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THE CONTINUITY OF MULTIPLICATION FOR TWO TOPOLOGIES ASSOCIATED WITH A SEMIFINITE TRACE ON VON NEUMANN ALGEBRA

(submitted by D. Mushtari)

Abstract. Let $\mathcal{M}$ be a semifinite von Neumann algebra in a Hilbert space $\mathcal{H}$ and τ be a normal faithful semifinite trace on $\mathcal{M}$. Let $\mathcal{M}^{\mathrm{pr}}$ denote the set of all projections in $\mathcal{M}$, e denote the unit of $\mathcal{M}$, and $\|\cdot\|$ denote the C^*-norm on $\mathcal{M}$.

The set of all τ-measurable operators $\widetilde{\mathcal{M}}$ with sum and product defined as the respective closures of the usual sum and product, is a *-algebra. The sets

$U(\varepsilon,\delta)=\{x\in\widetilde{\mathcal{M}}\colon \|xp\|\leq\varepsilon \text{ and } \tau(e-p)\leq\delta \text{ for some } p\in\mathcal{M}^{\mathrm{pr}}\}$, $\varepsilon>0$, $\delta>0$,

form a base at 0 for a metrizable vector topology t_τ on $\widetilde{\mathcal{M}}$, called *the measure topology*. Equipped with this topology, $\widetilde{\mathcal{M}}$ is a complete topological *-algebra. We will write $x_i \xrightarrow{\tau} x$ in case a net $\{x_i\}_{i\in I}\subset\widetilde{\mathcal{M}}$ converges to $x\in\widetilde{\mathcal{M}}$ for the measure topology on $\widetilde{\mathcal{M}}$. By definition, a net $\{x_i\}_{i\in I}\subset\widetilde{\mathcal{M}}$ *converges τ-locally to* $x\in\widetilde{\mathcal{M}}$ (notation: $x_i \xrightarrow{\tau l} x$) if $x_ip \xrightarrow{\tau} xp$ for all $p\in\mathcal{M}^{\mathrm{pr}}$, $\tau(p)<\infty$; and a net $\{x_i\}_{i\in I}\subset\widetilde{\mathcal{M}}$ *converges weak τ-locally to* $x\in\widetilde{\mathcal{M}}$ (notation: $x_i \xrightarrow{w\tau l} x$) if $px_ip \xrightarrow{\tau} pxp$ for all $p\in\mathcal{M}^{\mathrm{pr}}$, $\tau(p)<\infty$.

Theorem 1. *Let $x_i, x\in\widetilde{\mathcal{M}}$.*

1. *If $x_i \xrightarrow{\tau l} x$, then $x_iy \xrightarrow{\tau l} xy$ and $yx_i \xrightarrow{\tau l} yx$ for every fixed $y\in\widetilde{\mathcal{M}}$.*

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2. *If* $x_i \xrightarrow{w\tau l} x$, *then* $x_i y \xrightarrow{w\tau l} xy$ *and* $yx_i \xrightarrow{w\tau l} yx$ *for every fixed* $y \in \widetilde{\mathcal{M}}$.

Theorem 2. *If* $\{x_i\}_{i\in I} \subset \widetilde{\mathcal{M}}$ *is bounded in measure and if* $x_i \xrightarrow{\tau l} x \in \widetilde{\mathcal{M}}$, *then* $x_i y \xrightarrow{\tau} xy$ *for all* τ*-compact* $y \in \widetilde{\mathcal{M}}$.

Theorem 3. *Let* $x, y, x_i, y_i \in \widetilde{\mathcal{M}}$ *and let a set* $\{x_i\}_{i\in I}$ *be bounded in measure. If* $x_i \xrightarrow{\tau l} x$ *and* $y_i \xrightarrow{\tau l} y$, *then* $x_i y_i \xrightarrow{\tau l} xy$.

