Khayyam J. Math. 1 (2015), no. 2, 219–229

Khayyam Journal of Mathematics

URL: www.emis.de/journals/KJM/

# $n$-DUAL SPACES ASSOCIATED TO A NORMED SPACE

YOSAFAT E. P. PANGALELA

Communicated by M. Filali

Abstract. For a real normed space $X$, we study the $n$-dual space of $(X, \|\cdot\|)$ and show that the space is a Banach space. Meanwhile, for a real normed space $X$ of dimension $d \geq n$ which satisfies property (G), we discuss the $n$-dual space of $(X, \|\cdot, \dots, \cdot\|_G)$, where $\|\cdot, \dots, \cdot\|_G$ is the Gähler $n$-norm. We then investigate the relationship between the $n$-dual space of $(X, \|\cdot\|)$ and the $n$-dual space of $(X, \|\cdot, \dots, \cdot\|_G)$. We use this relationship to determine the $n$-dual space of $(X, \|\cdot, \dots, \cdot\|_G)$ and show that the space is also a Banach space.

## 1. Introduction

In the 1960's, the notion of $n$-normed spaces was introduced by Gähler [2, 3, 4, 5] as a generalisation of normed spaces. For every real normed space $X$ of dimension $d \geq n$, Gähler showed that $X$ can be viewed as an $n$-normed space by using the Gähler $n$-norm, which is denoted by $\|\cdot, \dots, \cdot\|_G$. This $n$-norm is defined by using the set of bounded linear functionals on $X$. Since then, many researchers have studied operators and functionals on $n$-normed space $X$ (see [1, 6, 9, 10, 11, 13, 14, 15]).

In [13], the author and Gunawan introduced the concept of $n$-dual spaces. For every real normed space $X$ of dimension $d \geq n$, there are two $n$-dual spaces associated to $X$. The first is the $n$-dual space of $(X, \|\cdot\|)$, and the other is the $n$-dual space of $(X, \|\cdot, \dots, \cdot\|_G)$. In case $X$ is the $l^p$ space for some $1 \leq p < \infty$, the author and Gunawan have investigated and given the relationship between both $n$-dual spaces [13]. Here we provide an analogues result on more general normed spaces.

For a real normed space $X$, we investigate the $n$-dual space of $(X, \|\cdot\|)$ by using the $(n-1)$-dual space of $(X, \|\cdot\|)$ (Theorem 3.2). We then focus on a real normed

*Date*: Received: 16 August 2015, Accepted: 04 December 2015.

2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 46C05, 46C15, 46B99, 46C99.

*Key words and phrases.* $n$-dual spaces, $n$-normed spaces, bounded linear functionals.

space $X$ of dimension $d \geq n$ which satisfies property (G) and discuss the relationship between the $n$-dual space of $(X, \|\cdot\|)$ and the $n$-dual space of $(X, \|\cdot, \ldots, \cdot\|_G)$ (Theorem 4.5). It is interesting to observe that both the $n$-dual space of $(X, \|\cdot\|)$ and the $n$-dual space of $(X, \|\cdot, \ldots, \cdot\|_G)$ are Banach spaces (Theorem 3.3 and Theorem 4.7).

## 2. Preliminaries

Let $n$ be a nonnegative integer and $X$ a real vector space of dimension $d \geq n$. We call a real-valued function $\|\cdot, \ldots, \cdot\|$ on $X^n$ an *$n$-norm* on $X$ if for all $x_1, \ldots, x_n, x' \in X$, we have

(1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
(2) $\|x_1, \ldots, x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$; and
(4) $\|x_1 + x', \ldots, x_n\| \leq \|x_1, \ldots, x_n\| + \|x', \ldots, x_n\|$.

We then call the pair $(X, \|\cdot, \ldots, \cdot\|)$ an *$n$-normed space*.

An example of an $n$-normed space is the $l^p$ space, where $1 \leq p < \infty$, equipped with

$$\|x_1, \ldots, x_n\|_p := \Big(\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det (x_{ij_k})|^p \Big)^{\frac{1}{p}}$$

for $x_1, \ldots, x_n \in l^p$ (see [7, Section 3]).

Another interesting example of $n$-normed spaces is the Gähler $n$-norm which was introduced in [3, 4, 5]. Let $X$ be a real normed space of dimension $d \geq n$, and $X^{(1)}$ the dual space of $X$. Gähler showed that the function $\|\cdot, \ldots, \cdot\|_G$ which is given by

$$\|x_1, \ldots, x_n\|_G := \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \left| \det \left[ f_j (x_i) \right]_{i,j} \right|$$

for all $x_1, \ldots, x_n \in X$, is an $n$-norm on $X$. Hence every real normed space $X$ can be viewed as an $n$-normed space $(X, \|\cdot, \ldots, \cdot\|_G)$.

