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MIXED HYBRID FINITE ELEMENT SCHEME FOR STEFAN PROBLEM WITH PRESCRIBED CONVECTION

Abstract. We construct a mixed hybrid finite element scheme of lowest order for the Stefan problem with prescribed convection and suggest and investigate an iterative method for its solution. In the iterative method we use a preconditioner constructed by using "standard" finite element approximation of Laplace operator on a finer grid.

The proposed approach develops the results of [1], where a spectrally equivalent preconditioner for the condensed matrix in mixed hybrid finite element approximation for linear elliptic equation was constructed.

1. Introduction

Stefan problem with prescribed convection serves as a mathematical model for the heat transfer and solidification process in the metal casting (see [2, 3]). Commonly used numerical methods of its solving are based on the implicit or semi-implicit mesh approximations in time variable with lowest order finite element approximation in space variables [4, 5].

Because in the applied problems both the temperature fields and fluxes are of practical interest, mixed and mixed hybrid finite element schemes appear as important method for its numerical solution.

Mixed and mixed hybrid finite element schemes are thoroughly investigated for the linear boundary-value problems (cf. [6, 7] and bibliography therein), while a few publications have concern with these methods for nonlinear problems, especially for free and moving boundary problems. We construct a mixed hybrid finite element scheme for a Stefan problem which is a case of moving boundary problem.

The main purpose of the article is to suggest an effective iterative algorithm to solve this finite element scheme. We construct and investigate an iterative

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process with a preconditioner, the rate of convergence of this algorithm does not depend on the mesh size. On the other hand, the implementation of the iterative method reduces to the solution of a "standard" finite-dimensional variational inequality, which can be made by using any of coordinate or gradient relaxation methods.

2. Mathematical model

Let $\Omega \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. We consider the following nonlinear problem: find $(u(x,t), \theta(x,t))$ such that

$$\begin{cases} \dfrac{\partial\theta}{\partial t} + v\dfrac{\partial\theta}{\partial x_1} - \Delta u = 0, & \text{in } \Omega, t > 0, \\ u = z(x,t) & \text{on } \Gamma_D, t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}} = g(x,t) & \text{on } \Gamma_N, t > 0, \\ \theta(x,t) \in H(u(x,t)) & \text{in } \Omega, t > 0, \\ \theta(x,0) = \theta_0(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the unit vector of outward normal, $v = \text{const} > 0$, z, g and θ_0 are given functions. We consider the case when the graph of $H : \mathbb{R}^1 \to \mathbb{R}^1$ monotonically increases and contains a vertical segment and suppose that the function H has a single values at points on Γ_D.

Problem (1) can serve as a simplified model of continuous casting process, where u is the temperature of casting metal, $\theta(u)$ is the enthalpy function and v is the casting speed in x_1 direction. The enthalpy function has a mentioned above property for example in the case of copper casting.

The existence and uniqueness of a weak solution for problem (1) are studied in [8, 9].

3. Semi-discretization

First, we introduce the semi-discretization of problem (1) using the characteristics of the first order differential operator and constant steps τ in time. Namely, if (x_1, x_2, t) is the point at the time level t we use the following approximation:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}\right) H \approx \frac{1}{\tau}\left(H(x_1,x_2,t) - H(\tilde{x}_1, x_2, t-\tau)\right), \quad \tilde{x}_1 = x_1 - v\tau.$$

If $x_1 - v\tau < 0$ then we put $\tilde{x}_1 = 0$.

After semi-discretization problem (1) on each time level can be formally written in the pointwise form as

$$\begin{cases} -\Delta u + Pu \ni f & \text{in } \Omega, \\ u = z & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \mathbf{n}} = g & \text{on } \Gamma_N, \end{cases} \tag{2}$$

where $Pu = H(u)/\tau$ is the multivalued maximal monotone nonlinear operator and the right-hand side $f = \tilde{H}(u)/\tau$ also arises due to semi-discretization.

