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LACUNARY SELF-SIMILAR FRACTAL SETS AND INTERSECTION OF CANTOR SETS

(submitted by M. A. Malakhaltsev)

Abstract. The problem on intersection of Cantor sets was examined in many papers. To solve this problem, we introduce the notion of lacunary self-similar set. The main difference to the standard (Hutchinson) notion of self-similarity is that the set of similarities used in the construction may vary from step to step in a certain way. Using a modification of method described in [3], [4], we find the Hausdorff dimension of a lacunary self-similar set.

In one form or another, the problem on intersection of Cantor sets was examined in many papers. In [5], [6], [8] the intersection of so-called thick Cantor sets was considered. In [2] it was proved that the intersection of two standard Cantor sets can be of any Hausdorff dimension from 0 to $\frac{\ln 2}{\ln 3}$. In [7] there was investigated the intersection of standard Cantor sets one of which is translated by an element of Cantor set.

To solve this problem, we introduce the notion of lacunary self-similar set. The main difference to the standard (Hutchinson) notion of self-similarity is that the set of similarities used in the construction may vary from step to step in a certain way. Using a modification of method described in [3], [4], we find a Hausdorff dimension of a lacunary self-similar set.

Let K be the (ordinary) Cantor set in $[0,1]$. It is shown that for almost all $a \in [0,1]$ (with respect to Lebesgue's measure) the set $(a+K)\bigcap K$ is the lacunary self-similar set and its Hausdorff dimension is equal to $\frac{\ln 2}{3\ln 3}$. Then we give some generalization of this fact.

1. Lacunary self-similar sets

Let us recall the definition of Hausdorff measure, dimension and metric. Let U be a nonempty set in $\mathbb{R}^n$, $|U| = \sup\{|x-y| : x, y \in U\}$. If $E \subset \bigcup\limits_i U_i$, and

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$0 < |U_i| \le \delta$ for each i, then $\{U_i\}$ is called a δ-covering of E. Given $E \subset \mathbb{R}^n$ and $s \ge 0$, for any $\delta > 0$ the outer measure $\mathcal{H}^s_\delta(E)$ of E is defined by the equality

$$\mathcal{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s,$$

where inf is taken over all countable δ-coverings $\{U_i\}$ of E.

The Hausdorff outer s-dimensional measure $\mathcal{H}^s$ on $\mathbb{R}^n$ is defined by the equality

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

This limit exists, but can be infinite because $\mathcal{H}^s_\delta$ does not decrease as $\delta \to 0$.

It is known that the σ-algebra of $\mathcal{H}^s$-measurable sets contains the Borel sets. And for any $E \subset \mathbb{R}^n$ there exists a unique number $\dim(E) \in [0, n]$ such that

$$\mathcal{H}^t(E) = \begin{cases} \infty, & t < \dim(E) \\ 0, & t > \dim(E) \end{cases} .$$

$\dim(E)$ is called the Hausdorff dimension (the fractal dimension) of E.

If $E \subset \mathbb{R}^n$, and $\delta \ge 0$, the δ-parallel body of E is the closed set

$$[E]_\delta = \{x \in \mathbb{R}^n : \inf_{y \in E} |x - y| \le \delta\}.$$

Let Γ be a collection of nonempty compact subset of $\mathbb{R}^n$. The Hausdorff metric d on Γ is

$$d(E, F) = \inf\{\delta : E \subset [F]_\delta, F \subset [E]_\delta\}.$$

It is known that Γ endowed with the Hausdorff metric d is a complete metric space .

Let us construct a lacunary self-similar set and find its Hausdorff dimension. Note that the collection of lacunary self-similar sets contains the well-known self-similar sets.

A mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called a similitude if $|\psi(x) - \psi(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^n$, where $r < 1$. Clearly, any similitude is a continuous mapping.

