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# THE EMBEDDING OF AN ORDERED SEMIGROUP INTO AN LE-SEMIGROUP

(submitted by M. M. Arslanov)

Abstract. In this paper we prove the following: If $S$ is an ordered semigroup, then the set $\mathcal{P}(S)$ of all subsets of $S$ with the multiplication "$\circ$" on $\mathcal{P}(S)$ defined by "$A \circ B := (AB]$ if $A, B \in \mathcal{P}(S)$, $A \neq \emptyset$, $B \neq \emptyset$ and $A \circ B := \emptyset$ if $A = \emptyset$ or $B = \emptyset$ is an le-semigroup having a zero element and $S$ is embedded in $\mathcal{P}(S)$.

If $(S, ., \leq)$ is an ordered semigroup, for $A \subseteq S$, we define $(A] := \{t \in S \mid t \leq a$ for some $a \in A\}$. For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. An element 0 of $S$ is called the zero element of $S$ if $0 \leq x$ and $0x = x0 = 0$ for all $x \in S$ [1]. Let $(S, ., \leq)$, $(T, \circ, \preceq)$ be ordered semigroups, $f : S \to T$ a mapping from $S$ into $T$.

The mapping $f$ is called isotone if $x, y \in S$, $x \leq y$ implies $f(x) \leq f(y)$. $f$ is called reverse isotone if $x, y \in S$, $f(x) \preceq f(y)$ implies $x \leq y$. [Each reverse isotone mapping is (1-1): Let $x, y \in S$, $f(x) = f(y)$. Since $f(x) \preceq f(y)$, we have $x \leq y$. Since $f(y) \preceq f(x)$, we have $y \leq x$.] $f$ is called a homomorphism if it is isotone and satisfies $f(xy) = f(x) \circ f(y)$ for all $x, y \in S$. $f$ is called an isomorphism if it is onto, homomorphism and reverse isotone. $S$ and $T$ are called isomorphic if there exists an isomorphism between them [3]. $S$ is embedded in $T$ if, by definition, $S$ is isomorphic to a subset of $T$, i.e., if there exists a mapping $f : S \to T$ which is homomorphism and reverse isotone [4]. An l-semigroup (: lattice ordered semigroup) is a semigroup $S$ at the same time a lattice satisfying the conditions $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [1]. By an le-semigroup we mean an l-semigroup having a greatest element "$e$" (i.e. $e \geq a$ for all $a \in S$) [2]. We denote by $\mathcal{P}(S)$ the set of all subsets of $S$.

**Theorem.** *Let $(S, ., \leq)$ be an ordered semigroup. We define a multiplication "$\circ$" on $\mathcal{P}(S)$ as follows:*

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$$\circ : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S) \mid (A, B) \to A \circ B$$

*where*

$$A \circ B := \begin{cases} (AB] & \text{if} \quad A, B \in \mathcal{P}(S)\backslash\{\emptyset\} \\ \emptyset & \text{if} \quad A = \emptyset \text{ or } B = \emptyset \end{cases}$$

*Then* $(\mathcal{P}(S), \circ, \subseteq)$ *is an le-semigroup having a zero element and* $(S, ., \leq)$ *is embedded in* $(\mathcal{P}(S), \circ, \subseteq)$*.*

*Proof.* First of all, the set $\mathcal{P}(S)$ in non-empty. The multiplication " $\circ$ " on $\mathcal{P}(S)$ is well defined. Moreover, we have the following:

1) The multiplication " $\circ$ " on $\mathcal{P}(S)$ is associative. In fact:

Let $A, B, C \in \mathcal{P}(S)$. If $A = \emptyset$ or $B = \emptyset$ or $C = \emptyset$, then $(A \circ B) \circ C = \emptyset$ and $A \circ (B \circ C) = \emptyset$, so $(A \circ B) \circ C = A \circ (B \circ C)$.

Let $A, B, C \in \mathcal{P}(S)\backslash\{\emptyset\}$. We have $A \circ B$, $B \circ C \in \mathcal{P}(S)\backslash\{\emptyset\}$. Let now $x \in (A \circ B) \circ C := ((A \circ B)C]$. Then $x \leq yc$ for some $y \in A \circ B$, $c \in C$. Since $y \in A \circ B := (AB]$, we have $y \leq ab$ for some $a \in A$, $b \in B$. Then

$$x \leq (ab)c = a(bc); \ a \in A, \ bc \in BC \subseteq (BC] := B \circ C,$$

so $x \in (A(B \circ C)] := A \circ (B \circ C)$. Similarly, $A \circ (B \circ C)] \subseteq (A \circ B) \circ C$.

2) $(\mathcal{P}(S), \circ, \subseteq)$ is an le-semigroup:
Let $A, B, C \in \mathcal{P}(S)$. Then $A \circ (B \cup C) = (A \circ B) \cup (A \circ C)$. Indeed:
If $A = \emptyset$, then $A \circ (B \cup C) = \emptyset$, $A \circ B = \emptyset$, $A \circ C = \emptyset$.

If $B = \emptyset$, then $A \circ (B \cup C) = A \circ C$, $(A \circ B) \cup (A \circ C) = A \circ C$.