Because of maximal monotonicity of function H and the property $\mathrm{Dom}(H) = \mathbb{R}$ the operator Pu is the subdifferential of convex continuous function $\varphi(u)$. The weak formulation of problem (2) can be written in the form of variational inequality: find $u(x) \in H^1(\Omega)$, $u(x) = z(x)$ on Γ_D such that

$$\int_\Omega \nabla u \nabla(q-u)dx + \varphi(q) - \varphi(u) \geqslant \int_\Omega f(q-u)dx + \int_{\Gamma_N} g(q-u)d\Gamma \quad (3)$$

$\forall q \in H^1(\Omega)$, $q(x) = z(x)$ on Γ_D.

It is well known that (3) has a unique solution [10].

4. Mixed hybrid formulation of the problem

Now, let $\mathbf{v} = \nabla u$ be so called flux function, then we get the following mixed formulation of (2):

$$\begin{cases} \mathbf{v} - \nabla u = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{v} - Pu \ni -f & \text{in } \Omega, \\ u = z & \text{on } \Gamma_D, \\ \mathbf{v}\cdot\mathbf{n} = g & \text{on } \Gamma_N. \end{cases} \quad (4)$$

Let $H(\operatorname{div},\Omega) = \{\mathbf{w} \in L_2(\Omega)^n : \operatorname{div}\mathbf{w} \in L_2(\Omega)\}$ with the norm $\|\mathbf{w}\|^2 = \int_\Omega (|\mathbf{w}|^2 + |\operatorname{div}\mathbf{w}|^2)dx$, $\mathcal{H}(\operatorname{div},\Omega) = \{\mathbf{w} \in H(\operatorname{div},\Omega) : \mathbf{w}\cdot\mathbf{n} \in L_2(\partial\Omega)\}$ with the norm $\|\mathbf{w}\|^2_{\mathcal{H}} = \|\mathbf{w}\|^2 + \int_{\partial\Omega}(\mathbf{w}\cdot\mathbf{n})^2 d\Gamma$ and subspaces $\mathcal{H}_N(\operatorname{div},\Omega) = \{\mathbf{w} \in \mathcal{H}(\operatorname{div},\Omega) : \mathbf{w}\cdot\bar{\mathbf{n}} = g$ a. e. on $\Gamma_N\}$, $\mathcal{H}^0_N(\operatorname{div},\Omega) = \{\mathbf{w} \in \mathcal{H}(\operatorname{div},\Omega) : \mathbf{w}\cdot\mathbf{n} = 0$ a. e. on $\Gamma_N\}$. Now by a weak solution of problem (4) we mean a triple $(u,\mathbf{v},\sigma) \in L_2(\Omega) \times \mathcal{H}_N \times L_2(\Omega)$, such that

$$\begin{cases} \int_\Omega \mathbf{v}\cdot\mathbf{w}\; dx + \int_\Omega u \operatorname{div}\mathbf{w} dx - \int_{\Gamma_D} z(\mathbf{w}\cdot\mathbf{n})d\Gamma = 0 \quad \forall \mathbf{w} \in \mathcal{H}^0_N, \\ \int_\Omega \operatorname{div}\mathbf{v}\; q dx - \int_\Omega \sigma(x) q dx = -\int_\Omega f q dx \quad \forall q \in L_2(\Omega), \\ \sigma(x) \in Pu(x) \text{ for a. e. } x \in \Omega. \end{cases} \quad (5)$$

Note, that by construction if u is solution of (5) then $(u, \nabla u, \sigma)$ is a solution of problem (5).

Let $\bar{\Omega} = \bigcup_{i=1}^{m} \bar{e}_i$ be a partitioning of the domain into m nonoverlapping subdomains, where e_i has a piecewise smooth boundary. Hereafter we suppose that the parts Γ_D and Γ_N of the boundary $\partial\Omega$ are composed by the whole sides ∂e_i. The common sides of elements e_i and e_j we denote by Γ_{ij}, where $i \neq j$, $i,j = \overline{1,m}$. We suppose that $\partial\Omega$ consists of s segments, which we denote as $\Gamma_1,\ldots,\Gamma_s$. Let the intersections ∂e_i with Γ_j, $j = \overline{1,s}$ be denoted by $\Gamma_{i,m+j}$, $i = \overline{1,m}$, $j = \overline{1,s}$ and let $\Gamma_{i,m+j}$ for $j = \overline{1,s_1}$, $s_1 < s$ compose Γ_N, while $\Gamma_{i,m+k}$, $i = \overline{1,m}$, $k = \overline{s_1+1,s}$ compose Γ_D.

