gamma_n\Big)\Big),
\end{aligned}
$$

which implies that $\Gamma=\Big(\cdots\big((\Gamma_1\triangle\Gamma_2)\triangle\Gamma_3\big)\triangle\cdots\Big)\triangle\Gamma_n$. □

We can define the skew product of n graphs $\Gamma_1,\Gamma_2,\ldots,\Gamma_n$ by

$$\Gamma_1\triangle\Big(\Gamma_2\triangle\Big(\Gamma_3\triangle\big(\cdots\triangle(\Gamma_{n-1}\triangle\Gamma_n)\cdots\big)\Big)\Big).$$

Then the vertex set will be again $V(\Gamma_1)\times\cdots\times V(\Gamma_n)$ and two vertices

$$(x_1,x_2,\ldots,x_n)\text{and}(y_1,y_2,\ldots,y_n)$$

are adjacent if and only if $x_ny_n\in E(\Gamma_n)$ and for each j, $1\le j\le n-1$, $x_j=y_j$ or $x_jy_j\in E(\Gamma_j)$.

Corollary 2.7. *Let $\Gamma_1=Cay(G_1,S_1)$, $\Gamma_2=Cay(G_2,S_2)$, …, $\Gamma_n=Cay(G_n,S_n)$ and $\Gamma=Cay(G,S)$, where $G=G_1\times G_2\times\cdots\times G_n$ and $S=\big\{(x_1,x_2,\ldots,x_n)\mid \forall 1\leqslant j\leqslant n-1: x_j\in S_j\vee x_j=e, x_n\in S_n\big\}$. Then*

$$\Gamma=\Gamma_1\triangle\Big(\Gamma_2\triangle\Big(\Gamma_3\triangle\big(\cdots\triangle(\Gamma_{n-1}\triangle\Gamma_n)\cdots\big)\Big)\Big).$$

Proof. We will present the proof for $n=4$. The proof in general is similar, but lengthy. We first notice that $V(\Gamma)=V\Big(\Gamma_1\triangle\big(\Gamma_2\triangle(\Gamma_3\triangle\Gamma_4)\big)\Big)=V(\Gamma_1)\times$

$V(\Gamma_2)\times V(\Gamma_3)\times V(\Gamma_4)$. So, it is enough to prove the equality of edge sets.

$$
\begin{array}{rcl}
E(\Gamma) & = & \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; (a_1b_1^{-1},a_2b_2^{-1},a_3b_3^{-1},a_4b_4^{-1}) \in S\Big\} \\
& = & \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; (a_1b_1^{-1}=e \wedge a_2b_2^{-1}=e \wedge a_3b_3^{-1}=e \\
& \wedge & a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}=e \wedge a_2b_2^{-1}=e \wedge a_3b_3^{-1}\in S_3 \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}=e \wedge a_2b_2^{-1}\in S_2 \wedge a_3b_3^{-1}\in S_3 \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}=e \wedge a_2b_2^{-1}\in S_2 \wedge a_3b_3^{-1}=e \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}\in S_1 \wedge a_2b_2^{-1}\in S_2 \wedge a_3b_3^{-1}\in S_3 \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}\in S_1 \wedge a_2b_2^{-1}\in S_2 \wedge a_3b_3^{-1}=e \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}\in S_1 \wedge a_2b_2^{-1}=e \wedge a_3b_3^{-1}=e \wedge a_4b_4^{-1}\in S_4) \\
& \vee & (a_1b_1^{-1}\in S_1 \wedge a_2b_2^{-1}=e \wedge a_3b_3^{-1}\in S_3 \wedge a_4b_4^{-1}\in S_4)\Big\} \\
& = & \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; \big(a_1=b_1 \wedge a_2=b_2 \wedge a_3=b_3 \\
& \wedge & a_4b_4\in E(\Gamma_4)\big) \vee \big(a_1=b_1 \wedge a_2=b_2 \wedge a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1=b_1 \wedge a_2b_2\in E(\Gamma_2) \wedge a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1=b_1 \wedge a_2b_2\in E(\Gamma_2) \wedge a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1b_1\in E(\Gamma_1) \wedge a_2b_2\in E(\Gamma_2) \wedge a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1b_1\in E(\Gamma_1) \wedge a_2b_2\in E(\Gamma_2) \wedge a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1b_1\in E(\Gamma_1) \wedge a_2=b_2 \wedge a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_1b_1\in E(\Gamma_1) \wedge a_2=b_2 \wedge a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big)\Big\} \\
& = & \Bigg\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Bigg|\; \Bigg((a_1=b_1 \wedge a_2=b_2) \\
& \wedge & \Big(\big(a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \vee \big(a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big)\Big)\Bigg) \\
& \vee & \Bigg(\big(a_1=b_1 \wedge a_2b_2\in E(\Gamma_2)\big) \wedge \Big(\big(a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big)\Big)\Bigg) \vee \Bigg(\big(a_1b_1\in E(\Gamma_1) \wedge a_2b_2\in E(\Gamma_2)\big) \\
& \wedge & \Big(\big(a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \vee \big(a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big)\Big)\Bigg) \\
& \vee & \Bigg((a_1b_1\in E(\Gamma_1) \wedge a_2=b_2) \wedge \Big(\big(a_3=b_3 \wedge a_4b_4\in E(\Gamma_4)\big) \\
& \vee & \big(a_3b_3\in E(\Gamma_3) \wedge a_4b_4\in E(\Gamma_4)\big)\Big)\Bigg)\Bigg\}.
\end{array}
$$

