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A NOTE ON SEMI-PSEUDOORDERS IN SEMIGROUPS

(submitted by M. M. Arslanov)

Abstract. An important problem for studying the structure of an ordered semigroup S is to know conditions under which for a given congruence ρ on S the set S/ρ is an ordered semigroup. In [1] we introduced the concept of pseudoorder in ordered semigroups and we proved that each pseudoorder on an ordered semigroup S induces a congruence σ on S such that S/σ is an ordered semigroup. In [3] we introduced the concept of semi-pseudoorder (also called pseudocongruence) in semigroups and we proved that each semi-pseudoorder on a semigroup S induces a congruence σ on S such that S/σ is an ordered semigroup. In this note we prove that the converse of the last statement also holds. That is each congruence σ on a semigroup $(S,.)$ such that S/σ is an ordered semigroup induces a semi-pseudoorder on S.

For a given ordered semigroup $(S,.,\leq)$ is essential to know if there exists a congruence ρ on S such that S/ρ be an ordered semigroup. This plays an important role for studying the structure of ordered semigroups. If S is a semigroup (resp. an ordered semigroup), by a congruence on S we mean an equivalence relation σ on S such that $(a,b)\in\sigma$ implies $(ac,bc)\in\sigma$ and $(ca,cb)\in\sigma$ for all $c\in S$. If S is a semigroup and σ a congruence on S, then the set $S/\sigma := \{(a)_\sigma \mid a\in S\}$ ($(a)_\sigma$ is the σ-class of S containing a ($a\in S$)) is a semigroup and the operation on S/σ is defined via the operation on S. The following question is natural: If $(S,.,\leq)$ is an ordered semigroup and σ a congruence on S, then is the set S/σ an ordered semigroup ? A probable order on S/σ could be the relation " $\preceq$ " on S/σ defined by means of the order " $\leq$ " on S, that is

$$\begin{aligned}\preceq: &= \{(t,z)\in S/\sigma\times S/\sigma \mid \exists\ (a,b)\in\leq \text{ such that } t=(a)_\sigma, z=(b)_\sigma\}\\ &= \{((x)_\sigma,(y)_\sigma) \mid \exists\ a\in(x)_\sigma, b=(y)_\sigma \text{ such that } (a,b)\in\leq\}.\end{aligned}$$

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But this relation is not an order, in general. An example can be found in [1]. The following question arises: Is there a congruence σ on S for which S/σ is an ordered semigroup ? This let to the concept of pseudoorder introduced by the same authors in [1]. Let $(S, ., \leq)$ be an ordered semigroup. A relation ρ on S is called pseudoorder if

(1) $\leq \subseteq \rho$
(2) $(a, b) \in \rho$ and $(b, c) \in \rho$ imply $(a, c) \in \rho$.
(3) $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$ for each $c \in S$.

According to Lemma 1 in [1], if $(S, ., \leq)$ is an ordered semigroup and σ a pseudoorder on S, then the relation $\bar{\sigma}$ on S defined by

$$\bar{\sigma} := \{(a, b) \in S \times S \ \mid \ (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}$$

is a congruence on S and the set
$S/\bar{\sigma}$ is an ordered semigroup. So according to [1],
each pseudoorder on an ordered semigroup S induces a congruence $\bar{\sigma}$ on S such that $S/\bar{\sigma}$ is an ordered semigroup. For a further study of pseudoorders in ordered semigroups we refer to [2]. On the other hand, the concept of pseudocongruences in semigroups has been introduced by the same authors in [3]. If $(S, .)$ is a semigroup, by a pseudocongruence on S we mean a relation ρ on S such that

(1) $(a, a) \in \rho \ \ \forall \, a \in S$
(2) $(a, b) \in \rho$ and $(b, c) \in \rho$ imply $(a, c) \in \rho$.
(3) $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$ for each $c \in S$.