Let $X$ be a real normed space of dimension $d \geq n$. Any real-valued function $f$ on $X^n$ is called an *$n$-functional* on $X$. An $n$-functional $f$ is *multilinear* if it satisfies two following properties:

(1) $f(x_1 + y_1, \ldots, x_n + y_n) = \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n} f(z_1, \ldots, z_n)$ and
(2) $f(\alpha_1 x_1, \ldots, \alpha_n x_n) = \alpha_1 \cdots \alpha_{n-1} f(x_1, \ldots, x_n)$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

For multilinear $n$-functionals $f, h$ on $X$, we define an $n$-functional $f + h$ by

$$(f + h)(x_1, \ldots, x_n) := f(x_1, \ldots, x_n) + h(x_1, \ldots, x_n)$$

for $x_1, \ldots, x_n \in X$. Then $f + h$ is also multilinear. On the other hand, we say $f = h$ if

$$f(x_1, \ldots, x_n) = h(x_1, \ldots, x_n)$$

for $x_1, \ldots, x_n \in X$.

We call an $n$-functional $f$ *bounded* on a real normed space $(X, \|\cdot\|)$ (respectively, an $n$-normed space $(X, \|\cdot, \ldots, \cdot\|)$) if there exists a constant $K > 0$ such that

$$|f(x_1, \ldots, x_n)| \leq K \|x_1\| \cdots \|x_n\| \ \text{(respectively, } |f(x_1, \ldots, x_n)| \leq K \|x_1, \ldots, x_n\| \text{)}$$

for all $x_1, \ldots, x_n \in X$.

Let $S_n$ denote the group of permutations of $(1, \ldots, n)$. Recall from [13] that every bounded multilinear $n$-functional $f$ on $(X, \|\cdot, \ldots, \cdot\|)$ is *antisymmetric* in the sense that

$$f(x_1, \ldots, x_n) = \operatorname{sgn}(\sigma) f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$$

for $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$. Here $\operatorname{sgn}(\sigma) = 1$ if $\sigma$ is an even permutation, and $\operatorname{sgn}(\sigma) = -1$ if $\sigma$ is an odd permutation. Note that if $f$ is antisymmetric then for any linearly dependent $x_1, \ldots, x_n \in X$, we have $f(x_1, \ldots, x_n) = 0$.

*Remark* 2.1. In general, we do not have the antisymmetric property for bounded multilinear $n$-functionals on $(X, \|\cdot\|)$.

The space of bounded multilinear $n$-functionals on $(X, \|\cdot\|)$ is called the *$n$-dual space* of $(X, \|\cdot\|)$ and denoted by $X^{(n)}$. For $n = 0$, we define $X^{(0)}$ as $\mathbb{R}$. The function $\|\cdot\|_{n,1}$ on $X^{(n)}$ where

$$\|f\|_{n,1} := \sup_{x_1, \ldots, x_n \neq 0} \frac{|f(x_1, \ldots, x_n)|}{\|x_1\| \cdots \|x_n\|}$$

for $f \in X^{(n)}$, defines a norm on $X^{(n)}$ and then $X^{(n)}$ is a normed space.

Meanwhile, the *$n$-dual space* of $(X, \|\cdot, \ldots, \cdot\|)$ is the space of all bounded multilinear $n$-functionals on $(X, \|\cdot, \ldots, \cdot\|)$. This space is also a normed space with the following norm

$$\|f\|_{n,n} := \sup_{\|x_1, \ldots, x_n\| \neq 0} \frac{|f(x_1, \ldots, x_n)|}{\|x_1, \ldots, x_n\|}.$$

Now let $X, Y$ be real normed spaces. We write $B(X, Y)$ to denote the set of bounded linear operators from $X$ into $Y$. The function $\|\cdot\|_{\text{op}}$ where

$$\|u\|_{\text{op}} := \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|}$$

for every $u \in B(X, Y)$, is a norm on $B(X, Y)$. For simplification, we write $B(X, Y)$ to denote the normed space $B(X, Y)$ equipped with $\|\cdot\|_{\text{op}}$. Otherwise, if $\|\cdot\|^*$ is a norm function on $B(X, Y)$, we write $(B(X, Y), \|\cdot\|^*)$ to denote the normed space $B(X, Y)$ equipped by the norm $\|\cdot\|^*$.

## 3. The $n$-dual space of $(X, \|\cdot\|)$

In this section, we first identify the bounded multilinear $n$-functionals on $(X, \|\cdot\|)$ (Proposition 3.1). We then identify the $n$-dual space of $(X, \|\cdot\|)$ by using the $(n-1)$-dual space of $(X, \|\cdot\|)$ (Theorem 3.2). Finally we show that the $n$-dual space of $(X, \|\cdot\|)$ is a Banach space (Theorem 3.3).