If $C = \emptyset$, then $A \circ (B \cup C) = A \circ B$, $(A \circ B) \cup (A \circ C) = A \circ B$.

Let $A, B, C \in \mathcal{P}(S)\backslash\{\emptyset\}$. We have

$$A \circ (B \cup C) := (A(B \cup C)], \ A \circ B := (AB], \ A \circ C := (AC].$$

Since $(AB], (AC] \subseteq (A(B \cup C)]$, we have $(AB] \cup (AC] \subseteq (A(B \cup C)]$.
Let now $t \in (A(B \cup C)]$. Then $t \leq ax$ for some $a \in A$, $x \in B \cup C$. If $x \in B$, then $t \in (AB] \subseteq (AB] \cup (AC]$. If $x \in C$, then $t \in (AC] \subseteq (AB] \cup (AC]$.
Similarly, for any $A, B, C \in \mathcal{P}(S)\backslash\{\emptyset\}$, we have $(A \cup B) \circ C = (A \circ C) \cup (B \circ C)$.
Finally, $S$ is the greatest element and $\emptyset$ the zero element of $\mathcal{P}(S)$.

3) We consider the mapping

$$f : (S, ., \leq) \to (\mathcal{P}(S), \circ, \subseteq) \ \mid \ a \to f(a) := (a].$$

The mapping $f$ is well defined. Moreover,

A) The mapping $f$ is a homomorphism. Indeed:
Let $a, b \in S$. We have $(a], (b] \in \mathcal{P}(S)\backslash\{\emptyset\}$ (since $a \in (a]$, $b \in (b]$).
Thus we have

$$f(a) \circ f(b) = (a] \circ (b] := ((a](b]] = (ab] := f(ab).$$

Let $a, b \in S$, $a \leq b$. Then $f(a) := (a] \subseteq (b] := f(b)$.

B) The mapping $f$ is reverse isotone: Let $a, b \in S$, $f(a) \subseteq f(b)$. Then $a \in (a] \subseteq (b]$, and $a \leq b$.

**Remark.** More generally, we have the following: If $A, B_i \in \mathcal{P}(S)$, $i \in I$, then

$$A \circ (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \circ B_i) \text{ and } (\bigcup_{i \in I} B_i) \circ A = \bigcup_{i \in I}(B_i \circ A).$$

In fact,

A) If $A = \emptyset$, then $A \circ (\bigcup_{i \in I} B_i) = \emptyset$, and $A \circ B_i = \emptyset$ for all $i \in I$, so $\bigcup_{i \in I}(A \circ B_i) = \emptyset$. Thus $A \circ (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \circ B_i)$.

B) If $A \neq \emptyset$, then

I) If $\bigcup_{i \in I} B_i = \emptyset$, then $A \circ (\bigcup_{i \in I} B_i) = \emptyset$. Since $\bigcup_{i \in I} B_i = \emptyset$, we have $B_i = \emptyset$ for all $i \in I$, then $A \circ B_i = \emptyset$ for all $i \in I$, and $\bigcup_{i \in I}(A \circ B_i) = \emptyset$. Then $A \circ (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \circ B_i)$.

II) Let $\bigcup_{i \in I} B_i \neq \emptyset$. We put $J := \{i \in I \mid B_i = \emptyset\}$, $K := \{i \in I \mid B_i \neq \emptyset\}$. Clearly $I = J \cup K$ and $J \cap K = \emptyset$. If $K = \emptyset$, then $I = J$, $B_i = \emptyset$ for all $i \in I$, and $\bigcup_{i \in I} B_i = \emptyset$. Impossible. Thus $K \neq \emptyset$.

$\alpha$) Let $J = \emptyset$. Then $I = K$, $B_i \neq \emptyset$ for all $i \in I$. Since $A \neq \emptyset$ and $B_i \neq \emptyset$ for all $i \in I$, we have $A \circ B_i := (AB_i]$ for all $i \in I$. Then $\bigcup_{i \in I}(A \circ B_i) = \bigcup_{i \in I}(AB_i]$.

Besides, $\bigcup_{i \in I}(AB_i] = (A(\bigcup_{i \in I} B_i)]$. Thus we have

$$\bigcup_{i \in I}(A \circ B_i) = (A(\bigcup_{i \in I} B_i)] \text{..........}(*)$$

Since $A \neq \emptyset$ and $(\bigcup_{i \in I} B_i) \neq \emptyset$, we have $A \circ (\bigcup_{i \in I} B_i) = (A(\bigcup_{i \in I} B_i)]$. Then, by (*), $A \circ (\bigcup_{i \in I} B_i) = \bigcup_{i \in I}(A \circ B_i)$.