Let further $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$, $u = (u_1, \ldots, u_m)$. Then system (4) can be written in the following form

$$\begin{cases} \mathbf{v}_i - \nabla u_i = 0, \quad \operatorname{div} \mathbf{v}_i - \sigma_i = -f_i, \quad \sigma_i \in P u_i \text{ in } e_i, \\ u_i - u_j = 0, \quad \mathbf{v}_i \cdot \mathbf{n}_{ij} - \mathbf{v}_j \cdot \mathbf{n}_{ij} = 0 \text{ on } \Gamma_{ij}, i, j = \overline{1, m}, \\ u_i = z \text{ on } \Gamma_{i,m+k}, k = \overline{s_1 + 1, s}, \\ \mathbf{v}_i \cdot \mathbf{n}_i = g \text{ on } \Gamma_{i,m+j}, j = \overline{1, s_1}, \end{cases} \tag{6}$$

where $\mathbf{n}_i$ is the outward normal vector to ∂e_i and $\mathbf{n}_{ij}$ is the unit normal vector to Γ_{ij} directed from e_i to e_j. Let us note that in semi-discretized problem the flux function $\mathbf{v} \cdot \mathbf{n}$ is continuous in Ω though in initial differential problem flux may have a jump.

On the basis of (6) we define a weak mixed hybrid formulation of problem (2). Let $U = \prod\limits_{1 \leqslant i \leqslant m} L_2(e_i)$, $V = \prod\limits_{1 \leqslant i \leqslant m} \mathcal{H}(\operatorname{div}, e_i)$ and $\Lambda = \prod\limits_{i > j} L_2(\Gamma_{ij})$. We introduce the bilinear forms $\mathcal{M} : V \times V \to \mathbb{R}$, $B : U \times V \to \mathbb{R}$, $C : \Lambda \times V \to \mathbb{R}$ and $\mathcal{D} : U \times U \to \mathbb{R}$ by the following equalities

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{m} \int_{e_i} \mathbf{v}_i \cdot \mathbf{w}_i dx, \quad \mathcal{B}(u, \mathbf{w}) = \sum_{i=1}^{m} \int_{e_i} u_i \operatorname{div} \mathbf{w}_i dx,$$

$$\mathcal{C}(\lambda, \mathbf{w}) = \sum_{i=1}^{m} \Big(\sum_{j=i+1}^{m} \int_{\Gamma_{ij}} \lambda_{ij} (\mathbf{w}_j - \mathbf{w}_i) \cdot \mathbf{n}_{ij} d\Gamma - \sum_{j=1}^{s_1} \int_{\Gamma_{i,m+j}} \lambda_{i,m+j} (\mathbf{w}_i \cdot \mathbf{n}_{i,m+j}) d\Gamma \Big),$$

$$\mathcal{D}(q, \sigma) = \sum_{i=1}^{m} \int_{e_i} \sigma_i q_i dx$$

and linear functionals

$$F(q) = \sum_{i=1}^{m} \int_{e_i} f q_i dx, \quad \ell(\mu) = \sum_{i=1}^{m} \sum_{j=1}^{s_1} \int_{\Gamma_{i,m+j}} g \mu_{i,m+j} d\Gamma,$$

$$r(\mathbf{w}) = \sum_{i=1}^{m} \sum_{k=s_1+1}^{s} \int_{\Gamma_{i,m+k}} z (\mathbf{w}_i \cdot \mathbf{n}_i) d\Gamma.$$