Therefore

$$\begin{aligned}
E(\Gamma) &= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \ \Big| \\
&\quad \Big(\big(a_1=b_1 \wedge a_2=b_2\big) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\Big) \\
&\vee \Big(\big(a_1=b_1 \wedge a_2b_2 \in E(\Gamma_2)\big)\big) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\Big) \\
&\vee \Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big)\big) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\Big) \\
&\vee \Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2=b_2\big)\big) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\Big)\Big\} \\
&= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \ \Big| \ \Big(a_1=b_1 \wedge \Big(\big(a_2=b_2 \wedge (a_3,a_4)(b_3,b_4) \\
&\in E(\Gamma_3 \triangle \Gamma_4)\big) \vee \big(a_2b_2 \in E(\Gamma_2) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\big)\Big)\Big) \\
&\vee \Big(a_1b_1 \in E(\Gamma_1) \wedge \Big(\big(a_2b_2 \in E(\Gamma_2) \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\big) \\
&\vee \big(a_2=b_2 \wedge (a_3,a_4)(b_3,b_4) \in E(\Gamma_3 \triangle \Gamma_4)\big)\Big)\Big)\Big\} \\
&= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \ \Big| \\
&\quad \Big(a_1=b_1 \wedge (a_1,a_2,a_3)(b_1,b_2,b_3) \in E\big(\Gamma_2 \vartriangle (\Gamma_3 \vartriangle \Gamma_4)\big)\Big) \\
&\vee \Big(a_1b_1 \in E(\Gamma_1) \wedge (a_1,a_2,a_3)(b_1,b_2,b_3) \in E\big(\Gamma_2 \vartriangle (\Gamma_3 \vartriangle \Gamma_4)\big)\Big)\Big\} \\
&= E\Big(\Gamma_1 \vartriangle \big(\Gamma_2 \vartriangle (\Gamma_3 \vartriangle \Gamma_4)\big)\Big).
\end{aligned}$$

□

Proposition 2.8. *Suppose that* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, $\overline{S_1} = \Big\{(x,e) \mid x \in S_1\Big\}$, $\overline{S_2} = \Big\{(t,z) \mid t \in S_1, z \in S_2\Big\}$ *and* $\Gamma = Cay(G,S)$. *If* $G = G_1 \times G_2$ *and* $S = \overline{S_1} \cup \overline{S_2}$ *then* $\Gamma \cong \Gamma_1 \triangledown \Gamma_2$.