Let $\{\psi_j\}_{j=1}^m$ be a set of similitudes with ratios $\{r_j\}_{j=1}^m$. We set $r = \max\limits_{1 \le j \le m} r_j$. Let $\mathcal{N}_k \subseteq \{1, \ldots, m\}$ be a nonempty set, for any positive integer k (if $\mathcal{N}_k = \{1, \ldots, m\}$ for all k, then the following construction gives a self-similar set). We set $\mathcal{L}^k = \prod\limits_{i=1}^{k} \mathcal{N}_i$, $\mathcal{L}^k_p = \prod\limits_{i=p}^{k} \mathcal{N}_i$ $(p \le k)$. For any sequence $(j_1 \ldots j_k) \in \mathcal{L}^k$ and any set F we set $F_{j_1 \ldots j_k} = (\psi_{j_1} \circ \ldots \circ \psi_{j_k})(F)$. Now, let Γ be a collection of nonempty compact sets, and $\psi^k : \Gamma \to \Gamma$,

$$\psi^k(F) = \bigcup_{\mathcal{L}^k} (\psi_{j_1} \circ \ldots \circ \psi_{j_k})(F) = \bigcup_{\mathcal{L}^k} F_{j_1 \ldots j_k} .$$

for any positive integer k.

Theorem 1. *There exists a unique set $E \in \Gamma$ such that for all $F \in \Gamma$*

$$\psi^k(F) \stackrel{k \to \infty}{\longrightarrow} E$$

with respect to the Hausdorff metric d.

$\diamond$ If $F, G \in \Gamma$, then, by the definition of d,

$$d(\psi^k(F), \psi^k(G)) \le \sup_{\mathcal{L}^k} d(F_{j_1\dots j_k}, G_{j_1\dots j_k}) \le r^k d(F, G)\ .$$

Take any $F \in \Gamma$ and set $M = \sup_{1\le j\le m} d(F, F_j)$. For any positive integers p and q ($p < q$),

$$d(\psi^p(F), \psi^q(F)) \le d(\psi^p(F), \psi^p(\bigcup_{\mathcal{L}^q_{p+1}} F_{j_{p+1}\dots j_q})) \le r^p d(F, \bigcup_{\mathcal{L}^q_{p+1}} F_{j_{p+1}\dots j_q})$$

$$\le r^p \sup_{\mathcal{L}^q_{p+1}} d(F, F_{j_{p+1}\dots j_q}) \le r^p \sup_{\mathcal{L}^q_{p+1}} (d(F, F_{j_{p+1}}) + d(F_{j_{p+1}}, F_{j_{p+1}j_{p+2}}) + \cdots$$

$$+ d(F_{j_{p+1}\dots j_{q-1}}, F_{j_{p+1}\dots j_q}) \le r^p (M + rM + r^2 M + \cdots r^{q-p-1} M) < r^p \frac{M}{1-r}\ .$$

Hence $d(\psi^p(F), \psi^q(F)) \overset{p\to\infty}{\longrightarrow} 0$. As Γ endowed with the Hausdorff metric d is a complete metric space, $\psi^k(F)$ converges to a nonempty compact set E.

For any $G \in \Gamma$

$$d(\psi^k(G), E) \le d(\psi^k(G), \psi^k(F)) + d(\psi^k(F), E) \overset{k\to\infty}{\longrightarrow} 0\ ,$$

and so E is unique. $\diamond$

Let $\mathcal{K}$ be the set of finite sequences $\{(j_1, \dots, j_k)\}_{\mathcal{L}^k, k}$. A finite subset $\mathcal{P} \subset \mathcal{K}$ is called a tree if for any $(j_1, \dots, j_k) \in \mathcal{K}$ there exists $(i_1, \dots, i_p) \in \mathcal{P}$ such that $j_q = i_q$ for all $q = 1, \dots, \min\{k, p\}$, moreover, if $p \le k$, then this $(i_1, \dots, i_p) \in \mathcal{P}$ is unique. Note that for any k the set $\mathcal{L}^k$ is a tree.

Given a tree $\mathcal{P}$, we set $p = \inf\{k : (j_1, \dots, j_k) \in \mathcal{P}\}$ (the length of the shortest branch of the tree) and $q = \sup\{k : (j_1, \dots, j_k) \in \mathcal{P}\}$ (the length of the longest branch of the tree).

Lemma 1. *For any tree $\mathcal{P}$ and any non-negative numbers $\{a_k\}_{k=1}^m$ we have*

$$\sum_{\mathcal{P}} a_{j_1} \dots a_{j_k} \ge \inf_{p\le k\le q} \sum_{\mathcal{L}^k} a_{j_1} \dots a_{j_k}\ ,$$

where p is the length of the shortest branch of $\mathcal{P}$, and q is the length of the longest one.