$\beta$) Let $J \neq \emptyset$. Then $B_i = \emptyset$ for all $i \in I$, $\bigcup\limits_{i\in I} B_i = \emptyset$, and

$$\emptyset \neq \bigcup_{i\in I} B_i = (\bigcup_{i\in J} B_i) \cup (\bigcup_{i\in K} B_i) = \bigcup_{i\in K} B_i.$$

Since $A \neq \emptyset$ and $\bigcup\limits_{i\in I} B_i \neq \emptyset$, we have

$$A \circ (\bigcup_{i\in I} B_i) = (A(\bigcup_{i\in I} B_i)] = (A(\bigcup_{i\in K} B_i)] = \bigcup_{i\in K} (AB_i].$$

Since $A \neq \emptyset$ and $B_i \neq \emptyset$ for all $i \in K$, we have $A \circ B_i := (AB_i]$ for all $i \in K$, and $\bigcup\limits_{i\in K} (A \circ B_i) = \bigcup\limits_{i\in K} (AB_i]$. Thus we have

$$A \circ (\bigcup_{i\in I} B_i) = \bigcup_{i\in K} (A \circ B_i) .......... (**)$$

Since $B_i = \emptyset$ for all $i \in J$, we have $A \circ B_i = \emptyset$ for all $i \in J$, then $\bigcup\limits_{i\in J} (A \circ B_i) = \emptyset$. Then

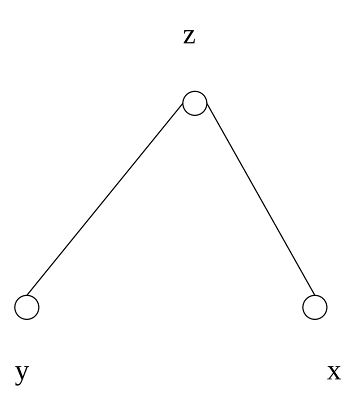
$$\bigcup_{i\in K} (A \circ B_i) \cup \bigcup_{i\in J} (A \circ B_i) = \bigcup_{i\in K} (A \circ B_i) .......... (***)$$

By (**) and (***), we have

$$A \circ (\bigcup_{i\in I} B_i) = \bigcup_{i\in K} (A \circ B_i) \cup \bigcup_{i\in J} (A \circ B_i) = \bigcup_{i\in K\cup J} (A \circ B_i) = \bigcup_{i\in J} (A \circ B_i). \quad \square$$

**Example.** We consider the ordered semigroup
$S = \{x, y, z\}$ defined by the multiplication and the figure below:

| . | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x$ | $x$ | $x$ | $z$ |
| $y$ | $x$ | $y$ | $z$ |
| $z$ | $x$ | $z$ | $z$ |



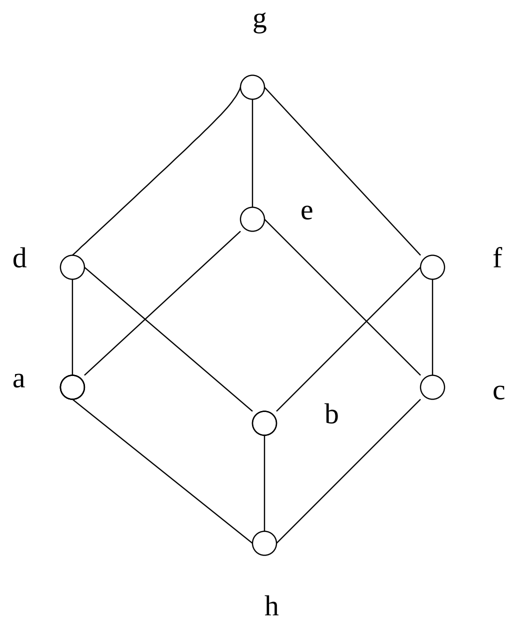

Applying the Theorem of this note, the ordered semigroup $(S, ., \leq)$ is embedded into the le-semigroup $L = \{a, b, c, d, e, f, g, h\}$, defined by the multiplication "."and the order " $\leq_L$ " below:

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $g$ | $a$ | $g$ | $g$ | $g$ | $h$ |
| $b$ | $a$ | $b$ | $g$ | $d$ | $g$ | $g$ | $g$ | $h$ |
| $c$ | $a$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $h$ |
| $d$ | $a$ | $d$ | $g$ | $d$ | $g$ | $g$ | $g$ | $h$ |
| $e$ | $a$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $h$ |
| $f$ | $a$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $h$ |
| $g$ | $a$ | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ | $h$ |
| $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ | $h$ |

$$\begin{aligned}\leq_L: \quad = \quad & \{(a,a),(a,d),(a,e),(a,g),(b,b),(b,d),(b,f),(b,g),(c,d),(c,e),\\ & (c,f),(c,g),(d,d),(d,g),(e,e),(e,g),(f,f),(f,g),(g,g),(h,a),\\ & (h,b),(h,c),(h,d),(h,e),(h,f),(h,g),(h,h)\}.\end{aligned}$$

We give the covering relation "$\prec$' and the figure of $S$.

$$\begin{aligned}\prec \quad = \quad & \{(a,d),(a,e),(b,d),(b,f),(c,e),(c,f),(d,g),(e,g),(f,g),\\ & (h,a),(h,b),(h,c)\}.\end{aligned}$$




The embedding is given by the mapping:

$$f : (S, ., \leq) \to (L, ., \leq_L) \left| \begin{array}{l} x \to a \\ y \to b \\ z \to g \end{array} \right.$$

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