If $(S, ., \leq)$ is an ordered semigroup, then each pseudoorder on
S is a pseudocongruence on S. Indeed, if ρ is a
pseudoorder on S and $a \in S$, then $(a, a) \in \leq \subseteq \rho$. Pseudocongruences can be also called semi-pseudoorders, and from now on we will keep that terminology of semi-pseudoorders. We have seen in [3], that each semi-pseudoorder on a semigroup S induces a congruence ρ on S such that S/ρ is an ordered semigroup. In this paper we prove that the converse of this statement also holds. For a semigroup $(S, .)$ we define a multiplication " $*$ " on on S/ρ defined by $(a)_\rho * (b)_\rho := (ab)_\rho$. If $(S, .)$ is a semigroup and ρ a congruence on S and if there exists an order relation "∇" on S/ρ such that the $(S/\rho, *, \nabla)$ is an ordered semigroup, then there exists a semi-pseudoorder σ on S such that $\rho = \bar{\sigma}$. So each congruence ρ on a semigroup $(S, .)$ such that S/ρ is an ordered semigroup induces a semi-pseudoorder on S.

If $(S, .)$ is a semigroup and σ a semi-pseudoorder on S, we define

$$\bar{\sigma} := \sigma \cap \sigma^{-1}.$$

The relation $\bar{\sigma}$ is a congruence on S. Indeed: If $a \in S$, then $(a, a) \in \sigma$, then $(a, a) \in \sigma^{-1}$, so $(a, a) \in \sigma \cap \sigma^{-1} := \bar{\sigma}$. If $(a, b) \in \bar{\sigma}$, then $(a, b) \in \sigma$ and $(a, b) \in \sigma^{-1}$, then $(b, a) \in \sigma^{-1}$ and $(b, a) \in \sigma$, so $(b, a) \in \sigma^{-1} \cap \sigma := \bar{\sigma}$. If $(a, b) \in \bar{\sigma}$ and $(b, c) \in \bar{\sigma}$, then $(a, b) \in \sigma$, $(a, b) \in \sigma^{-1}$, $(b, c) \in \sigma$, $(b, c) \in \sigma^{-1}$, then $(a, c) \in \sigma$ and $(a, c) \in \sigma^{-1}$, thus $(a, c) \in \sigma \cap \sigma^{-1} := \bar{\sigma}$. Let $(a, b) \in \bar{\sigma}$ and $c \in S$. We have $(a, b) \in \sigma$ and $(a, b) \in \sigma^{-1}$. Since $(a, b) \in \sigma$, $c \in S$, we have $(ac, bc) \in \sigma$, $(ca, cb) \in \sigma$. Since $(a, b) \in \sigma^{-1}$, we have $(b, a) \in \sigma$, then $(bc, ac) \in \sigma$, $(ca, cb) \in \sigma$, hence $(ac, bc) \in \sigma^{-1}$, $(ca, cb) \in \sigma^{-1}$. Then we have $(ac, bc) \in \sigma \cap \sigma^{-1} := \bar{\sigma}$ and $(ca, cb) \in \sigma \cap \sigma^{-1} := \bar{\sigma}$.

[It might be also noted that $\bar{\sigma} = \{(a, b) \in S \times S \mid (a, b) \in \sigma \text{ and } (b, a) \in \sigma\}$. Hence $\bar{\sigma}$ is a congruence on S (cf. [3])]. Since $\bar{\sigma}$ is a congruence on S, the set $S/\bar{\sigma}$ with the operation " $*$ " on $S/\bar{\sigma}$ defined by $(a)_{\bar{\sigma}} * (b)_{\bar{\sigma}} := (ab)_{\bar{\sigma}}$ is a semigroup (It is known).