**Proposition 3.1.** *Let $X$ be a real normed space of dimension $d \geq n$ and $f$ a bounded multilinear $n$-functional on $(X, \|\cdot\|)$. Then there exists $u_f \in B(X, X^{(n-1)})$ such that for $x_1, \ldots, x_{n-1}, z \in X$,*

$$f(x_1, \ldots, x_{n-1}, z) = (u_f(z))(x_1, \ldots, x_{n-1}).$$

*Furthermore, $\|f\|_{n,1} = \|u_f\|_{\text{op}}$.*

*Proof.* Take $z \in X$ and define an $(n-1)$-functional $f_z$ on $X$ with

$$f_z(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, z)$$

for $x_1, \ldots, x_{n-1} \in X$. We show $f_z \in X^{(n-1)}$. Note that for $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \in X$ and $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$, we have

$$\begin{aligned} f_z\left(x_1+y_1, \ldots, x_{n-1}+y_{n-1}\right) &= f\left(x_1+y_1, \ldots, x_{n-1}+y_{n-1}, z\right) \\ &= \sum_{z_i \in\{x_i, y_i\}, 1 \leq i \leq n-1} f\left(z_1, \ldots, z_{n-1}, z\right) \\ &= \sum_{z_i \in\{x_i, y_i\}, 1 \leq i \leq n-1} f_z\left(z_1, \ldots, z_{n-1}\right), \end{aligned}$$

$$\begin{aligned} f_z\left(\alpha_1 x_1, \ldots, \alpha_{n-1} x_{n-1}\right) &= f\left(\alpha_1 x_1, \ldots, \alpha_{n-1} x_{n-1}, z\right) \\ &= \alpha_1 \cdots \alpha_{n-1} f\left(x_1, \ldots, x_{n-1}, z\right) \\ &= \alpha_1 \cdots \alpha_{n-1} f_z\left(x_1, \ldots, x_{n-1}\right), \end{aligned}$$

and

$$\left|f_z\left(x_1, \ldots, x_{n-1}\right)\right| = \left|f\left(x_1, \ldots, x_{n-1}, z\right)\right| \leq \|f\|_{n,1}\|z\|\left(\left\|x_1\right\| \cdots\left\|x_{n-1}\right\|\right)$$

since $f$ is bounded on $(X, \|\cdot\|)$. Hence $f_z : X^{n-1} \rightarrow \mathbb{R}$ is multilinear and bounded; and then $f_z \in X^{(n-1)}$.

Now define $u_f : X \rightarrow X^{(n-1)}$ with $u_f(z) := f_z$ for $z \in X$. We have to show $u_f \in B(X, X^{(n-1)})$. First we show that $u_f$ is linear. Take $z_1, z_2 \in X$ and $\alpha, \beta \in \mathbb{R}$. For every $x_1, \ldots, x_{n-1} \in X$, we have

$$\begin{aligned} \left(u_f\left(\alpha z_1+\beta z_2\right)\right)\left(x_1, \ldots, x_{n-1}\right) &= f_{\alpha z_1+\beta z_2}\left(x_1, \ldots, x_{n-1}\right) \\ &= f\left(x_1, \ldots, x_{n-1}, \alpha z_1+\beta z_2\right) \\ &= f\left(x_1, \ldots, x_{n-1}, \alpha z_1\right)+f\left(x_1, \ldots, x_{n-1}, \beta z_2\right) \\ &= \alpha f\left(x_1, \ldots, x_{n-1}, z_1\right)+\beta f\left(x_1, \ldots, x_{n-1}, z_2\right) \\ &= \alpha f_{z_1}\left(x_1, \ldots, x_{n-1}\right)+\beta f_{z_2}\left(x_1, \ldots, x_{n-1}\right) \\ &= \left(\alpha u_f\left(z_1\right)\right)\left(x_1, \ldots, x_{n-1}\right)+\left(\beta u_f\left(z_2\right)\right)\left(x_1, \ldots, x_{n-1}\right) \\ &= \left(\alpha u_f\left(z_1\right)+\beta u_f\left(z_2\right)\right)\left(x_1, \ldots, x_{n-1}\right) \end{aligned}$$

and

$$u_f\left(\alpha z_1+\beta z_2\right) = \alpha u_f\left(z_1\right)+\beta u_f\left(z_2\right).$$

Hence $u_f$ is linear.

Next we show the boundedness of $u_f$. Take $z \in X$. Then for $x_1, \ldots, x_{n-1} \in X$, we have

$$\begin{aligned} \left|\left(u_f(z)\right)\left(x_1, \ldots, x_{n-1}\right)\right| &= \left|f_z\left(x_1, \ldots, x_{n-1}\right)\right| = \left|f\left(x_1, \ldots, x_{n-1}, z\right)\right| \\ &\leq \|f\|_{n,1}\left\|x_1\right\| \cdots\left\|x_{n-1}\right\|\|z\| \quad (f \text{ is bounded on } (X,\|\cdot\|)) \end{aligned}$$

and then

$$\left\|u_f(z)\right\| = \sup_{x_1, \ldots, x_{n-1} \neq 0} \frac{\left|\left(u_f(z)\right)\left(x_1, \ldots, x_{n-1}\right)\right|}{\left\|x_1\right\| \cdots\left\|x_{n-1}\right\|} \leq \|f\|_{n,1}\|z\|$$

which is finite. This implies

$$\sup_{z \neq 0} \frac{\left\|u_f(z)\right\|}{\|z\|} \leq \|f\|_{n,1}$$

which is finite. Therefore $u_f$ is bounded and $\left\|u_f\right\|_{\mathrm{op}} \leq \|f\|_{n,1}$.