$\diamond$ Let $\mathcal{P}_{j_1\dots j_p} = \{(i_1, \dots, i_k) \in \mathcal{P} : (j_1, \dots, j_p) = (i_1, \dots, i_p)\}$, where $(j_1, \dots, j_p) \in \mathcal{L}^p$. If $(l_1, \dots, l_p) \in \mathcal{L}^p$, then

$$\mathcal{P}' = \bigcup_{\mathcal{L}^p} \bigcup_{\mathcal{P}_{l_1\dots l_p}} (j_1, \dots, j_p, l_{p+1}, \dots, l_k) \tag{1}$$

is a tree. Note that the first p elements of $(j_1, \dots, j_p, l_{p+1}, \dots, l_k)$ run through $\mathcal{L}^p$ and the remaining elements run through the ends of $(l_1, \dots, l_k) \in \mathcal{P}_{l_1\dots l_p}$.

We set

$$\mu_{j_1\dots j_p} = \begin{cases} \sum\limits_{\mathcal{P}_{j_1\dots j_p}} a_{j_{p+1}} \dots a_{j_k} & \text{if} \quad (j_1, \dots, j_p) \notin \mathcal{P} \\ 1 & \text{if} \quad (j_1, \dots, j_p) \in \mathcal{P}\ , \end{cases}$$

so

$$\sum_{\mathcal{P}} a_{j_1} \dots a_{j_k} = \sum_{\mathcal{L}^p} a_{j_1} \dots a_{j_p} \mu_{j_1\dots j_p}\ . \tag{2}$$

Let $\mu = \inf_{\mathcal{L}^p} \mu_{j_1 \dots j_p}$. From (2) it follows that

$$\sum_{\mathcal{P}} a_{j_1} \dots a_{j_k} \geq \sum_{\mathcal{L}^p} a_{j_1} \dots a_{j_k}$$

if $\mu = 1$.

If $\mu < 1$, then there exists $(l_1, \dots, l_p) \notin \mathcal{P}$ such that $\mu = \mu_{l_1 \dots l_p}$. Thus, using (2), we get

$$\sum_{\mathcal{P}} a_{j_1} \dots a_{j_k} \geq \mu_{l_1 \dots l_p} \sum_{\mathcal{L}^p} a_{j_1} \dots a_{j_p}$$

$$= \sum_{\mathcal{L}^p} a_{j_1} \dots a_{j_p} \sum_{\mathcal{P}_{l_1 \dots l_p}} a_{l_{p+1}} \dots a_{l_k} = \sum_{\mathcal{P}'} a_{j_1} \dots a_{j_k} \ .$$

Thus it follows from (1) that $\mathcal{P}'$ is a tree. Clearly, the length of the shortest branch of $\mathcal{P}'$ is equal to $p' > p$, and the length of the longest branch of $\mathcal{P}'$ is equal to $q' \leq q$.

We proceed in this way until, after a finite number of steps, we reach a tree $\mathcal{L}^k$, where $p \leq k \leq q$. This completes the proof. $\diamond$

It is clear that $\liminf_{k \to \infty} \sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t$ does not increase as t increases from 0 to ∞. Furthermore, if $t < t'$, then

$$\inf_{l \geq k} \sum_{\mathcal{L}^l} (r_{j_1} \dots r_{j_l})^t = \inf_{l \geq k} \sum_{\mathcal{L}^l} (r_{j_1} \dots r_{j_l})^{t' + (t-t')} \geq (r^k)^{(t-t')} \inf_{l \geq k} \sum_{\mathcal{L}^l} (r_{j_1} \dots r_{j_l})^{t'} \ .$$

Hence there exists a unique number $s \geq 0$ such that

$$\liminf_{k \to \infty} \sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t = \begin{cases} \infty, & t < s \\ 0, & t > s \end{cases} \ . \tag{3}$$

Lemma 2. *Let $\{V_i\}$ be a collection of disjoint open subset of $\mathbb{R}^n$ such that each V_i contains a ball of radius $c_1\rho$ and is contained in a ball of radius $c_2\rho$. Then any ball B of radius ρ intersects, at most, $(1+2c_2)^n c_1^{-n}$ of the sets $\bar{V}_i$ (the bar denotes closure).*