If $\mathcal{M}$ is abelian, then the weak τ-local and τ-local convergencies on $\widetilde{\mathcal{M}}$ coincides with the familiar convergence locally in measure. If $\tau(e) = \infty$, then the boundedness condition cannot be omitted in Theorem 2.

If $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ with standard trace, then Theorem 2 for sequences is a "Basic lemma"of the theory of projection methods: *If y is compact and $x_n \to x$ strongly, then $x_n y \to xy$ uniformly, i.e.* $\|x_n y - xy\| \to 0$ as $n \to \infty$. Theorem 3 means that the mapping

$$(x, y) \mapsto xy : \ (\mathcal{B}(\mathcal{H})_1 \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}))$$

is strong-operator continuous ($\mathcal{B}(\mathcal{H})_1$ denotes the unit ball of $\mathcal{B}(\mathcal{H})$).

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1. Introduction

Let $\mathcal{M}$ be a semifinite von Neumann algebra of operators in a Hilbert space $\mathcal{H}$ and τ be a distinguished normal faithful semifinite trace on $\mathcal{M}$. Let $\mathcal{M}^{\mathrm{pr}}$ denote the lattice of all projections in $\mathcal{M}$, e denote the identity, and $\mathcal{M}_1$ denote the unit ball of $\mathcal{M}$ in the C^*-norm $\|\cdot\|$ on $\mathcal{M}$. The closed, densely defined linear operator x in $\mathcal{H}$ with domain $\mathcal{D}(x)$ is said to be *affiliated* with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$ then x is said to be τ*-measurable* if and only if, for every $\varepsilon > 0$ there exists a projection $p \in \mathcal{M}^{\mathrm{pr}}$ for which $p(\mathcal{H}) \subseteq \mathcal{D}(x)$ and $\tau(e - p) < \varepsilon$. We denote by $\widetilde{\mathcal{M}}$ the set of all τ-measurable operators. With sum and product defined as the respective closures of the usual sum and product, $\widetilde{\mathcal{M}}$ is a *-algebra. The sets

$$U(\varepsilon, \delta) = \{x \in \widetilde{\mathcal{M}} : \|xp\| \leq \varepsilon \text{ and } \tau(e - p) \leq \delta \text{ for some } p \in \mathcal{M}^{\mathrm{pr}}\},$$

where $\varepsilon > 0$, $\delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $\widetilde{\mathcal{M}}$, called *the measure topology* ([8]; [11, p. 18]). Equipped with this topology, $\widetilde{\mathcal{M}}$ is a complete topological *-algebra in which $\mathcal{M}$ is dense. We will write $x_i \xrightarrow{\tau} x$ in case a net $\{x_i\}_{i\in I} \subset \widetilde{\mathcal{M}}$ converges to $x \in \widetilde{\mathcal{M}}$ for the measure topology on $\widetilde{\mathcal{M}}$.

A subset X of $\widetilde{\mathcal{M}}$ is *bounded in measure*, if it is bounded with respect to this topology on the vector space of $\widetilde{\mathcal{M}}$, that is in case for every neighborhood U of 0 there is an $\alpha > 0$ such that $\alpha X \subset U$ [8, p. 106].

If $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$, the von Neumann algebra of all bounded linear operators in $\mathcal{H}$ is equipped with the usual standard trace, then $\widetilde{\mathcal{M}}$ coincides with $\mathcal{M}$ and in this case the measure topology coincides with the $\|\cdot\|$-topology. If $\mathcal{M}$ is abelian, then $\mathcal{M}$ may be identified with $L^\infty(\Omega, \mu)$ and $\tau(f) = \int_\Omega f \, \mathrm{d}\mu$ where (Ω, μ) is a localizable measure space. In this case, $\widetilde{\mathcal{M}}$ is the space $S_0(\Omega)$ consisting of those measurable complex-valued functions on Ω which are bounded except on a set of finite measure and the measure topology on $\widetilde{\mathcal{M}}$ may be identified simply with the familiar topology of convergence in measure.