Proof. By the definition of converse skew product of graphs, $V(\Gamma) = V(\Gamma_1 \triangledown \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$. On the other hand,

$$\begin{aligned}
E(\Gamma) &= \Big\{(a,b)(c,d) \ \Big| \ (a,b)(c^{-1},d^{-1}) \in S\Big\} \\
&= \Big\{(a,b)(c,d) \ \Big| \ (ac^{-1},bd^{-1}) \in S\Big\} \\
&= \Big\{(a,b)(c,d) \ \Big| \ (ac^{-1},bd^{-1}) \in \overline{S_1} \vee (ac^{-1},bd^{-1}) \in \overline{S_2}\Big\} \\
&= \Big\{(a,b)(c,d) \ \Big| \ (ac^{-1} \in S_1 \wedge bd^{-1} = e) \vee (ac^{-1} \in S_1 \wedge bd^{-1} \in S_2)\Big\} \\
&= \Big\{(a,b)(c,d) \ \Big| \ (ac \in E(\Gamma_1) \wedge b = d) \vee (ac \in E(\Gamma_1) \wedge bd \in E(\Gamma_2))\Big\} \\
&= E(\Gamma_1 \triangledown \Gamma_2).
\end{aligned}$$

Thus, $\Gamma = \Gamma_1 \triangledown \Gamma_2$, which completes our argument. $\square$

Notice that the set S in Proposition 2.8 is not uniquely determined. To see this, we assume that K_4 denotes the Klein four−group generated by a and b. Then

$$Cay\Big(V_4 \times \mathbb{Z}_2, \{(a,1),(e,1),(b,0),(c,0)\}\Big) \cong Cay(V_4,\{a,b\}) \triangledown Cay(\mathbb{Z}_2,\{1\}),$$

and $\{(a,1),(b,1),(a,0),(b,0)\} \neq \{(a,1),(e,1),(b,0),(c,0)\}$.

Suppose that Γ_1, Γ_2, …, Γ_n is a sequence of graphs. The converse skew product

$$\Gamma_1 \triangledown \Big(\Gamma_2 \triangledown \Big(\Gamma_3 \triangledown \big(\cdots \triangledown (\Gamma_{n-1} \triangledown \Gamma_n)\cdots\big)\Big)\Big),$$

has $\{(x_1,x_2,\ldots,x_n) \mid x_i \in V(\Gamma_i)\ ;\ 1 \le i \le n\}$ as vertex set, and two vertices $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_n)$ are adjacent if and only if there exists m, $1 \le m \le n$, such that for each i, $1 \le i \le m-1$, $x_iy_i \in E(\Gamma_i)$ and for each j, $m \le j \le n$, $x_j = y_j$.

Corollary 2.9. *Let* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, …, $\Gamma_n = Cay(G_n,S_n)$ *and* $\Gamma = Cay(G,S)$, *where* $G = G_1 \times G_2 \times \cdots \times G_n$. *Define:*

$$\begin{aligned}
\overline{S_1} &= \big\{(x_1,e,\ldots,e) \mid x_1 \in S_1\big\},\\
\overline{S_2} &= \big\{(x_1,x_2,e,\ldots,e) \mid x_1 \in S_1 \wedge x_2 \in S_2\big\},\\
&\ \vdots\\
\overline{S_{n-1}} &= \big\{(x_1,x_2,,\ldots,x_{n-1},e) \mid x_i \in S_i\ ;\ 1 \le i \le n-1\big\},\\
\overline{S_n} &= \big\{(x_1,x_2,\ldots,x_{n-1},x_n) \mid x_i \in S_i\ ;\ 1 \le i \le n\big\},\\
S &= \overline{S_1} \cup \overline{S_2} \cup \cdots \cup \overline{S_n}.
\end{aligned}$$

Then $\Gamma \cong \Gamma_1 \triangledown \Big(\Gamma_2 \triangledown \Big(\Gamma_3 \triangledown \big(\cdots \triangledown (\Gamma_{n-1} \triangledown \Gamma_n)\cdots\big)\Big)\Big)$.