$\diamond$ If $\bar{V}_i \bigcap B \neq \emptyset$, $\bar{V}_i$ is contained in a ball concentric with B and of radius $(1+2c_2)\rho$. Let h elements of the collection $\{\bar{V}_i\}$ intersect B, then summing up the volumes of the corresponding interior balls, we get $h(c_1\rho)^n \leq (1+2c_2)^n \rho^n$, which proves the Lemma. $\diamond$

We say that the set of similitudes $\{\psi_j\}_{j=1}^m$ satisfies the open set condition if there exists a nonempty bounded open set $V \subset \mathbb{R}^n$ such that

$$\psi_j(V) \subset V \text{ and } \psi_j(V) \bigcap \psi_i(V) = \emptyset \tag{4}$$

for any $i, j = \overline{1, m},\ j \neq i$. Thus the sets $\{V_{j_1 \dots j_k}\}_{\mathcal{L}^k, k}$ form a net in the sense that any two of sets from the collection are either disjoint, or one set is included into the other one. The collection $\{V_{j_1 \dots j_k}\}_{\mathcal{P}}$ is disjoint for any tree $\mathcal{P}$.

If (4) holds, then $\{\psi^k(\bar{V})\}_k$ is a decreasing sequence of compact sets, which converges to E with respect to the Hausdorff metric by Theorem 1. It follows from the definition of Hausdorff metric that $E = \bigcap_{k=1}^{\infty} \psi^k(\bar{V})$.

The set E from Theorem 1 is called lacunary self-similar, if the open set conditions holds true.

Theorem 2. *The Hausdorff dimension of the lacunary self-similar set E is equal to* $\dim(E) = s$, *where s is determined by (3).*

$\diamond$ For any $\mathcal{L}^k$ (moreover, for any tree $\mathcal{P}$) the collection $\{\bar{V}_{j_1\dots j_k}\}_{\mathcal{L}^k}$ is a cover of E. As

$$\liminf_{k\to\infty}\sum_{\mathcal{L}^k}|\bar{V}_{j_1\dots j_k}|^t = |\bar{V}|\liminf_{k\to\infty}\sum_{\mathcal{L}^k}(r_{j_1}\dots r_{j_k})^t = \begin{cases} \infty, & t<s \\ 0, & t>s \end{cases}$$

and $|\bar{V}_{j_1\dots j_k}| \le r^k|\bar{V}| \overset{k\to\infty}{\longrightarrow} 0$, we get $\dim(E)\le s$.

To prove the opposite inequality we show that, if $t\in[0,s)$, then $\mathcal{H}^t(E)=\infty$. Since E is compact, it is sufficient to prove that

$$\liminf_{\delta\to 0}\sum|U_i|^t = \infty\ ,$$

where the infimum is taken over all finite δ-covers $\{U_i\}$ of E. Given any δ-cover $\{U_i\}_{i=1}^N$ of E, we can cover E by balls $\{B_i\}_{i=1}^N$ with $|B_i|\le 2|U_i|$, then

$$\sum_{i=1}^N|U_i|^t \ge 2^{-t}\sum_{i=1}^N|B_i|^t\ . \tag{5}$$

Suppose that an open set V such that (4) holds true contains a ball of radius c_1 and is contained in a ball of radius c_2. Take any $\rho\in(0,1)$. For each infinite sequence $(j_1,j_2,\dots)$ with $(j_1,\dots,j_k)\in\mathcal{L}^k$ for all k, curtail the sequence at the least value of k such that

$$(\min_{1\le j\le m} r_j)\rho \le r_{j_1}\dots r_{j_k}\le\rho\ , \tag{6}$$

and let us denote by $\mathcal{P}$ the set of finite sequences obtained in this way. It is clear that $\mathcal{P}$ is a tree.

Each $V_{j_1\dots j_k}$ contains a ball of radius $c_1r_{j_1}\dots r_{j_k}$ and hence a ball of radius $c_1\rho(\min\limits_{1\le j\le m} r_j)$, by (6), and is contained in a ball of radius $c_2r_{j_1}\dots r_{j_k}$ and therefore of radius $c_2\rho$. By Lemma 2, any ball B of radius ρ intersects, at most, $h=(c_1+2c_2)^n(c_1\min\limits_{1\le j\le m} r_j)^{-n}$ sets of collection $\{\bar{V}_{j_1\dots j_k}\}_{\mathcal{P}}$. Note that h does not depend on ρ.