Finally we claim that $\|u_f\|_{\text{op}} = \|f\|_{n,1}$. Recall that we already have $\|u_f\|_{\text{op}} \leq \|f\|_{n,1}$. To show the reverse inequality, note that for $z \in X$, $u_f(z) = f_z$ is bounded. Then for $x_1, \dots, x_{n-1}, z \in X$,

$$\begin{aligned} |f(x_1, \dots, x_{n-1}, z)| &= |f_z(x_1, \dots, x_{n-1})| \\ &\leq \|f_z\|_{n,1} \|x_1\| \cdots \|x_{n-1}\| \\ &\quad (f \text{ is bounded on } (X, \|\cdot\|)) \\ &= \|u_f(z)\| \|x_1\| \cdots \|x_{n-1}\| \\ &\leq (\|u_f\|_{\text{op}} \|z\|) \|x_1\| \cdots \|x_{n-1}\| \end{aligned}$$

since $u_f$ is bounded. Hence

$$\|f\|_{n,1} = \sup_{x_1, \dots, x_{n-1}, z \neq 0} \frac{|f(x_1, \dots, x_{n-1}, z)|}{\|x_1\| \cdots \|x_{n-1}\| \|z\|} \leq \|u_f\|_{\text{op}}$$

and $\|f\|_{n,1} \leq \|u_f\|_{\text{op}}$. Therefore $\|u_f\|_{\text{op}} = \|f\|_{n,1}$, as claimed. □

**Theorem 3.2.** *Let $X$ be a real normed space of dimension $d \geq n$. Then the $n$-dual space of $(X, \|\cdot\|)$ is $B(X, X^{(n-1)})$.*

*Proof.* For a bounded multilinear $n$-functional $f$ on $(X, \|\cdot\|)$, let $u_f \in B(X, X^{(n-1)})$ be as in Proposition 3.1. Define a map $\theta$ from the $n$-dual space of $(X, \|\cdot\|)$ to $B\left(X, X^{(n-1)}\right)$ with

$$\theta(f) := u_f$$

for $f \in X^{(n)}$. We have to show that $\theta$ is isometric and bijective.

The isometricness of $\theta$ follows from Proposition 3.1.

Next we show the injectivity of $\theta$. Let $f, h$ be bounded multilinear $n$-functionals on $(X, \|\cdot\|)$ such that $\theta(f) = \theta(h)$. Then $u_f = u_h$ and for every $x_1, \dots, x_{n-1}, x_n \in X$, we have

$$\begin{aligned} f(x_1, \dots, x_{n-1}, x_n) &= (u_f(x_n))(x_1, \dots, x_{n-1}) \\ &= (u_h(x_n))(x_1, \dots, x_{n-1}) \\ &= h(x_1, \dots, x_{n-1}, x_n). \end{aligned}$$

Hence $f = h$ and $\theta$ is injective.

To show that $\theta$ is surjective, we take $u \in B\left(X, X^{(n-1)}\right)$ and have to show that there exists a bounded multilinear $n$-functional $f_u$ on $(X, \|\cdot\|)$ such that $\theta(f_u) = u$. Now we define $f_u$ an $n$-functional on $X$ where

$$f_u(x_1, \dots, x_{n-1}, x_n) := (u(x_n))(x_1, \dots, x_{n-1})$$

for $x_1, \dots, x_{n-1}, x_n \in X$. We claim that $f_u$ is multilinear and bounded on $(X, \|\cdot\|)$.

First we show that $f_u$ is multilinear. Take $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. We have

$$\begin{aligned} f_u(x_1 + y_1, \ldots, x_n + y_n) &= (u(x_n + y_n))(x_1 + y_1, \ldots, x_{n-1} + y_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \le i \le n-1} (u(x_n + y_n))(z_1, \ldots, z_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \le i \le n-1} (u(x_n) + u(y_n))(z_1, \ldots, z_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \le i \le n-1} (f(z_1, \ldots, z_{n-1}, x_n) + f(z_1, \ldots, z_{n-1}, y_n)) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \le i \le n} f(z_1, \ldots, z_{n-1}, z_n) \end{aligned}$$

and

$$\begin{aligned} f_u(\alpha_1 x_1, \ldots, \alpha_n x_n) &= (u(\alpha_n x_n))(\alpha_1 x_1, \ldots, \alpha_{n-1} x_{n-1}) \\ &= \alpha_1 \cdots \alpha_{n-1} (u(\alpha_n x_n))(x_1, \ldots, x_{n-1}) \ \ (u(\alpha_n x_n) \text{ is multilinear}) \\ &= \alpha_1 \cdots \alpha_{n-1} \alpha_n (u(x_n))(x_1, \ldots, x_{n-1}) \ \ (u \text{ is linear}) \\ &= \alpha_1 \cdots \alpha_{n-1} \alpha_n f_u(x_1, \ldots, x_{n-1}, x_n). \end{aligned}$$

Hence $f_u$ is multilinear.