Proof. By the definition, $V(\Gamma) = \Big(\Gamma_1 \triangledown \Big(\Gamma_2 \triangledown \Big(\cdots \triangledown (\Gamma_{n-1} \triangledown \Gamma_n)\cdots\Big)\Big)\Big) = V(\Gamma_1) \times \cdots \times V(\Gamma_n)$. On the other hand,

$$\begin{aligned}
E(\Gamma) &= \Big\{(a_1,a_2,\ldots,a_n)(b_1,b_2,\ldots,b_n) \;\Big|\; (a_1b_1^{-1},a_2b_2^{-1},\ldots,a_nb_n^{-1}) \in S\Big\}\\
&= \Big\{(a_1,a_2,\ldots,a_n)(b_1,b_2,\ldots,b_n) \;\Big|\; (a_1b_1^{-1},a_2b_2^{-1},\ldots,a_nb_n^{-1}) \in \overline{S_1}\\
&\vee\ \ (a_1b_1^{-1},a_2b_2^{-1},\ldots,a_nb_n^{-1}) \in \overline{S_2} \vee \cdots \vee (a_1b_1^{-1},a_2b_2^{-1},\ldots,a_nb_n^{-1}) \in \overline{S_n}\Big\}\\
&= \Big\{(a_1,\ldots,a_n)(b_1,\ldots,b_n) \;\Big|\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} = e \wedge \ldots \wedge a_nb_n^{-1} = e)\\
&\vee\ \ (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge a_3b_3^{-1} = e \wedge \ldots \wedge a_nb_n^{-1} = e)\\
&\vee\ \ \cdots \vee (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge \ldots \wedge a_{n-1}b_{n-1}^{-1} \in S_{n-1} \wedge a_nb_n^{-1} = e)\\
&\vee\ \ (a_1b_1^{-1} \in S_1 \wedge \ldots \wedge a_{n-1}b_{n-1}^{-1} \in S_{n-1} \wedge a_nb_n^{-1} \in S_n)\Big\}.
\end{aligned}$$

So

$$
\begin{aligned}
E(\Gamma) &= \Big\{(a_1,\dots,a_n)(b_1,\dots,b_n) \;\Big|\; (a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \dots \wedge a_n = b_n) \\
&\vee \; (a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge \dots \wedge a_n = b_n) \\
&\vee \; \cdots \vee (a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-1}b_{n-1} \in E(\Gamma_{n-1}) \wedge a_n = b_n) \\
&\vee \; (a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-1}b_{n-1} \in E(\Gamma_{n-1}) \wedge a_nb_n \in E(\Gamma_n))\Big\} \\
&= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; (a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \dots \\
&\wedge \; a_n = b_n) \vee (a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge \dots \wedge a_n = b_n) \\
&\vee \; \cdots \\
&\vee \; \Big(\big(a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-2}b_{n-2} \in E(\Gamma_{n-2})\big) \\
&\wedge \; \Big(\big(a_{n-1}b_{n-1} \in E(\Gamma_{n-1}) \wedge a_n = b_n\big) \\
&\vee \; \big(a_{n-1}b_{n-1} \in E(\Gamma_{n-1}) \wedge a_nb_n \in E(\Gamma_n)\big)\Big)\Big)\Big\} \\
&= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \dots \\
&\wedge \; a_n = b_n\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge \dots \wedge a_n = b_n\big) \\
&\vee \; \cdots \vee \big(a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-2}b_{n-2} \in E(\Gamma_{n-2}) \wedge a_{n-1} = b_{n-1} \wedge a_n = b_n\big) \\
&\vee \; \big(a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-2}b_{n-2} \in E(\Gamma_{n-2}) \\
&\wedge \; (a_{n-1},a_n)(b_{n-1},b_n) \in E(\Gamma_{n-1} \triangledown \Gamma_n)\big)\Big\} \\
&= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \dots \\
&\wedge \; a_n = b_n\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge \dots \wedge a_n = b_n\big) \\
&\vee \; \cdots \\
&\vee \; \Big(\big(a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-3}b_{n-3} \in E(\Gamma_{n-3})\big) \\
&\wedge \; \Big(\big(a_{n-2}b_{n-2} \in E(\Gamma_{n-2}) \wedge a_{n-1} = b_{n-1} \wedge a_n = b_n\big) \\
&\vee \; \big(a_{n-2}b_{n-2} \in E(\Gamma_{n-2}) \wedge (a_{n-1},a_n)(b_{n-1},b_n) \in E(\Gamma_{n-1} \triangledown \Gamma_n)\big)\Big)\Big)\Big\} \\
&= \Big\{(a_1,a_2,\dots,a_n)(b_1,b_2,\dots,b_n) \;\Big|\; \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \dots \\
&\wedge \; a_n = b_n\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge \dots \wedge a_n = b_n\big) \\
&\vee \; \cdots \\
&\vee \; \Big(a_1b_1 \in E(\Gamma_1) \wedge \dots \wedge a_{n-3}b_{n-3} \in E(\Gamma_{n-3}) \\
&\wedge \; (a_{n-2},a_{n-1},a_n)(b_{n-2},b_{n-1},b_n) \in E\big(\Gamma_{n-2} \triangledown (\Gamma_{n-1} \triangledown \Gamma_n)\big)\Big)\Big\} \\
&= \; \cdots .
\end{aligned}
$$