The weak mixed hybrid formulation is as follows: find $(u, \mathbf{v}, \lambda, \sigma) \in U \times V \times \Lambda \times U$ such that

$$\begin{cases} \mathcal{M}(\mathbf{v}, \mathbf{w}) + \mathcal{B}(u, \mathbf{w}) + \mathcal{C}(\lambda, \mathbf{w}) = r(\mathbf{w}) \quad \forall \mathbf{w} \in V, \\ \mathcal{B}(q, \mathbf{v}) + \mathcal{D}(q, \sigma) = -F(q) \quad \forall q \in U, \\ \mathcal{C}(\mu, \mathbf{v}) = -\ell(\mu) \quad \forall \mu \in \Lambda, \\ \sigma(u) \in \bar{P} u, \end{cases} \tag{7}$$

where $\bar{P} u = (H(u_1)/\tau, \ldots, H(u_m)/\tau)$ for $u \in U$.

Proposition 1. *The problems (3) and (7) are equivalent in the following sense:*

if u is the solution of (3), then $(u, \mathbf{v}, \lambda, \sigma)$ with $u_i = u|_{e_i}$ – restriction of u to e_i, $\mathbf{v}_i = \nabla u_i$ a. e. in e_i, $\lambda_{ij} = u_i$ a. e. on Γ_{ij} and $\sigma \in \bar{P}(u)$ is the solution of (7);

backwards, if $(u, \mathbf{v}, \lambda, \sigma)$ is the solution of problem (7), then the function $u : u|_{e_i} = u_i$ is a solution to problem (3).

5. Approximation

Let further Ω be a polygonal domain and $\tau_h = \{e_1, e_2, \dots, e_m\}$ be its conforming triangulation [11]. We assume that all e_i are convex polygons.

Let U_h, V_h and Λ_h be finite element subspaces of the corresponding spaces

$$U_h = \prod_{1 \leqslant i \leqslant m} U_{ih}, \quad V_h = \prod_{1 \leqslant i \leqslant m} V_{ih}$$

and refer to Λ_h as the space of vector-functions with constant components. Here V_{ih} is a finite dimensional subspace in $\mathcal{H}(\mathrm{div}, e_i)$ of dimension n_i consisting of vector-functions $\mathbf{v}_i \in \mathcal{H}(\mathrm{div}, e_i)$ such that $\mathbf{v}_i \cdot \mathbf{n}_{ij}$ is a constant on the corresponding interface Γ_{ij} and U_{ih} is a subspace of U consisting of vector-functions such that each its coordinate is a constant on each subdomain e_i. The value of n_i is equal to the total number of interfaces Γ_{ij} belonging to the boundary of e_i, $i = \overline{1, m}$. Thus, the dimension of V_h is equal to $\hat{n} = \sum_{i=1}^{m} n_i$, the dimension of U_h is equal to m, and the dimension of Λ_h is equal to $\check{n}$ where $\check{n}$ is the total number of interfaces.

The finite element approximation of (7) reads as follows: find $(u_h, \mathbf{v}_h, \lambda_h, \sigma_h) \in U_h \times V_h \times \Lambda_h \times U_h$ satisfying the following relations:

$$\begin{cases} \mathcal{M}(\mathbf{v}_h, \mathbf{w}) + \mathcal{B}(u_h, \mathbf{w}) + \mathcal{C}(\lambda_h, \mathbf{w}) = r(\mathbf{w}) \quad \forall \mathbf{w} \in V_h, \\ \mathcal{B}(q, \mathbf{v}_h) + \mathcal{D}(q, \sigma(u_h)) = -F(q) \quad \forall q \in U_h, \\ \mathcal{C}(\mu, \mathbf{v}_h) = -\ell(\mu) \quad \forall \mu \in \Lambda_h, \\ \sigma_h \in \bar{P}u. \end{cases} \tag{8}$$