Let B be a ball of radius ρ. We denote

$$\mathcal{D}=\{(j_1,\dots,j_k)\in\mathcal{P}\ :\ \bar{V}_{j_1\dots j_k}\bigcap B\ne\emptyset\}\ .$$

The set $\mathcal{D}$ contains, at most, h elements. Hence

$$\sum_{\mathcal{D}}(r_{j_1}\dots r_{j_k})^t\le h\rho^t = h|B|^t\ ,$$

hence

$$|B|^t\ge h^{-1}\sum_{\mathcal{D}}(r_{j_1}\dots r_{j_k})^t\ . \tag{7}$$

Suppose that for any ball B_i from (5) we have constructed a tree $\mathcal{P}_i$ and a subset $\mathcal{D}_i \subset \mathcal{P}_i$. It follows from (7) that

$$\sum_{i=1}^{N} |B_i|^t \geq h^{-1} \sum_{i=1}^{N} \sum_{\mathcal{D}_i} (r_{j_1} \dots r_{j_k})^t . \tag{8}$$

Let $\mathcal{S} = \bigcup_{i=1}^{N} \mathcal{D}_i$. From the construction of the $\mathcal{S}$ we see that the collection $\{\bar{V}_{j_1 \dots j_k}\}_{\mathcal{S}}$ is a $(2c_2\delta)$-cover of E. We set

$$\mathcal{P} = \{(j_1, \dots, j_k) \in \mathcal{S} \ : \ \text{for any } l < k \ (j_1, \dots, j_l) \notin \mathcal{S}\} .$$

Clearly, $\mathcal{P} \subset \mathcal{S}$ is a tree, hence the collection $\{\bar{V}_{j_1 \dots j_k}\}_{\mathcal{P}}$ is a $(2c_2\delta)$-cover of E.

It follows from Lemma 1 that

$$\sum_{i=1}^{N} \sum_{\mathcal{D}_i} (r_{j_1} \dots r_{j_k})^t \geq \sum_{\mathcal{P}} (r_{j_1} \dots r_{j_k})^t \geq \inf_{p \leq k \leq q} \sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t , \tag{9}$$

where p and q are the lengths of the shortest and the longest branches of the tree $\mathcal{P}$, respectively.

Thus by (5), (8), (9),

$$\sum_{i=1}^{N} |U_i|^t \geq 2^{-t} h^{-1} \inf_{p \leq k \leq q} \sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t .$$

As $|B_i| \to 0$ as $\delta \to 0$, we have $p \to \infty$. It follows from the equality $\liminf\limits_{k \to \infty} \sum\limits_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t = \infty$, $t < s$, that $\mathcal{H}^t(E) = \infty$. $\diamond$

Let $\{\mathcal{M}_l\}_{l=1}^{2^m-1}$ be the collection of all nonempty subsets of $\{1, \dots, m\}$. We will denote by $\chi_{\mathcal{M}_l}$ the characteristic function of $\mathcal{M}_l$

$$\chi_{\mathcal{M}_l}(\mathcal{N}_k) = \begin{cases} 1 & \text{if} \quad \mathcal{M}_l = \mathcal{N}_k \\ 0 & \text{if} \quad \mathcal{M}_l \neq \mathcal{N}_k \end{cases}$$

for any positive integer k and $l = 1, \dots, (2^m-1)$. Set $f_l(k) = \sum\limits_{j=1}^{k} \chi_{\mathcal{M}_l}(\mathcal{N}_j)$.

Corollary 1. *Suppose that* $\lim\limits_{k \to \infty} \frac{f_l(k)}{k} = P_l$ *for any* $l = 1, \dots, (2^m-1)$ *and* s *is the number such that*

$$\prod_{l=1}^{2^m-1} \left(\sum_{\mathcal{M}_l} r_j^s \right)^{P_l} = 1 . \tag{10}$$

Then the Hausdorff dimension of the lacunary self-similar set E *is equal to* $\dim(E) = s$.