Therefore

$$\begin{aligned} E(\Gamma) &= \Big\{(a_1,a_2,\ldots,a_n)(b_1,b_2,\ldots,b_n) \,\Big|\, \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge \ldots \\ &\wedge \;\; a_n = b_n\big) \vee \Big(a_1b_1 \in E(\Gamma_1) \wedge (a_2,\ldots,a_n)(b_2,\ldots,b_n) \\ &\in \;\; E\Big(\Gamma_2\triangledown\Big(\Gamma_3\triangledown\Big(\Gamma_4\triangledown\big(\cdots\triangledown(\Gamma_{n-1}\triangledown\Gamma_n)\cdots\big)\Big)\Big)\Big)\Big)\Big\} \\ &= \;\; E\Big(\Gamma_1\triangledown\Big(\Gamma_2\triangledown\Big(\Gamma_3\triangledown\big(\cdots\triangledown(\Gamma_{n-1}\triangledown\Gamma_n)\cdots\big)\Big)\Big)\Big). \end{aligned}$$

which implies that $\Gamma = \Gamma_1\triangledown\Big(\Gamma_2\triangledown\Big(\Gamma_3\triangledown\big(\cdots\triangledown(\Gamma_{n-1}\triangledown\Gamma_n)\cdots\big)\Big)\Big)$. □

We can define the converse skew product of n graphs $\Gamma_1,\Gamma_2,\ldots,\Gamma_n$ by

$$\Big(\cdots\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big)\triangledown\cdots\Big)\triangledown\Gamma_n.$$

Then the vertex set will be again $V(\Gamma_1)\times\cdots\times V(\Gamma_n)$ and two vertices

$$(x_1,x_2,\ldots,x_n)\text{and } (y_1,y_2,\ldots,y_n)$$

are adjacent if and only if $x_1y_1 \in E(\Gamma_1)$ and for each i, $2 \leq i \leq n$, $x_i = y_i$ or $x_iy_i \in E(\Gamma_i)$.

Corollary 2.10. *Suppose* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, $\ldots$, $\Gamma_n = Cay(G_n,S_n)$ *and* $\Gamma = Cay(G,S)$, *where* $G = G_1\times G_2\times\cdots\times G_n$ *and*

$$S = \big\{(x_1,x_2,\ldots,x_n) \mid x_1 \in S_1, \forall 2 \leqslant i \leqslant n : x_i \in S_i \vee x_i = e\big\}.$$

Then $\Gamma = \Big(\cdots\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big)\triangledown\cdots\Big)\triangledown\Gamma_n$.