Let now v_{ij} be the degrees of freedom for vector-function $\mathbf{v}_{ih}$, associated with Γ_{ij}, u_i be the degrees of freedom for the function u_h, associated with e_i, and λ_{ij} be the degrees of freedom for λ_h, associated with Γ_{ij}, $j > i$. The algebraic formulation of (8) is: to find $(\bar{v}, \bar{u}, \bar{\lambda}, \bar{\sigma})$ such that

$$\begin{cases} \widehat{\mathcal{M}}(\bar{v}, \bar{w}) + \widehat{\mathcal{B}}(\bar{u}, \bar{w}) + \widehat{\mathcal{C}}(\bar{\lambda}, \bar{w}) = \widehat{r}(\bar{w}), \\ \widehat{\mathcal{B}}(\bar{q}, \bar{v}) + \widehat{\mathcal{D}}(\bar{q}, \sigma(\bar{u})) = -\widehat{F}(\bar{q}), \\ \widehat{\mathcal{C}}(\bar{\mu}, \bar{v}) = -\widehat{\ell}(\bar{\mu}), \\ \bar{\sigma} \in \bar{P}\bar{u}, \end{cases} \tag{9}$$

$\forall (\bar{w}, \bar{q}, \bar{\mu}) \in \mathbb{R}^{\hat{n}} \times \mathbb{R}^m \times \mathbb{R}^{\check{n}}$. Here

$$\widehat{\mathcal{M}}(\bar{v}, \bar{w}) = \sum_{i=1}^{m} (M_i \bar{v}_i, \bar{w}_i), \quad \bar{v}_i, w_i \in \mathbb{R}^{n_i},$$

$$\widehat{\mathcal{B}}(\bar{u}, \bar{w}) = \sum_{i=1}^{m} u_i \Big(-\sum_{j=1}^{i-1} w_{ij} |\Gamma_{ij}| + \sum_{j=i+1}^{m+s} w_{ij} |\Gamma_{ij}|\Big),$$

$$\widehat{\mathcal{C}}(\bar{\lambda},\bar{w}) = \sum_{i=1}^{m}\Big(\sum_{j=i+1}^{m}\lambda_{ij}(w_{ji}-w_{ij})|\Gamma_{ij}| - \sum_{i=1}^{s_1}\lambda_{i,m+j}w_{i,m+j}|\Gamma_{i,m+j}|\Big),$$

$$\widehat{\mathcal{D}}(\bar{q},\bar{\sigma}) = \sum_{i=1}^{m}\sigma_i q_i \mathrm{mes}(e_i)$$

are the bilinear forms on $\mathbb{R}^{\hat{n}}\times\mathbb{R}^{\hat{n}}$, $\mathbb{R}^{m}\times\mathbb{R}^{\hat{n}}$, $\mathbb{R}^{\check{n}}\times\mathbb{R}^{\hat{n}}$ and $\mathbb{R}^{m}\times\mathbb{R}^{\hat{n}}$, respectively, and

$$\widehat{r}(\bar{w}) = \sum_{i=1}^{m}\sum_{j=s_1+1}^{s} z_i(\mathbf{w}_i\cdot\mathbf{n}_i)d\Gamma, \quad \int\limits_{\Gamma_{i,m+j}} z d\Gamma, \quad i=\overline{1,m}, j=\overline{s_1+1,s},$$

$$\widehat{F}(\bar{q}) = \sum_{i=1}^{m} f_i q_i, \quad f_i = \int\limits_{e_i} f dx, \quad i=\overline{1,m},$$

$$\widehat{\ell}(\bar{\mu}) = \sum_{i=1}^{m}\Big(\sum_{j=1}^{s_1} g_{ij}\mu_{i,m+j}\Big), \quad g_{ij} = \int\limits_{\Gamma_{i,m+j}} g d\Gamma, \quad i=\overline{1,m}, j=\overline{1,s_1}$$

are linear forms defined on $\mathbb{R}^{\check{n}}$, $\mathbb{R}^{m}$ and $\mathbb{R}^{\check{n}}$.

The entries $m_{k,l}^{(i)}$ of matrices M_i are defined by the standard way using the $L_2(e_i)$ scalar products of the nodal basis functions of the subspaces V_{ih}.