$\diamond$ We have

$$\sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t = \prod_{i=1}^{k} \left(\sum_{\mathcal{N}_i} r_j^t \right) = \prod_{l=1}^{2^m-1} \left(\sum_{\mathcal{M}_l} r_j^t \right)^{f_l(k)}$$

$$= \left(\prod_{l=1}^{2^m-1} \left(\sum_{\mathcal{M}_l} r_j^t \right)^{(\frac{f_l(k)}{k} - P_l)} \prod_{l=1}^{2^m-1} \left(\sum_{\mathcal{M}_l} r_j^t \right)^{P_l} \right)^k$$

for any positive integer k. Hence

$$\liminf_{k\to\infty} \sum_{\mathcal{L}^k} (r_{j_1} \dots r_{j_k})^t = \begin{cases} \infty, & t < s \\ 0, & t > s \end{cases} . \quad \diamond$$

For instance, let $\Omega = \{\omega = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \dots)\}$ be the set of sequences of independent random nonempty subsets of $\{1, \dots, m\}$, where $\mathcal{N}_k = \mathcal{M}_l$ with probability P_l, $l = 1, \dots, (2^m - 1)$. We will denote by P the probability measure on Ω. Using Theorem 1 we construct the lacunary self-similar set E_ω for any $\omega \in \Omega$, if the open set conditions holds true. From the strong law of large numbers it may be concluded that

$$P\{\omega \in \Omega : \lim_{k\to\infty} \frac{f_l(k)}{k} = P_l\} = 1 ,$$

and so

$$P\{\omega \in \Omega : \dim(E_\omega) = s\} = 1 ,$$

where s is determined by (10).

2. The intersection of Cantor's sets

Let us set $K_0 = [0, 1]$, $K_1 = [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$, $K_2 = [0, \frac{1}{9}] \bigcup [\frac{2}{9}, \frac{1}{3}] \bigcup [\frac{2}{3}, \frac{7}{9}] \bigcup [\frac{8}{9}, 1]$, etc., K_{i+1} is obtained from K_i by cutting out the open middle part of each interval in K_i. Thus, K_i consists of 2^i closed intervals of length 3^{-i}. Taking the intersection of all sets K_i, we obtain the Cantor set $K = \bigcap\limits_{i=0}^{\infty} K_i$.

Consider the similitudes of the real line $\psi_1(x) = \frac{x}{3}$, $\psi_2(x) = \frac{x}{3} + \frac{2}{3}$. Since

$$K_i = \bigcup_{j_1 \dots j_i} (\psi_{j_1} \circ \dots \circ \psi_{j_i})([0, 1]) ,$$

where the sum is taken over all i-tuples $\{j_1 \dots j_i\}$, the Cantor set is the self-similar set (it is the particular case of a lacunary self-similar set). It is known that the Hausdorff dimension of the Cantor set is $\dim(K) = \frac{\ln 2}{\ln 3}$.

For $a \in [0, 1]$, we set $K + a = \{x : x - a \in K\}$, $E_a = K \bigcap (K + a)$.

Theorem 3. *For almost all $a \in [0, 1]$ with respect to Lebesgue's measure, E_a is a lacunary self-similar set and its Hausdorff dimension is equal to $\frac{\ln 2}{3 \ln 3}$.*

$\diamond$ For any non-negative integer i the set $S_{a,i} = K_i \bigcap (K_i + a)$ is a covering of E_a . It is clear that $\{S_{a,i}\}$ is a decreasing sequence of sets, and $E_a = \bigcap\limits_{i=0}^{\infty} S_{a,i}$. From the definition of δ-parallel body we conclude that $S_{a,i} \subset [S_{a,i+1}]_{3^{-i}}$. Hence

$$d(S_{a,i}, E_a) \xrightarrow{i\to\infty} 0 . \tag{11}$$

Let $a = 0, a_1 a_2 a_3 \dots$ be the triadic expansion of $a \in [0, 1]$. We can exclude from the consideration the set of a for which the triadic expansion is ambiguously determined, because Lebesgue's measure of this set, is equal to zero (for these a the set E_a is either finite or similar to Cantor's set). We set

$$\lambda_a(k) = \begin{cases} 0 & \text{if } \sum_{i=0}^{k-1} a_i = 0 \pmod 2, \\ 1 & \text{if } \sum_{i=0}^{k-1} a_i = 1 \pmod 2, \end{cases}$$

for any positive integer k (we state $a_0 = 0$). Let us consider the nonempty sets $\mathcal{N}_{a,k} \subset \{1, 2\}$

$$\mathcal{N}_{a,k} = \begin{cases} \{1\} & \text{if } \ a_k \neq 2, \ \lambda_a(k) = 1, \\ \{2\} & \text{if } \ a_k \neq 0, \ \lambda_a(k) = 0, \\ \{1, 2\} & \text{if } \ a_k = 2, \ \lambda_a(k) = 1 \text{ or } a_k = 0, \ \lambda_a(k) = 0 \ . \end{cases}$$