If $(S, .)$ is a semigroup and σ a semi-pseudoorder on S, we define a relation "∇" on $S/\bar{\sigma}$ as follows: $(a)_{\bar{\sigma}} \nabla (b)_{\bar{\sigma}}$ if and only if $(a, b) \in \sigma$. The relation "∇" on $S/\bar{\sigma}$ is well defined. Indeed: Let $(a)_{\bar{\sigma}} = (c)_{\bar{\sigma}}$, $(b)_{\bar{\sigma}} = (d)_{\bar{\sigma}}$ and $(a)_{\bar{\sigma}} \preceq (b)_{\bar{\sigma}}$. Since $(a)_{\bar{\sigma}} \nabla (b)_{\bar{\sigma}}$, we have $(a, b) \in \sigma$. Since $(a)_{\bar{\sigma}} = (c)_{\bar{\sigma}}$, we have $(a, c) \in \bar{\sigma} := \sigma \cap \sigma^{-1} \subseteq \sigma^{-1}$, then $(c, a) \in \sigma$. Since $(b)_{\bar{\sigma}} = (d)_{\bar{\sigma}}$, we have $(b, d) \in \bar{\sigma} := \sigma \cap \sigma^{-1} \subseteq \sigma$, then $(b, d) \in \sigma$. Then $(c, d) \in \sigma$, and $(c)_{\bar{\sigma}} \nabla (d)_{\bar{\sigma}}$. (Cf. also [3]).

Theorem. *Let (S, .) be a semigroup. If σ is a semi-pseudoorder on S, then the set $(S/\bar{\sigma}, *, \nabla)$ is an ordered semigroup. Let ρ be a congruence on S and suppose there exists an order relation " $\preceq$ " on S/ρ such that $(S/\bar{\rho}, *, \nabla)$ be an ordered semigroup. Then there exists a semi-pseudoorder σ on S such that*

$$\rho = \bar{\sigma} \text{ and } \preceq = \nabla.$$

Proof. For the first part of the Theorem we refer to the Theorem in [3]. Let now ρ be a congruence on S and " $\preceq$ " an order on S/ρ such that $(S/\rho, *, \preceq)$ be an ordered semigroup. Let σ be the relation on S defined by

$$\sigma := \{(a, b) \in S \times S \mid (a)_\rho \preceq (b)_\rho\}.$$

1) σ is a semi-pseudoorder on S. In fact:
Let $a \in S$. Since $(a)_\rho \preceq (a)_\rho$, we have $(a, a) \in \sigma$. Let $(a, b) \in \sigma$, $(b, c) \in \sigma$. Then $(a)_\rho \preceq (b)_\rho$, $(b)_\rho \preceq (c)_\rho$, then $(a)_\rho \preceq (c)_\rho$, and $(a, c) \in \sigma$. Let $(a, b) \in \sigma$ and $c \in S$. Then $(a)_\rho \preceq (b)_\rho$ and $(c)_\rho \in S/\rho$. Since $(S/\rho, *, \preceq)$ is an ordered semigroup, we have $(a)_\rho * (c)_\rho \preceq (b)_\rho * (c)_\rho$, then $(ac)_\rho \preceq (bc)_\rho$, and $(ac, bc) \in \sigma$. Similarly $(a, b) \in \sigma$ and $c \in S$, imply $(ca, cb) \in \sigma$.

2) $\rho = \bar{\sigma}$. Indeed: We have

$$\begin{aligned}
(a, b) \in \rho \quad &\Leftrightarrow \quad (a)_\rho = (b)_\rho \\
&\Leftrightarrow \quad (a)_\rho \preceq (b)_\rho \text{ and } (b)_\rho \preceq (a)_\rho \\
&\Leftrightarrow \quad (a, b) \in \sigma \text{ and } (b, a) \in \sigma \\
&\Leftrightarrow \quad (a, b) \in \bar{\sigma}.
\end{aligned}$$

3) $\preceq = \nabla$. Indeed:
Let $(a)_\rho \preceq (b)_\rho$. Since $(a, b) \in \sigma$, we have $(a)_{\bar{\sigma}} \nabla (b)_{\bar{\sigma}}$. By 2), $\rho = \bar{\sigma}$. So $(a)_{\bar{\sigma}} = (a)_\rho$ and $(b)_{\bar{\sigma}} = (b)_\rho$. Then $(a)_\rho \nabla (b)_\rho$.

Let $(a)_\rho \nabla (b)_\rho$. Since $\rho = \bar{\sigma}$, we have $(a)_\rho = (a)_{\bar{\sigma}}$ and $(b)_\rho = (b)_{\bar{\sigma}}$. Then $(a)_{\bar{\sigma}} \nabla (b)_{\bar{\sigma}}$, hence $(a,b) \in \sigma$, and $(a)_\rho \preceq (b)_\rho$.