In matrix-vector form problem (9) can be written as follows:

$$A\begin{pmatrix}\bar{v}\\ \bar{u}\\ \bar{\lambda}\end{pmatrix} + \begin{pmatrix}0\\ -\bar{P}(\bar{u})\\ 0\end{pmatrix} \ni \begin{pmatrix}\bar{r}\\ -\bar{f}\\ -\bar{g}\end{pmatrix}, \quad A = \begin{pmatrix}M & B^T & C^T\\ B & 0 & 0\\ C & 0 & 0\end{pmatrix}. \tag{10}$$

We rewrite the system as

$$\begin{cases} M\bar{v} + B^T\bar{u} + C^T\bar{\lambda} = \bar{r},\\ B\bar{v} - \bar{P}\bar{u} \ni -\bar{f},\\ C\bar{v} = -\bar{g}\end{cases} \tag{11}$$

and eliminate $\bar{v}$ from the system. After that we obtain a reduced system which can be written in block form by

$$\begin{pmatrix}BM^{-1}B^T & BM^{-1}C^T\\ CM^{-1}B^T & CM^{-1}C^T\end{pmatrix}\begin{pmatrix}\bar{u}\\ \bar{\lambda}\end{pmatrix} + \begin{pmatrix}\bar{P}\bar{u}\\ 0\end{pmatrix} \ni \begin{pmatrix}f + BM^{-1}\bar{r}\\ g + CM^{-1}\bar{r}\end{pmatrix}.$$

Using notations

$$S = \begin{pmatrix}BM^{-1}B^T & BM^{-1}C^T\\ CM^{-1}B^T & CM^{-1}C^T\end{pmatrix}, \quad \mu = \begin{pmatrix}\bar{u}\\ \bar{\lambda}\end{pmatrix},$$

$$F = \begin{pmatrix}f + BM^{-1}\bar{r}\\ g + CM^{-1}\bar{r}\end{pmatrix}, \quad \tilde{P} = \begin{pmatrix}\bar{P} & 0\\ 0 & 0\end{pmatrix},$$

we finally obtain the following finite-dimensional problem:

$$S\mu + \tilde{P}\mu \ni F. \tag{12}$$

Note that Schur complement matrix S is a symmetric and positive definite matrix [1], while $\tilde{P}$ is a maximal monotone operator. Owing to this fact problem (12) has a unique solution, whence problem (11) has also a unique solution.

6. Iterative method

For the sake of simplicity we analyze only the case of rectangular meshes. We suppose that Ω is the unit square and construct mesh with step h in both directions which defines the partitioning of Ω into elements e_i.

To obtain preconditioner for (12) we construct finer grid in Ω with the step $h/2$. We denote by $\tau_{h/2}$ the new partitioning of Ω. Let further $W_{h/2}$ be the piecewise finite element subspace of $H^1(\Omega)$ and $\mathcal{A}$ be the stiffness matrix, corresponding to the approximation in this subspace of Laplace operator with Dirichlet boundary conditions on Γ_D. Let the nodes of $\tau_{h/2}$ consist of two groups: the first group contains the nodes of τ_h, while the second one contains all others (called by the fictitious ones). Then matrix $\mathcal{A}$ can be represented in the corresponding block form:

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix}.$$

It is shown in [1] that the operator S is spectrally equivalent to the Schur complement $S_{\mathcal{A}}$:

$$S_{\mathcal{A}} = \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$$

with constants of equivalence which do not depend on mesh size:

$$\alpha(S_{\mathcal{A}}\mu, \mu) \leqslant (S\mu, \mu) \leqslant \beta(S_{\mathcal{A}}\mu, \mu) \quad \forall \mu.$$

In the case under consideration we get $\alpha = 1$, $\beta = 6$.

This observation allows us to use the matrix $S_{\mathcal{A}}$ as a preconditioner in iterative process for solving (12):

$$S_{\mathcal{A}}\frac{\mu^{n+1} - \mu^n}{\tau} + S\mu^n + \tilde{P}\mu^{n+1} \ni F. \tag{13}$$

The following statement is valid.