If x is any self-adjoint operator in $\mathcal{H}$ and if

$$x = \int_{\mathbb{R}} \lambda \, \mathrm{d}e^x_\lambda$$

is its spectral representation, we will write $\chi_T(x)$ for the spectral projection of x corresponding to the Borel subset $T \subset \mathbb{R}$. In particular $e^x_\lambda = \chi_{(-\infty,\lambda]}(x)$. If x is closed, densely defined linear operator affiliated with $\mathcal{M}$ and $|x| = \sqrt{x^*x}$, then the spectral resolution $\chi_\bullet(|x|)$ is contained in $\mathcal{M}$ and $x \in \widetilde{\mathcal{M}}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\tau(\chi_{(\lambda,\infty)}(|x|)) < \infty$.

For $p, q \in \mathcal{M}^{\mathrm{pr}}$ we write $p \sim q$ (*the Murray – von Neumann equivalence*), if $u^*u = p$ and $uu^* = q$ for some $u \in \mathcal{M}$.

A linear set $\mathcal{D}$ in $\mathcal{H}$ is said to be *associated* with $\mathcal{M}$ if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary operator u in $\mathcal{M}'$. If $\mathcal{D}$ is a closed linear manifold then $\mathcal{D}$ is associated with $\mathcal{M}$ if and only if the projection onto $\mathcal{D}$ lies in $\mathcal{M}$ [9, p. 403]. For every $x \in \widetilde{\mathcal{M}}$ the projection onto the closure of the range of x lies in $\mathcal{M}$. It is equal to the left support projection

$$s_l(x) = \wedge\{q \in \mathcal{M}^{\mathrm{pr}} : \; qx = x\}$$

and $s_l(x) \sim s_l(x^*)$.

The two-sided ideal of τ-*compact* operators

$$\widetilde{\mathcal{M}}_0 = \{x \in \widetilde{\mathcal{M}} : \; \tau(\chi_{(\lambda,\infty)}(|x|)) < \infty \text{ for all } \lambda > 0\}$$

is closed in measure topology [12]. If $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ with standard trace, then $\widetilde{\mathcal{M}}_0$ is precisely the ideal of compact operators. Let

$$\mathcal{M}^{\mathrm{pr}}_0 = \widetilde{\mathcal{M}}_0 \bigcap \mathcal{M}^{\mathrm{pr}} = \{p \in \mathcal{M}^{\mathrm{pr}} : \; \tau(p) < \infty\}.$$

Definition 1 (cf. [3, p. 114]). *A net* $\{x_i\}_{i\in I} \subset \widetilde{\mathcal{M}}$ *is said to converge* τ*-locally to* $x \in \widetilde{\mathcal{M}}$ *(notation:* $x_i \xrightarrow{\tau l} x$*) if* $x_i p \xrightarrow{\tau} xp$ *for all* $p \in \mathcal{M}_0^{\mathrm{pr}}$.

Definition 2 (cf. [3, p. 114]; [5, p. 746]). *A net* $\{x_i\}_{i\in I} \subset \widetilde{\mathcal{M}}$ *is said to converge weak* τ*-locally to* $x \in \widetilde{\mathcal{M}}$ *(notation:* $x_i \xrightarrow{w\tau l} x$*) if* $px_ip \xrightarrow{\tau} pxp$ *for all* $p \in \mathcal{M}_0^{\mathrm{pr}}$.

It is clear that

$$x_i \xrightarrow{\tau} x \Longrightarrow x_i \xrightarrow{\tau l} x \Longrightarrow x_i \xrightarrow{w\tau l} x \text{ for } x_i, x \in \widetilde{\mathcal{M}}.$$

If $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ with standard trace, then τ-local (respectively, weak τ-local) convergence coincides with strong-operator (respectively, weak-operator) convergence. If $\tau(e) < \infty$, then $\widetilde{\mathcal{M}}$ consists of all densely defined closed linear operators affiliated with $\mathcal{M}$ and weak τ-local convergence is precisely the convergence in measure topology on $\widetilde{\mathcal{M}}$. Moreover, the measure topology is a minimal one in the class of all topologies which are Hausdorff, metrizable, and compatible with the ring structure of $\widetilde{\mathcal{M}}$ [1, Theorem 2].