Proof. The proof in general case is tedious and so similar to Corollary 2.7. We will prove the result for $n = 4$. We first notice that

$$V(\Gamma) = V\Big(\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big)\triangledown\Gamma_4\Big) = (\Gamma_1)\times V(\Gamma_2)\times V(\Gamma_3)\times V(\Gamma_4).$$

So, it is enough to prove the equality of edge sets.

$$\begin{aligned} E(\Gamma) &= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \,\Big|\, (a_1b_1^{-1},a_2b_2^{-1},a_3b_3^{-1},a_4b_4^{-1}) \in S\Big\} \\ &= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \,\Big|\, (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} = e \wedge a_3b_3^{-1} = e \wedge a_4b_4^{-1} = e) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge a_3b_3^{-1} = e \wedge a_4b_4^{-1} = e) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} = e \wedge a_3b_3^{-1} \in S_3 \wedge a_4b_4^{-1} = e) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} = e \wedge a_3b_3^{-1} = e \wedge a_4b_4^{-1} \in S_4) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge a_3b_3^{-1} \in S_3 \wedge a_4b_4^{-1} = e) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge a_3b_3^{-1} = e \wedge a_4b_4^{-1} \in S_4) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} = e \wedge a_3b_3^{-1} \in S_3 \wedge a_4b_4^{-1} \in S_4) \\ &\vee \;\; (a_1b_1^{-1} \in S_1 \wedge a_2b_2^{-1} \in S_2 \wedge a_3b_3^{-1} \in S_3 \wedge a_4b_4^{-1} \in S_4)\Big\}. \end{aligned}$$

So

$$
\begin{aligned}
E(\Gamma) &= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge a_3 = b_3 \\
&\wedge\ a_4 = b_4\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge a_4 = b_4\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4 = b_4\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge a_3 = b_3 \wedge a_4b_4 \in E(\Gamma_4)\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4 = b_4\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3 = b_3 \wedge a_4b_4 \in E(\Gamma_4)\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2 \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4b_4 \in E(\Gamma_4)\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2) \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4b_4 \in E(\Gamma_4)\big)\Big\} \\
&= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; \Big(\Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2\big) \\
&\vee\ \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big)\Big) \wedge \Big(a_3 = b_3 \wedge a_4 = b_4\Big)\Big) \\
&\vee\ \Big(\Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big)\Big) \\
&\wedge\ \Big(a_3b_3 \in E(\Gamma_3) \wedge a_4 = b_4\Big)\Big) \\
&\vee\ \Big(\Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big)\Big) \\
&\wedge\ \Big(a_3 = b_3 \wedge a_4b_4 \in E(\Gamma_4)\Big)\Big) \\
&\vee\ \Big(\Big(\big(a_1b_1 \in E(\Gamma_1) \wedge a_2 = b_2\big) \vee \big(a_1b_1 \in E(\Gamma_1) \wedge a_2b_2 \in E(\Gamma_2)\big)\Big) \\
&\wedge\ \Big(a_3b_3 \in E(\Gamma_3) \wedge a_4b_4 \in E(\Gamma_4)\Big)\Big)\Big\} \\
&= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; \Big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3 = b_3 \\
&\wedge\ a_4 = b_4\Big) \vee \Big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4 = b_4\Big) \\
&\vee\ \Big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3 = b_3 \wedge a_4b_4 \in E(\Gamma_4)\Big) \\
&\vee\ \Big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3b_3 \in E(\Gamma_3) \wedge a_4b_4 \in E(\Gamma_4)\Big)\Big\} \\
&= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \;\Big|\; \Big(\Big(\big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3 = b_3\big) \\
&\vee\ \big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3b_3 \in E(\Gamma_3)\big)\Big) \\
&\wedge\ a_4 = b_4\Big) \vee \Big(\Big(\big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3 = b_3\big) \\
&\vee\ \big((a_1,a_2)(b_1,b_2) \in E(\Gamma_1 \triangledown \Gamma_2) \wedge a_3b_3 \in E(\Gamma_3)\big)\Big) \wedge a_4b_4 \in E(\Gamma_4)\Big)\Big\}.
\end{aligned}
$$