Proposition 2. *The iterative method (13) converges for any $\tau \in (0, 2/\beta)$ and for $\tau = 2/(\alpha + \beta)$ the following estimate holds:*

$$(S_{\mathcal{A}}(\mu^{n+1} - \mu), \mu^{n+1} - \mu)^{1/2} \leqslant \frac{\beta - \alpha}{\beta + \alpha}(S_{\mathcal{A}}(\mu^n - \mu), \mu^n - \mu)^{1/2}.$$

To avoid the explicit calculation of $S_{\mathcal{A}}$ on each step of process (13) we use the following trick. We complete the system (12) by the equations in fictitious nodes, so that the algebraic size of resulting system

$$\begin{cases} S\mu + \tilde{P}\mu \ni F, \\ 0 = 0 \end{cases}$$

is equal to number of fine grid nodes. After we write iterative process with operator $\mathcal{A}$ as a preconditioner for this system using block representation of

$\mathcal{A}$:

$$\begin{cases} \mathcal{A}_{11}\dfrac{\mu^{n+1}-\mu^n}{\tau} + \mathcal{A}_{12}\dfrac{\tilde{\mu}^{n+1}-\tilde{\mu}^n}{\tau} + S\mu^n + \tilde{P}\mu^{n+1} \ni F, \\ \mathcal{A}_{21}\dfrac{\mu^{n+1}-\mu^n}{\tau} + \mathcal{A}_{22}\dfrac{\tilde{\mu}^{n+1}-\tilde{\mu}^n}{\tau} + 0 = 0, \end{cases} \tag{14}$$

where $\tilde{\mu}$ are fictitious components. Eliminating from the second equation fictitious values we can verify that it is an equivalent form of process (13) for solving (12) with $S_{\mathcal{A}}$ as preconditioner.

For a fixed n problem (14) is the finite element variational inequality with positive definite and symmetric matrix $\mathcal{A}$, so, it can be solved, for example, by using any coordinate relaxation or gradient relaxation method.

On each step of constructed method we need to calculate $F - S\mu$. It is easy to do if we take into account the identity

$$F - S\mu^n = \begin{pmatrix} B\bar{v}^n \\ C\bar{v}^n \end{pmatrix}$$

with $\bar{v}^n = M^{-1}(\bar{r} - B^T\bar{u}^n - C^T\bar{\lambda}^n)$.

Thus, to solve (14) we get the the following

Algorithm

(1) Define $\mu^0 = (\bar{u}^0, \bar{\lambda}^0)$.
(2) For $n \geqslant 0$ on each element e_i calculate $\bar{v}^n$ by formula
$$\bar{v}_i^n = M_i^{-1}(\bar{r}_i - B_i^T\bar{u}_i^n - C^T\bar{\lambda}_i^n).$$
(3) Calculate $\mu^{n+1} = (\bar{u}^{n+1}, \bar{\lambda}^{n+1})$ by solving (14).
(4) $n := n+1$ goto step 2.

7. Numerical results

In numerical test we take $\Omega = (0,1)\times(0,1)$, $\Gamma_D = \{(x_1,x_2)\in\partial\Omega : x_2 = 0\}\cup\{(x_1,x_2)\in\partial\Omega : x_2 = 1\}$, $\Gamma_N = \partial\Omega\setminus\Gamma_D$. We solve problem in time interval [0,1] using constant time step $\tau = 0.05$ and various grids in space to compare number of iterations. As H we take the function

$$H(u) = \begin{cases} 0.5u & \text{if } u < 0, \\ [0,1] & \text{if } u = 0, \\ u+1 & \text{if } u > 0. \end{cases}$$

On the top of square Ω we put $g = -3$ in the boundary condition, on bottom we use value $g = 0$, on the left side $z = 2$, and on the right side $z = -1$. As initial condition we take $u_0 = -1$.