2. Main Results

Further we assume that $\tau(e) = \infty$.

Theorem 1. *Let* $x_i, x \in \widetilde{\mathcal{M}}$.

1. *If* $x_i \xrightarrow{\tau l} x$*, then* $x_iy \xrightarrow{\tau l} xy$ *and* $yx_i \xrightarrow{\tau l} yx$ *for every fixed* $y \in \widetilde{\mathcal{M}}$.

2. *If* $x_i \xrightarrow{w\tau l} x$*, then* $x_iy \xrightarrow{w\tau l} xy$ *and* $yx_i \xrightarrow{w\tau l} yx$ *for every fixed* $y \in \widetilde{\mathcal{M}}$.

Proof. Let $x_i, x, y \in \widetilde{\mathcal{M}}$ and let $p \in \mathcal{M}_0^{\mathrm{pr}}$. Since $s_l(yp) \sim s_l(py^*) \le p$, one has $s_l(yp) \in \mathcal{M}_0^{\mathrm{pr}}$.

1. Suppose that $x_i \xrightarrow{\tau l} x$. One has

$$yx_i \xrightarrow{\tau l} yx \text{ and } x_iyp = x_is_l(yp)yp \xrightarrow{\tau} xs_l(yp)yp = xyp,$$

since the multiplication operations $z \mapsto yz$ ($\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$) and $z \mapsto zyp$ ($\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$) are continuous in the measure topology.

2. One has $r = p \vee q \in \mathcal{M}_0^{\mathrm{pr}}$ for $p, q \in \mathcal{M}_0^{\mathrm{pr}}$, since $p \vee q - p \sim q - p \wedge q$ [8, p. 105]. By [3, p. 114] $x_i \xrightarrow{w\tau l} x$ if and only if $px_iq \xrightarrow{\tau} pxq$ for all $p, q \in \mathcal{M}_0^{\mathrm{pr}}$. Indeed, from $rx_ir \xrightarrow{\tau} rxr$ it follows that

$$px_iq = p \cdot rx_ir \cdot q \xrightarrow{\tau} p \cdot rxr \cdot q = pxq.$$

Therefore,

$$px_iyp = px_is_l(yp)yp \xrightarrow{\tau} pxs_l(yp)yp = pxyp.$$

Now the convergence $yx_i \xrightarrow{w\tau l} yx$ follows from the fact that the mapping $z \mapsto z^*$ ($\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$) is weak τ-local continuous and by taking adjoints.

Theorem 2. *If $\{x_i\}_{i\in I} \subset \widetilde{\mathcal{M}}$ is bounded in measure and if $x_i \xrightarrow{\tau l} x \in \widetilde{\mathcal{M}}$, then $x_i y \xrightarrow{\tau} xy$ for all $y \in \widetilde{\mathcal{M}}_0$.*

Proof. Step 1. Without loss of generality we may assume that $y \in \widetilde{\mathcal{M}}_0$ is self-adjoint and non-negative. Indeed, let $y \in \widetilde{\mathcal{M}}_0$ and $y^* = u|y^*|$ be the polar decomposition of y^*. Then $y = |y^*|u^*$ and from $x_i|y^*| \xrightarrow{\tau} x|y^*|$ it follows that $x_i y \xrightarrow{\tau} xy$, since the multiplication operation $z \mapsto zu^*$ ($\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$) is continuous in the measure topology.