Next we show that $f_u$ is bounded on $(X, \|\cdot\|)$. Take $x_1, \ldots, x_n \in X$. Then

$$\begin{aligned} |f_u(x_1, \ldots, x_{n-1}, x_n)| &= |(u(x_n))(x_1, \ldots, x_{n-1})| \\ &\le \|u(x_n)\| \|x_1\| \cdots \|x_{n-1}\| \ \ (u(x_n) \text{ is bounded}) \\ &\le (\|u\|_{\text{op}} \|x_n\|) \|x_1\| \cdots \|x_{n-1}\| \ \ (u \text{ is bounded}) \end{aligned}$$

and $f_u$ is bounded.

Hence $f_u$ is multilinear and bounded on $(X, \|\cdot\|)$, as claimed. Note that $\theta(f_u) = u_{f_u}$. Take $x_1, \ldots, x_n \in X$ and we have

$$(u(x_n))(x_1, \ldots, x_{n-1}) = f_u(x_1, \ldots, x_{n-1}, x_n) = ((u_{f_u})(x_n))(x_1, \ldots, x_{n-1}).$$

Then $u(x_n) = u_{f_u}(x_n)$ for $x_n \in X$, and

$$u = u_{f_u} = \theta(f_u).$$

Therefore, $\theta$ is surjective and a bijection, as required. □

Recall from [8, Theorem 2.10-2] that for normed spaces $X, Y$, the normed space $B(X, Y)$ is a Banach space if $Y$ is a Banach space. Since $\mathbb{R}$ is a Banach space, then for every normed space $X$, $X^{(1)}$ is also a Banach space. Hence Theorem 3.2 with $n = 2$ implies that $X^{(2)}$ is also a Banach space. Therefore, by induction and Theorem 3.2, we get the following theorem.

**Theorem 3.3.** *Let $X$ be a real normed space of dimension $d \ge n$. Then the $n$-dual space of $(X, \|\cdot\|)$ is a Banach space.*

## 4. The $n$-dual space of $(X, \|\cdot, \cdots, \cdot\|_G)$

In this section, we focus on normed spaces of dimension $d \ge n$ which satisfy property (G). On this space, we investigate the relationship between bounded multilinear $n$-functionals on $(X, \|\cdot, \cdots, \cdot\|_G)$ and bounded multilinear $n$-functionals on $(X, \|\cdot\|)$

(Lemma 4.3). We then use it to determine the $n$-dual space of $(X, \|\cdot, \cdots, \cdot\|_G)$ (Theorem 4.5) and show that the space is a Banach space (Theorem 4.7).

First we recall the functional $g$ and property (G) introduced by Miličić in [12]. The functional $g : X^2 \to \mathbb{R}$ is defined by

$$g(x, y) := \frac{\|x\|}{2} \left(\tau_- (x, y) + \tau_+ (x, y)\right)$$

where

$$\tau_\pm (x, y) := \lim_{t \to \pm 0} t^{-1} \left(\|x + ty\| - \|x\|\right).$$

The functional $g$ satisfies the following properties: for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$

(G1) $g(x, x) = \|x\|^2$;
(G2) $g(\alpha x, \beta y) = \alpha\beta g(x, y)$;
(G3) $g(x, x + y) = \|x\|^2 + g(x, y)$; and
(G4) $|g(x, y)| \leq \|x\| \|y\|$.

We say that a real normed space $X$ satisfies *property (G)* if the functional $g(x, y)$ is linear with respect to $y \in X$. In that case, we then call $g$ a *semi-inner product* on $X$. For example, for $1 \leq p < \infty$, the $l^p$ space satisifes property (G) (see [16]).

By using the semi-inner product $g$, we define an orthogonal relation on $X$ as follows:

$$x \perp_g y \Leftrightarrow g(x, y) = 0.$$

Let $x \in X$ and $Y = \{y_1, \ldots, y_n\} \subseteq X$. We write $\Gamma(y_1, \ldots, y_n)$ to denote the Gram determinant $\det [g(y_i, y_j)]_{i,j}$. If $\Gamma(y_1, \ldots, y_n) \neq 0$, then the vector

$$x_Y := -\frac{1}{\Gamma(y_1, \ldots, y_n)} \det \begin{bmatrix} 0 & y_1 & \cdots & y_n \\ g(y_1, x) & g(y_1, y_1) & \cdots & g(y_1, y_n) \\ \vdots & \vdots & & \vdots \\ g(y_n, x) & g(y_n, y_1) & \cdots & g(y_n, y_n) \end{bmatrix}$$

is called the Gram-Schimdt projection of the vector $x$ on $Y$.