To solve inequality with matrix $\mathcal{A}$ on each step of iterative process (14) we use SOR-method, the stopping criterion was $\|\mu^{k+1} - \mu^k\| \leqslant 10^{-13}$ and the stopping criterion of outer process was $\|\mu^{n+1} - \mu^n\| \leqslant 10^{-12}$. As iterative parameter in outer process we take $\tau = 2/7$, which we get using estimates of equivalence of matrices S and $S_{\mathcal{A}}$.

In the second column of table 1 we show the number of iteration which we need to solve problem on the first time level (on the next levels the number of iterations becomes smaller) and in the third column – the total number of iteration, which is equal to the sum of iterations on all time steps. From the table we can see that the number of iterations does not depend on mesh size.

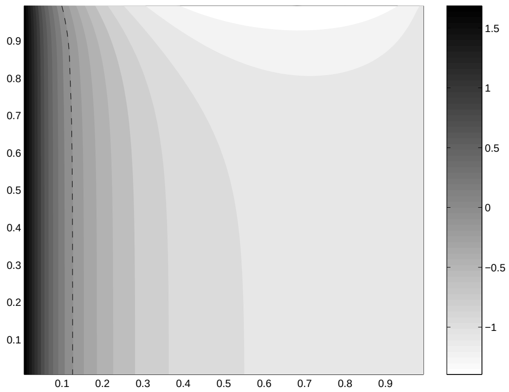

FIGURE 1. Temperature distribution on $t = 0.0125$

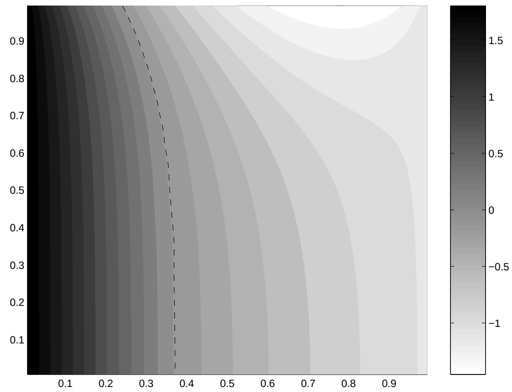

FIGURE 2. Temperature distribution on $t = 0.0750$

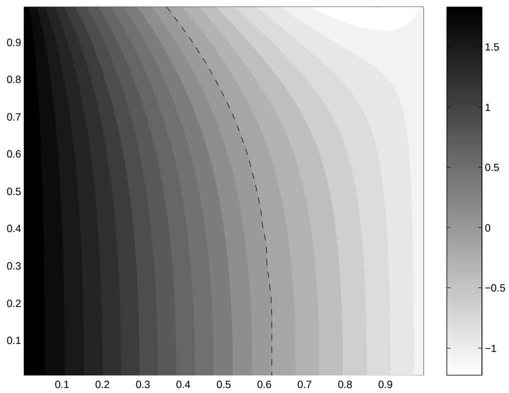

FIGURE 3. Temperature distribution on $t = 0.0225$

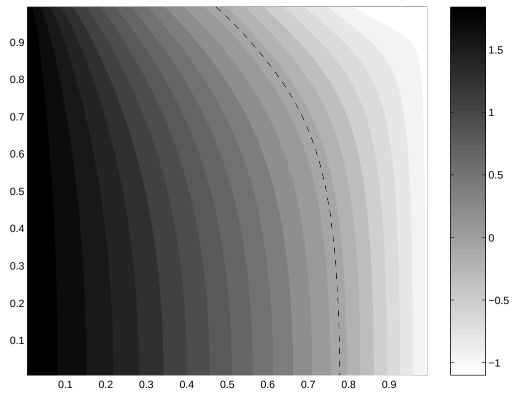

FIGURE 4. Temperature distribution on $t = 1$

On the figures we show temperature distributions at several time levels.

Grid size	Iter 1	Total Iter
11×11	81	1195
21×21	82	1183
41×41	83	1189
81×81	83	1189
161×161	83	1171

TABLE 1. Number of iterations

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