Step 2. Fix non-negative $y \in \widetilde{\mathcal{M}}_0$ and $\varepsilon, \delta > 0$. A subset X of $\widetilde{\mathcal{M}}$ is bounded in measure if and only if for every $d > 0$ there exists a constant $c < \infty$ such that $X \subset U(c, d)$ [8, p. 106]. Let $n \in \mathbb{N}$ and

$$y_{1,n} = \int_{[0,n^{-1})} \lambda \, \mathrm{d}e_\lambda^y, \quad y_{2,n} = \int_{[n^{-1},n)} \lambda \, \mathrm{d}e_\lambda^y, \quad y_{3,n} = \int_{[n,\infty)} \lambda \, \mathrm{d}e_\lambda^y.$$

Then $y = y_{1,n} + y_{2,n} + y_{3,n}$ and for $z_i = x_i - x$ one has

$$x_i y - xy = z_i y_{1,n} + z_i y_{2,n} + z_i y_{3,n}, \quad i \in I. \tag{1}$$

The set $\{z_i\}_{i\in I}$ is bounded in measure. There exists a constant $c > 0$ such that

$$\{z_i\}_{i\in I} \subset U(c, \delta). \tag{2}$$

Let

$$n_1 = \min\{k \in \mathbb{N} : \ 2\varepsilon k \geq c\}.$$

Since $\|y_{1,n}\| < n^{-1}$, one has $y_{1,n} \xrightarrow{\tau} 0$ as $n \to \infty$. Since

$$\tau(s_l(y_{3,n})) = \tau(\chi_{[n,\infty)}(y)) \to 0 \ \text{ as } n \to \infty,$$

one has $y_{3,n} \xrightarrow{\tau} 0$ as $n \to \infty$. Therefore $y_{1,n} + y_{3,n} \xrightarrow{\tau} 0$ as $n \to \infty$. Then there exists $m \in \mathbb{N}$ such that

$$y_{1,n} + y_{3,n} \subset U({n_1}^{-1}, \delta) \text{ for all } n \geq m. \tag{3}$$

Recall that

$$U(\varepsilon_1, \delta_1)U(\varepsilon_2, \delta_2) \subset U(\varepsilon_1\varepsilon_2, \delta_1 + \delta_2) \text{ for all } \varepsilon_1, \delta_1, \varepsilon_2, \delta_2 > 0 \tag{4}$$

by [8, p. 107], [11, p. 18]. Now by (2) and (3) one has

$$z_i y_{1,n} + z_i y_{3,n} \in U(2\varepsilon, 2\delta) \text{ for all } i \in I, \ n \geq m. \tag{5}$$

Step 3. Let m be as above, $\lambda_k > 0$ and $p_k \in \mathcal{M}_0^{\mathrm{pr}}$ $(k = 1, \ldots, j)$, $p_k p_l = 0$ for $k \neq l$, such that

$$y_{2,m}^2 = \int_{[m^{-1},m)} \lambda^2 \, \mathrm{d}e_\lambda^y \leq \sum_{k=1}^{j} \lambda_k^2 p_k$$

(one can choose p_k as spectral projections of y). There exists $z \in \mathcal{M}_1$ such that

$$y_{2,m} = (\sum_{k=1}^{j} \lambda_k p_k) \cdot z$$

[4, Chap. 1, Sect. 1, Lemma 2]. Since $z_i \xrightarrow{\tau l} 0$, one has $z_i p_k \xrightarrow{\tau} 0$ for all $k = 1, \ldots, j$. Now

$$z_i y_{2,m} = \sum_{k=1}^{j} \lambda_k z_i p_k z \xrightarrow{\tau} 0,$$

because the multiplication operation $t \mapsto tz$ $(\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}})$ is continuous in the measure topology. Therefore, there exists $i_0 \in I$ such that

$$z_i y_{2,m} \in U(\varepsilon, \delta) \text{ for all } i \in I,\ i \geq i_0. \tag{6}$$

Step 4. Recall that

$$U(\varepsilon_1, \delta_1) + U(\varepsilon_2, \delta_2) \subset U(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2) \text{ for all } \varepsilon_1, \delta_1, \varepsilon_2, \delta_2 > 0 \tag{7}$$

by [8, p. 107], [11, p. 18]. The assertion of Theorem 2 follows from (1), (5) and (6), since

$$x_i y - xy \in U(3\varepsilon, 3\delta) \text{ for all } i \in I,\ i \geq i_0.$$