Next let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in $X$. As in [12], we call $x_1^\circ, \ldots, x_n^\circ$ the *left $g$-orthogonal sequence* where $x_1^\circ := x_1$ and for $i = 2, \ldots, n$,

$$x_i^\circ := x_i - (x_i)_{S_{i-1}},$$

where $S_{i-1} := \operatorname{span}\{x_1, \ldots, x_{i-1}\}$. Note that if $i < j$, then $x_i^\circ \perp_g x_j^\circ$ and $g(x_i^\circ, x_j^\circ) = 0$.

**Proposition 4.1.** *Let $X$ be a real normed space of dimension $d \geq n$ which satisfies property (G). Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in $X$. Then*

$$\|x_1^\circ\| \cdots \|x_n^\circ\| \leq \|x_1, \ldots, x_n\|_G \leq n! \, \|x_1\| \cdots \|x_n\|.$$

*Proof.* First we show the right inequality. Note that

$$\begin{aligned}
\|x_1,\dots,x_n\|_G &= \sup_{\substack{f_i\in X^{(1)},\|f_i\|\le 1\\ 1\le i\le n}} \left|\det\left[f_j\left(x_i\right)\right]_{i,j}\right| \\
&= \sup_{\substack{f_i\in X^{(1)},\|f_i\|\le 1\\ 1\le i\le n}} \left|\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} f_{\sigma(i)}\left(x_i\right)\right| \text{ (by the Leibniz formula)} \\
&\le \sup_{\substack{f_i\in X^{(1)},\|f_i\|\le 1\\ 1\le i\le n}} \sum_{\sigma\in S_n}\left|\prod_{i=1}^{n} f_{\sigma(i)}\left(x_i\right)\right| \text{ (by the triangle inequality)} \\
&\le \sup_{\substack{f_i\in X^{(1)},\|f_i\|\le 1\\ 1\le i\le n}} \sum_{\sigma\in S_n}\left(\prod_{i=1}^{n}\left\|f_{\sigma(i)}\right\|\left\|x_i\right\|\right) \text{ (each } f_i \text{ is bounded)} \\
&\le \sum_{\sigma\in S_n}\left(\prod_{i=1}^{n}\left\|x_i\right\|\right) \\
&= n!\left\|x_1\right\|\cdots\left\|x_n\right\|,
\end{aligned}$$

as required.

To show the left inequality, we first show that for a fixed $x\in X$, the functional $g_x$ on $X$ defined by

$$g_x(y) := \frac{g(x,y)}{\|x\|}$$

for $y\in X$, is bounded and linear. The linearity follows since $X$ satisfies property (G). Now take $y\in X$, by (G4), we have

$$\left|g_x(y)\right| = \left|\frac{g(x,y)}{\|x\|}\right| \le \|y\|$$

and $g_x$ is bounded, as required. Hence for $x\in X$, $g_x\in X^{(1)}$. Furthermore, $\|g_x\|\le 1$.

Now note that $\|x_1,\dots,x_n\|_G = \|x_1^\circ,\dots,x_n^\circ\|_G$. This implies

$$\begin{aligned}
\|x_1,\dots,x_n\|_G = \|x_1^\circ,\dots,x_n^\circ\|_G &= \sup_{\substack{f_i\in X^{(1)},\|f_i\|\le 1\\ 1\le i\le n}} \left|\det\left[f_j\left(x_i^\circ\right)\right]_{i,j}\right| \qquad (4.1)\\
&\ge \left|\det[g_{x_j^\circ}(x_i^\circ)]_{i,j}\right| = \frac{1}{\|x_1^\circ\|\cdots\|x_n^\circ\|}\left|\det\left[g(x_j^\circ,x_i^\circ)\right]_{i,j}\right|.
\end{aligned}$$

Since $x_1^\circ,\dots,x_n^\circ$ is the left $g$-orhogonal sequence, then $g(x_i^\circ,x_j^\circ)=0$ if $i<j$. By (G1), we get $g(x_i^\circ,x_i^\circ)=\|x_i^\circ\|^2$ for $i=1,\dots,n$. This implies

$$\left|\det\left[g(x_j^\circ,x_i^\circ)\right]_{i,j}\right| = \|x_1^\circ\|^2\cdots\|x_n^\circ\|^2$$

and (4.1) become

$$\|x_1,\dots,x_n\|_G \ge \|x_1^\circ\|\cdots\|x_n^\circ\|,$$

as required. □

*Remark* 4.2. Proposition 4.1 is a generalisation of Theorem 2.2 in [16]. In [16, Theorem 2.2], Wibawa-Kusumah and Gunawan only proved Proposition 4.1 for $l^p$ spaces where $1\le p<\infty$.