Therefore, we have:

$$\begin{aligned} E(\Gamma) &= \Big\{(a_1,a_2,a_3,a_4)(b_1,b_2,b_3,b_4) \Big| \Big((a_1,a_2,a_3)(b_1,b_2,b_3) \in E\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big) \\ &\wedge \quad a_4 = b_4\Big) \vee \Big((a_1,a_2,a_3)(b_1,b_2,b_3) \in E\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big) \wedge a_4b_4 \in E(\Gamma_4)\Big)\Big\} \\ &= E\Big(\big((\Gamma_1\triangledown\Gamma_2)\triangledown\Gamma_3\big)\triangledown\Gamma_4\Big), \end{aligned}$$

which completes our argument. □

If $B = \{(1,0),(0,1)\}$ then the NEPS of graphs Γ_1 and Γ_2 is just its Cartesian product of these graphs. The Cartesian product of Cayley graphs were considered in [1, Lemma 2.6]. The case of $B = \{(1,1)\}$ leads to the tensor product of graphs. The tensor product of Cayley graphs were studied in [6, Proposition 2.1]. So, it is natural to consider the general case of the NPES of Cayley graphs.

Proposition 2.11. *Suppose* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, $\Gamma = Cay(G,S)$ *and* $B = \{(1,0)\}$, *where* $G = G_1 \times G_2$ *and* $S = \Big\{(x,e) \mid x \in S_1\Big\}$. *Then* $\Gamma = \Gamma_1\ NEPS\ \Gamma_2$.

Proof. By the definition $V(\Gamma) = V(\Gamma_1\ NEPS\ \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$. On the other hand,

$$\begin{aligned} E(\Gamma) &= \{(x_1,x_2)(y_1,y_2) \mid (x_1,x_2)(y_1^{-1},y_2^{-1}) \in S\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid (x_1y_1^{-1},x_2y_2^{-1}) \in S\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid x_1y_1^{-1} \in S_1 \wedge x_2y_2^{-1} = e\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in E(\Gamma_1) \wedge x_2 = y_2\} \\ &= E(\Gamma_1\ NEPS\ \Gamma_2). \end{aligned}$$

Hence $\Gamma = \Gamma_1\ NEPS\ \Gamma_2$, se desired. □

Proposition 2.12. *Suppose* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, $\Gamma = Cay(G,S)$ *and* $B = \{(0,1)\}$, *where* $G = G_1 \times G_2$ *and* $S = \Big\{(e,y) \mid y \in S_2\Big\}$. *Then* $\Gamma = \Gamma_1\ NEPS\ \Gamma_2$.

Proof. By the definition $V(\Gamma) = V(\Gamma_1\ NEPS\ \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$. On the other hand,

$$\begin{aligned} E(\Gamma) &= \{(x_1,x_2)(y_1,y_2) \mid (x_1,x_2)(y_1^{-1},y_2^{-1}) \in S\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid (x_1y_1^{-1},x_2y_2^{-1}) \in S\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid x_1y_1^{-1} = e \wedge x_2y_2^{-1} \in S_2\} \\ &= \{(x_1,x_2)(y_1,y_2) \mid x_1 = y_1 \wedge x_2y_2 \in E(\Gamma_2)\} \\ &= E(\Gamma_1\ NEPS\ \Gamma_2). \end{aligned}$$