Theorem 3. *Let $x, y, x_i, y_i \in \widetilde{\mathcal{M}}$ and let a set $\{x_i\}_{i\in I}$ be bounded in measure. If $x_i \xrightarrow{\tau l} x$ and $y_i \xrightarrow{\tau l} y$, then $x_i y_i \xrightarrow{\tau l} xy$.*

Proof. For every $p \in \mathcal{M}_0^{\mathrm{pr}}$ one has

$$x_i y_i p - xyp = x_i(y_i p - yp) + (x_i - x)yp, \quad i \in I. \tag{8}$$

Fix $\varepsilon, \delta > 0$. By assumption of the theorem, there exists a constant $c > 0$ such that

$$\{x_i\}_{i\in I} \subset U(c, \delta). \tag{9}$$

Since $y_i p - yp \xrightarrow{\tau} 0$, there exists $i_1 \in I$ such that

$$y_i p - yp \in U(2\varepsilon c^{-1}, \delta) \text{ for all } i \in I, i \geq i_1. \tag{10}$$

Now by (9), (10) and (4) one has

$$x_i(y_i p - yp) \in U(2\varepsilon, 2\delta) \text{ for all } i \in I, i \geq i_1. \tag{11}$$

Since $x_i - x \xrightarrow{\tau l} 0$ and $yp \in \widetilde{\mathcal{M}}_0$, it follows by Theorem 2 that there exists $i_2 \in I$ such that

$$(x_i - x)yp \in U(\varepsilon, \delta) \text{ for all } i \in I, i \geq i_2. \tag{12}$$

There exists $i_0 \in I$ such that $i_0 \geq i_1$ and $i_0 \geq i_2$. Now by (8), (11), (12) and (7) one has

$$x_i y_i p - xyp \in U(3\varepsilon, 3\delta) \text{ for all } i \in I, i \geq i_0.$$

This proves the theorem.

Example 1. If $\mathcal{M}$ is abelian, then the weak τ-local and τ-local convergencies on $\widetilde{\mathcal{M}}$ coincides with the familiar convergence locally in measure (i.e., in other words, convergence in measure on every set of finite measure). The boundedness condition for $\{x_i\}_{i\in I}$ cannot be omitted in Theorem 2. Indeed, let $\Omega = (0, \infty)$ equipped with the Lebesgue measure μ. Define the functions

$$y(t) = \min\{1, t^{-1}\};\ x_n(t) = t\, \chi_{[n,2n]}(t) \quad (t \in (0, \infty),\ n \in \mathbb{N}).$$

Then
i) $x_n \xrightarrow{\tau l} 0$ as $n \to \infty$;
ii) $\{x_n\}_{n=1}^{\infty}$ is not bounded in measure;
iii) $y \in \widetilde{\mathcal{M}}_0 \bigcap \mathcal{M}_1$;
iv) since $(x_n y)(t) = \chi_{[n,2n]}(t)$ for every $t \in (0, \infty)$, $n \in \mathbb{N}$, $x_n y$ does not converge in measure topology.

Example 2. If $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ with standard trace, then Theorem 2 for sequences is a "Basic lemma" of the projection methods [2, pp. 18–19] (the boundedness condition for $\{x_n\}_{n=1}^{\infty}$ follows from *the principle of uniform boundedness*):

If y is compact and $x_n \to x$ strongly, then $x_n y \to xy$ uniformly, i.e. $\|x_n y - xy\| \to 0$ *as* $n \to \infty$.

Theorem 3 means that the mapping

$$(x, y) \mapsto xy : \ (\mathcal{B}(\mathcal{H})_1 \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}))$$

is strong-operator continuous [7, pp. 115–117].

Remark. The second part of Theorem 1 was already used in [6] and [10].

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