Remark 1. If $(S,.,\leq)$ is an ordered semigroup and ρ a pseudoorder on S, then the mapping

$$f\ (S,.,\leq)\ \rightarrow\ (S/\bar{\rho},*,\nabla)\ \mid\ a \rightarrow\ (a)_{\bar{\rho}}$$

is a homomorphism. In fact, if $a,b \in S$, then

$$f(ab) := (ab)_{\bar{\rho}} := (a)_{\bar{\rho}} * (b)_{\bar{\rho}} = f(a) * f(b).$$

Let now $a \leq b$. Since $(a,b) \in \leq \subseteq \rho$, we have $(a,b) \in \rho$. Then, since ρ is a semipseudoorder on S, we have $(a)_{\bar{\rho}} \nabla f(b)$, and $f(a) \nabla f(b)$. $\square$

For a semigroup S, we denote by $\mathcal{SP}(S)$ the set of semi-pseudoorders on S and by $\mathcal{C}(S)$ the set of congruences on S. Let " $\approx$ " be the equivalence relation on S defined as follows:

$$\rho \approx \sigma \text{ if and only if } \bar{\rho} = \bar{\sigma}.$$

Remark 2. If S is a semigroup and ρ a semi-pseudoorder on S, then the mapping

$$f\ :\ \mathcal{SP}(S)/\!\approx\ \rightarrow\ \mathcal{C}(S)\ \mid\ (\rho)_\approx\ \rightarrow \bar{\rho}$$

is (1-1) and onto. In fact: The mapping f is well defined: If ρ is a semi-pseudoorder on S, then $\bar{\rho}$ is a congruence on S. Let $\rho, \sigma \in \mathcal{SP}(S)$ and $(\rho)_\approx = (\sigma)_\approx$. Then we have $\rho \approx \sigma$, and $\bar{\rho} = \bar{\sigma}$.
f is (1-1): Let $\rho, \sigma \in \mathcal{SP}(S)$ such that $\bar{\rho} = \bar{\sigma}$. Then $\rho \approx \sigma$, and $(\rho)_\approx = (\sigma)_\approx$.
f is onto: Let $\rho \in \mathcal{C}(S)$. Then $\rho = \rho^{-1}$ and ρ is a semi-pseudoorder on S. Thus $\rho \in \mathcal{SP}(S)$ and

$$f((\rho)_\approx) := \bar{\rho} := \rho \cap \rho^{-1} = \rho \cap \rho = \rho.$$

$\square$

For a semigroup S, we denote by $\mathcal{OC}(S)$ the set of all congruences ρ on S for which there exists an order relation "∇" on S/ρ such that $(S/\rho,*,\nabla)$ is an ordered semigroup.

Remark 3. If S is a semigroup, then the mapping

$$f : \mathcal{SP}(S)/\!\approx\ \rightarrow \mathcal{OC}(S)\ \mid\ (\rho)_\approx \rightarrow \bar{\rho}$$

is (1-1) and onto. In fact: The mapping f is well defined: If ρ is a semi-pseudoorder on S, then $\bar{\rho}$ is a congruence on S. Then, by the Theorem, the set $(S/\bar{\rho},*,\nabla)$ is an ordered semigroup. Which means that $\bar{\rho} \in \mathcal{OC}(S)$.
Let $\rho, \sigma \in \mathcal{SP}(S)$ and $(\rho)_\approx = (\sigma)_\approx$. Then we have $\rho \approx \sigma$, and $\bar{\rho} = \bar{\sigma}$.
f is (1-1): Let $\rho, \sigma \in \mathcal{SP}(S)$ and $\bar{\rho} = \bar{\sigma}$. Then $\rho \approx \sigma$, and $(\rho)_\approx = (\sigma)_\approx$.
f is onto: Let $\rho \in \mathcal{OC}(S)$. By the Theorem, there exists a semi-pseudoorder σ on S such that $\rho = \bar{\sigma}$. Then $\sigma \in \mathcal{SP}(S)$, and $f((\sigma))_\approx := \bar{\sigma} = \rho$.

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