**Lemma 4.3.** *Let $X$ be a real normed space of dimension $d \geq n$ which satisfies property (G). Let $f$ be a multilinear $n$-functional on $X$. Then $f$ is antisymmetric and bounded on $(X, \|\cdot\|)$ if and only if $f$ is bounded on $(X, \|\cdot, \cdots, \cdot\|_G)$. Furthermore*

$$\|f\|_{n,n} \leq \|f\|_{n,1} \leq n! \, \|f\|_{n,n} .$$

*Proof.* First suppose that $f$ is antisymmetric bounded on $(X, \|\cdot\|)$. Take linearly independent $x_1, \ldots, x_n \in X$. Then

$$f(x_1, \ldots, x_n) = f(x_1^\circ, \ldots, x_n^\circ)$$

and by the left inequality in Proposition 4.1,

$$\begin{aligned} \frac{|f(x_1, \ldots, x_n)|}{\|x_1, \ldots, x_n\|_G} &\leq \frac{|f(x_1, \ldots, x_n)|}{\|x_1^\circ\| \cdots \|x_n^\circ\|} = \frac{|f(x_1^\circ, \ldots, x_n^\circ)|}{\|x_1^\circ\| \cdots \|x_n^\circ\|} \\ &\leq \|f\|_{n,1} \ (f \text{ is bounded on } (X, \|\cdot\|)) \end{aligned}$$

which is finite. Hence $f$ is bounded on $(X, \|\cdot, \cdots, \cdot\|_G)$ and

$$\|f\|_{n,n} \leq \|f\|_{n,1} . \tag{4.2}$$

Next suppose that $f$ is bounded on $(X, \|\cdot, \cdots, \cdot\|_G)$. Then $f$ is antisymmetric. To show the boundedness of $f$ on $(X, \|\cdot\|)$, we take linearly independent $x_1, \ldots, x_n \in X$. Then by the right inequality in Proposition 4.1,

$$\begin{aligned} \frac{|f(x_1, \ldots, x_n)|}{\|x_1\| \cdots \|x_n\|} &\leq n! \frac{|f(x_1, \ldots, x_n)|}{\|x_1, \ldots, x_n\|_G} \\ &\leq n! \, \|f\|_{n,n} \ (f \text{ is bounded on } (X, \|\cdot, \cdots, \cdot\|_G)) \end{aligned}$$

which is finite. Hence $f$ is bounded on $(X, \|\cdot\|)$ and

$$\|f\|_{n,1} \leq n! \, \|f\|_{n,n} . \tag{4.3}$$

Finally, by (4.2) and (4.3), we get

$$\|f\|_{n,n} \leq \|f\|_{n,1} \leq n! \, \|f\|_{n,n} ,$$

as required. □

Now we say $u \in B(X, X^{(n-1)})$ *antisymmetric* if for $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$,

$$(u(x_n))(x_1, \ldots, x_{n-1}) = \operatorname{sgn}(\sigma) \left(u\left(x_{\sigma(n)}\right)\right) \left(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\right)$$

and then define $B_{\text{as}}(X, X^{(n-1)})$ as the collection of antisymmetric elements of $B(X, X^{(n-1)})$. Note that $B_{\text{as}}(X, X^{(n-1)})$ is also a normed space with the norm inherited from $B(X, X^{(n-1)})$ which is $\|\cdot\|_{\text{op}}$.

Note that Theorem 3.2 and Lemma 4.3 imply that every bounded multilinear $n$-functional on $(X, \|\cdot, \cdots, \cdot\|_G)$ can be identified as an element of $B_{\text{as}}(X, X^{(n-1)})$ and vice versa. Therefore Lemma 4.3 implies the following corollary and theorem.

**Corollary 4.4.** *Let $X$ be a real normed space of dimension $d \geq n$ which satisfies property (G). The function $\|\cdot\|_G$ on $B_{\text{as}}(X, X^{(n-1)})$ where*

$$\|u\|_G := \sup_{\|x_1, \ldots, x_n\|_G \neq 0} \frac{|(u(x_n))(x_1, \ldots, x_{n-1})|}{\|x_1, \ldots, x_n\|_G}$$

*for $u \in B(X, X^{(n-1)})$, defines a norm on $B_{\text{as}}(X, X^{(n-1)})$. Furthermore, $\|\cdot\|_G$ and $\|\cdot\|_{\text{op}}$ are equivalent norms on $B_{\text{as}}(X, X^{(n-1)})$ with*

$$\|u\|_G \leq \|u\|_{\text{op}} \leq n! \, \|u\|_G$$

*for* $u \in B(X, X^{(n-1)})$.

**Theorem 4.5.** *Let* $X$ *be a real normed space of dimension* $d \geq n$ *which satisfies property (G). Then the* $n$*-dual space of* $(X, \|\cdot, \cdots, \cdot\|_G)$ *is* $\left(B_{\text{as}}(X, X^{(n-1)}), \|\cdot\|_G\right)$.

The rest of this section is devoted to show that for $n \in \mathbb{N}$, the $n$-dual space of $(X, \|\cdot, \cdots, \cdot\|_G)$ is a Banach space.

**Theorem 4.6.** *Let* $X$ *be a real normed space of dimension* $d \geq n$ *which satisfies property (G). Then* $B_{\text{as}}(X, X^{(n-1)})$ *is a Banach space.*

*Proof.* Since every closed subspace of a Banach space is also a Banach space, then by Theorem 3.3, it suffices to show that $B_{\text{as}}(X, X^{(n-1)})$ is a closed subspace of $B(X, X^{(n-1)})$.