The similitudes $\psi_1(x) = \frac{x}{3}$, $\psi_2(x) = \frac{x}{3} + \frac{2}{3}$ with ratios $r_1 = r_2 = \frac{1}{3}$ and sets $\mathcal{N}_{a,i}$ generate (see. Section 1) the lacunary self-similar set G_a (the open set condition holds for the open interval $(0, 1)$). As in Section 1 we set

$$\mathcal{L}_a^k = \prod_{i=1}^{k} \mathcal{N}_{a,i} \ , \quad \psi_a^k(F) = \bigcup_{\mathcal{L}_a^k} (\psi_{j_1} \circ \ldots \circ \psi_{j_k})(F) = \bigcup_{\mathcal{L}_a^k} F_{j_1 \ldots j_k} \ ,$$

where F is a nonempty compact set. By Theorem 1,

$$d(G_a, \psi_a^k([0, 1])) \stackrel{k\to\infty}{\longrightarrow} 0 \ . \tag{12}$$

It follows from the construction that $S_{a,k} \subset \psi_a^k([0, 1])$ and $\psi_a^k([0, 1]) \subset [S_{a,k}]_{3^{-k}}$, hence

$$d(\psi_a^k([0, 1]), S_{a,k}) \stackrel{k\to\infty}{\longrightarrow} 0 \ . \tag{13}$$

Thus by (11), (12), (13),

$$d(G_a, E_a) \leq d(G_a, \psi_a^k([0, 1])) + d(\psi_a^k([0, 1]), S_{a,k}) + d(S_{a,k}, E_a) \stackrel{k\to\infty}{\longrightarrow} 0 \ ,$$

so $E_a = G_a$.

To find the Hausdorff dimension of E_a it is sufficient to calculate s_a for which

$$\liminf_{k\to\infty} \sum_{\mathcal{L}_a^k} (r_{j_1} \ldots r_{j_k})^t = \liminf_{k\to\infty} 3^{-tk} M_a(k) = \begin{cases} \infty, & t < s_a \\ 0, & t > s_a \end{cases} \tag{14}$$

where $M_a(k)$ is the number of elements of $\mathcal{L}_a^k$.

Let us denote by χ_Z the characteristic function of a set Z. If Z is an integer, we consider Z as the subset of the set of integers. By construction we have

$$M_a(k) = 2^{f_a(k)}, \quad \text{where} \quad f_a(k) = \sum_{i=1}^{k} (\chi_0(a_i)\chi_0(\lambda_a(i)) + \chi_2(a_i)\chi_1(\lambda_a(i))) \ . \tag{15}$$

It is clear that, given a triadic expansion of a, we can calculate the Hausdorff dimension of E_a with the use of (14) and (15). However, for an arbitrary irrational number we cannot even calculate explicitly number characteristics of its triadic expansion, e.g. the proportion of digits $0, 1, 2$ in its triadic expansion. We only known [1] that almost all numbers in $[0, 1]$ (with respect to Lebesgue's measure) have the property that the proportion of digits $0, 1, 2$ in their triadic expansion equals $\frac{1}{3}$.

Let $\Omega = \{\omega = (a_1, a_2, a_3, \ldots)\}$, where a_k are independent random variables taking values $0, 1, 2$ with probability $\frac{1}{3}$. Let us consider the mapping $a : \Omega \to [0,1]$, $a = a(\omega) = \sum\limits_{k=1}^{\infty} \frac{a_k}{3^k}$, then ω is the triadic expansion of a. It is known [1] that for any Borel set $B \subset [0,1]$ we have $P\{\omega : a(\omega) \in B\} = \mathrm{mes(B)}$, where P is a probability measure on Ω and "mes" is the Lebesgue measure.