Therefore, $\Gamma = \Gamma_1\ NEPS\ \Gamma_2$, which completes our argument. □

Proposition 2.13. *Suppose* $\Gamma_1 = Cay(G_1,S_1)$, $\Gamma_2 = Cay(G_2,S_2)$, $\Gamma = Cay(G,S)$ *and* $B = \{(1,0),(0,1),(1,1)\}$, *where* $G = G_1 \times G_2$, $\overline{S_1} = \Big\{(x,e) \mid x \in S_1\Big\}$, $\overline{S_2} = \Big\{(e,y) \mid y \in S_2\Big\}$ *and* $S = \overline{S_1} \cup \overline{S_2} \cup S_1 \times S_2$. *Then* $\Gamma = \Gamma_1\ NEPS\ \Gamma_2 = \Gamma_1 \boxtimes \Gamma_2$.

Proof. By the definition of NEPS and Strong product of graphs

$$V(\Gamma) = V(\Gamma_1 NEPS \Gamma_2) = V(\Gamma_1 \boxtimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2).$$

On the other hand,

$$\begin{aligned}
E(\Gamma) &= \{(x_1, x_2)(y_1, y_2) \mid (x_1, x_2)(y_1^{-1}, y_2^{-1}) \in S\} \\
&= \{(x_1, x_2)(y_1, y_2) \mid (x_1 y_1^{-1}, x_2 y_2^{-1}) \in S\} \\
&= \{(x_1, x_2)(y_1, y_2) \mid (x_1 y_1^{-1} \in S_1 \wedge x_2 y_2^{-1} = e) \\
&\vee\ (x_1 y_1^{-1} \in S_1 \wedge x_2 y_2^{-1} = e) \vee (x_1 y_1^{-1} \in S_1 \wedge x_2 y_2^{-1} \in S_2)\} \\
&= \{(x_1, x_2)(y_1, y_2) \mid (x_1 y_1 = e \wedge x_2 y_2 \in E(\Gamma_2)) \\
&\vee\ (x_1 y_1 \in E(\Gamma_1) \wedge x_2 = y_2) \vee (x_1 y_1 \in E(\Gamma_1) \wedge x_2 y_2 \in E(\Gamma_2))\} \\
&= E(\Gamma_1\ NEPS\ \Gamma_2) = E(\Gamma_1 \boxtimes \Gamma_2).
\end{aligned}$$

Thus, $\Gamma = \Gamma_1\ NEPS\ \Gamma_2$. □

3. Concluding Remarks

In an earlier paper the present authors [6] investigated the behavior of Cayley graphs under graph operations: tensor product, composition, symmetric difference, disjunction and splice of Cayley graphs. In this paper, we consider some new operations containing $NEPS$, corona, hierarchical, strong , skew and converse skew product of Cayley graphs. Some conditions are obtained such that the Cayley graphs under these graph operations are again Cayley graphs.

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References

1. Y.G. Baik, Y.-Q. Feng, H.S. Sim and M.Y. Xu, *On the normality of Cayley graphs of Abelian groups*, Algebra Colloq. **5** (1998), 297–304.
2. L. Barriére, C. Dafló, M.A. Fiol and M. Mitjana, *The generalized hierarchical product of graphs* , Discrete Math. **309** (2009), 3871–3881.
3. L. Barriére, F. Comellas, C. Dafló and M.A. Fiol, *On the spectra of hypertrees*, Linear Algebra Appl. **428** (2008), 1499–1510.
4. N. Biggs, *Algeraic Graph Theory*, *Cambridge University Press*, Cambridge, 1974.
5. R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Second edition, Taylor & Francis Group, 2011.
6. N. Malekmohammadi and A.R. Ashrafi, *Cayley graphs under graph operations*, submitted.
7. M.M. Petrović, *Finite type graphs and some graph operations II*, Discrete Math. **58** (1986), 269–274.
8. Z. Shao, R.K. Yeh and D. Zhang, *The L(2, 1)-labeling on the skew and converse skew products of graphs*, Appl. Math. Lett. **20** (2007), 59–64.
9. Y.-N. Yeh and I. Gutman, *On the sum of all distances in composite graphs*, Discrete Math. **135** (1994), 359–365.

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