Take a sequence $\{u_m\} \subseteq B_{\text{as}}(X, X^{(n-1)})$ such that $u_m \to u$. We have to show $u \in B_{\text{as}}(X, X^{(n-1)})$. In other words, for $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$, we have to show

$$(u(x_n))(x_1, \ldots, x_{n-1}) = \operatorname{sgn}(\sigma) \left(u\left(x_{\sigma(n)}\right)\right)\left(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\right).$$

Take $x_1, \ldots, x_n \in X$ and $\sigma \in S_n$. First note that for $m \in \mathbb{N}$, we have

$$\|u(x_n) - u_m(x_n)\| = \|(u - u_m)(x_n)\| \leq \|u - u_m\|_{\text{op}} \|x_n\| \tag{4.4}$$

since $u - u_m$ is bounded. Since $u(x_n), u_m(x_n) \in X^{(n-1)}$, then $(u - u_m)(x_n)$ is bounded and for $y_1, \ldots, y_{n-1} \in X$, we have

$$\|((u - u_m)(x_n))(y_1, \ldots, y_{n-1})\| \leq \|u(x_n) - u_m(x_n)\| \|y_1\| \cdots \|y_{n-1}\|. \tag{4.5}$$

Since $u_m \to u$, then by (4.4) and (4.5), we get

$$(u_m(x_n))(y_1, \ldots, y_{n-1}) \to (u(x_n))(y_1, \ldots, y_{n-1}) \tag{4.6}$$

for $y_1, \ldots, y_{n-1} \in X$. Since $u_m$ is antisymmetric for every $m \in \mathbb{N}$, then (4.6) implies

$$(u(x_n))(x_1, \ldots, x_{n-1}) = \operatorname{sgn}(\sigma) \left(u\left(x_{\sigma(n)}\right)\right)\left(x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}\right),$$

as required. Thus $B_{\text{as}}(X, X^{(n-1)})$ is closed and then a Banach space. □

Furthermore, since $\|\cdot\|_G$ and $\|\cdot\|_{\text{op}}$ are equivalent norms on $B_{\text{as}}(X, X^{(n-1)})$, then by Theorem 4.5 and Theorem 4.6, we get the following theorem.

**Theorem 4.7.** *Let* $X$ *be a real normed space of dimension* $d \geq n$ *which satisfies property (G). Then the* $n$*-dual space of* $(X, \|\cdot, \cdots, \cdot\|_G)$ *is a Banach space.*

## References


1. H. Batkunde, H. Gunawan, and Y.E.P. Pangalela, *Bounded linear functionals on the* $n$*-normed space of* $p$*-summable sequences*, Acta Univ. M. Belii Ser. Math. **21** (2013), 66–75.
2. S. Gähler, *Lineare* 2*-normierte räume*, Math. Nachr. **28** (1964), 1–43.
3. S. Gähler, *Untersuchungen über verallgemeinerte* $m$*-metrische räume I*, Math. Nachr. **40** (1969), 165–189.
4. S. Gähler, *Untersuchungen über verallgemeinerte* $m$*-metrische räume II*, Math. Nachr. **40** (1969), 229–264.
5. S. Gähler, *Untersuchungen über verallgemeinerte* $m$*-metrische räume III*, Math. Nachr. **41** (1969), 23–36.
6. S.M. Gozali, H. Gunawan and O. Neswan, *On* $n$*-norms and* $n$*-bounded linear functionals in a Hilbert space*, Ann. Funct. Anal. **1** (2010), 72–79.
7. H. Gunawan, *The space of* $p$*-summable sequences and its natural* $n$*-norm*, Bull. Aust. Math. Soc. **64** (2001), 137–147.
8. E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, Inc. 1978.

9. S.N. Lal and M. Das, 2-*functionals and some extension theorems in linear spaces*, Indian J. Pure Appl. Math. **13** (1982), 912–919.
10. Z. Lewandowska, *Bounded* 2-*linear operators on* 2-*normed sets*, Glas. Mat. Ser. III **39** (2004), 301–312.
11. R. Malcheski and Z. Cvetkovski, *Dual space of the space of bounded linear $n$-functionals*, Mat. Bilten **36** (2012), 47–53.
12. P.M. Miličić, *On the Gram-Schmidt projection in normed spaces*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. **4** (1993), 85–92.
13. Y.E.P. Pangalela and H. Gunawan, *The $n$-dual space of the space of $p$-summable sequences*, Math. Bohem. **138** (2013), 439–448.
14. A.L. Soenjaya, *On $n$-bounded and $n$-continuous operator in $n$-normed space*, J. Indones. Math. Soc. **18** (2012), 45–56.
15. A. White, 2-*Banach spaces*, Math. Nachr. **42** (1969), 43–60.
16. R.A. Wibawa-Kusumah and H. Gunawan, *Two equivalent $n$-norms on the space of $p$-summable sequences*, Period. Math. Hungar. **67** (2013), 63–69.

Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand

*E-mail address*: yosafat.pangalela@maths.otago.ac.nz