Now let us consider the mapping $\gamma : \Omega \to \hat{\Omega}$, $\gamma(\omega) = \hat{\omega} = (\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$, where

$$\begin{array}{lll}
\hat{a}_k = 0 & \text{if} & a_k = 0,\ \lambda_a(k) = 0; \\
\hat{a}_k = 1 & \text{if} & a_k = 1,\ \lambda_a(k) = 0; \\
\hat{a}_k = 2 & \text{if} & a_k = 2,\ \lambda_a(k) = 0; \\
\hat{a}_k = 3 & \text{if} & a_k = 0,\ \lambda_a(k) = 1; \\
\hat{a}_k = 4 & \text{if} & a_k = 1,\ \lambda_a(k) = 1; \\
\hat{a}_k = 5 & \text{if} & a_k = 2,\ \lambda_a(k) = 1\ .
\end{array}$$

The probability P on Ω induces the probability $\hat{P} = \gamma_{\#}P$ on $\hat{\Omega}$. With respect to P the sequence of random variables $\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots$ is a homogeneous Markov chain with the starting distribution $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$ and the matrix of transition probabilities

$$\|p_{i,j}\| = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}.$$

It follows from the strong law of large numbers that

$$P\left\{\omega : \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \chi_0(a_i) = \frac{1}{3}\right\} = 1\ .$$

Hence

$$\hat{P}\left\{\hat{\omega} : \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (\chi_0(\hat{a}_i) + \chi_3(\hat{a}_i)) = \frac{1}{3}\right\} = 1\ . \tag{16}$$

Since the starting distribution and the matrix of transition probability do not change if we interchange the events $a_k = 3$ and $a_k = 5$, and the set in the braces in (16) depends only on the starting distribution and the matrix of transition probability, we have

$$\hat{P}\left\{\hat{\omega} : \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (\chi_0(\hat{a}_i) + \chi_5(\hat{a}_i)) = \frac{1}{3}\right\} = 1\ .$$

It follows from $\chi_0(\hat{a}_i) = \chi_0(a_i)\chi_0(\lambda_a(i))$ and $\chi_5(\hat{a}_i) = \chi_2(a_i)\chi_1(\lambda_a(i))$ that

$$\begin{aligned}
\mathrm{mes}&\left\{\mathrm{a} \in [0,1] : \lim_{\mathrm{k}\to\infty} \frac{\mathrm{f_a(k)}}{\mathrm{k}} = \frac{1}{3}\right\} \\
&= P\left\{\omega : \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} (\chi_0(a_i)\chi_0(\lambda_a(i)) + \chi_2(a_i)\chi_1(\lambda_a(i))) = \frac{1}{3}\right\}
\end{aligned}$$

$$= \hat{P}\left\{\hat{\omega} : \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}(\chi_0(\hat{a}_i)+\chi_5(\hat{a}_i)) = \frac{1}{3}\right\} = 1 \ .$$

From this and Corollary 1 we conclude that for almost all $a \in [0,1]$ the Hausdorff dimension of E_a equals s, where s is determined by

$$3^{-\frac{2}{3}s}(3^{-s}+3^{-s})^{\frac{1}{3}} = 1\frac{\ln 2}{3\ln 3} \ . \quad \diamond$$

The next Theorem gives a generalization of the previous one. Let us denote by K^p the set of numbers in $[0,1]$ which admit base p-expansion without odd digits. Note that $K^2 = \{0\}$ and K^3 is the middle third Cantor set. Given $a \in [0,1]$, we set $K^p + a = \{x : x - a \in K^p\}, E^p_a = K^p \bigcap (K^p + a)$.

Theorem 4. *If p is an even number, then for almost all $a \in [0,1]$ with respect to Lebesgue's measure, $E^p_a = \emptyset$.*

If p is an odd number, then for almost all $a \in [0,1]$ with respect to Lebesgue's measure, E^p_a is a lacunary self-similar set and

$$\dim(E^p_a) = \frac{2\ln((\frac{p+1}{2})!) - \ln(\frac{p+1}{2})}{p\ln p} \ .$$

$\diamond$ The proof of the first statement is trivial. The second statement can be proved just in the same way as Theorem 3. $\diamond$

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