

Lobachevskii Journal of Mathematics

<http://ljm.ksu.ru>

Vol. 14, 2004, 25–32

© A. I. Fedotov

*A. I. Fedotov*

**LEBESGUE CONSTANT ESTIMATION IN  
MULTIDIMENSIONAL SOBOLEV SPACE**

(submitted by F. Avkhadiev)

ABSTRACT. The norm estimation of the Lagrange interpolation operator is obtained. It is shown that the rate of convergence of the interpolative polynomials depends on the choice of the sequence of multiindices and, for some sequences, is equal to the rate of the best approximation of the interpolated function.

INTRODUCTION

In the paper [1] the collocation method for singular integral equations and periodic pseudodifferential equations in 1-dimensional Sobolev space was justified. The crucial role in the justification and error estimation plays the fact (Lemma 4) that the Lagrange interpolation operator in this space is bounded. To generalize this results for the multidimensional case the norm estimation (i.e. estimation of the Lebesgue constant) of the Lagrange interpolation operator in multidimensional Sobolev spaces is needed.

Here, we show that in  $m$ -dimensional Sobolev space  $H^s$  ( $s > m/2$ ) the norm of  $\mathbf{n}$ -order ( $\mathbf{n} = (n_1, n_2, \dots, n_m)$ ) Lagrange interpolation operator depends of the function  $M(\mathbf{n}, s)$  which, w.r.t. the choice of the sequence of multiindices  $(\mathbf{n})$ ,  $\mathbf{n} \rightarrow \infty$ , is either bounded, or grows infinitely.

---

*2000 Mathematical Subject Classification.* 65D05.

*Key words and phrases.* Lagrange interpolation operator, Sobolev space.

## 1. FORMULATION OF THE PROBLEM

Let's fix the natural  $m \in \mathbf{N}$  and denote by  $\mathbf{N} = N^m$ ,  $\mathbf{N}_0 = N_0^m$ ,  $\mathbf{Z} = Z^m$ ,  $\mathbf{R} = R^m$ ,  $\mathbf{\Delta} = \Delta^m$  Cartesian degrees of the sets of natural  $N$ , natural with zero added  $N_0$ , integer  $Z$ , real  $R$  numbers and the interval  $\Delta = (-\pi; \pi] \subset R$  correspondingly. For the elements of these sets ( $m$ -components vectors) besides the usual operations of addition, subtraction and multiplication to the number we'll define the following operations

$$\mathbf{l} \cdot \mathbf{k} = \sum_{j=1}^m l_j k_j, \quad \mathbf{l}^2 = \sum_{j=1}^m l_j^2, \quad \mathbf{l} * \mathbf{k} = (l_1 k_1, l_2 k_2, \dots, l_m k_m), \quad [\mathbf{l}] = \prod_{j=1}^m l_j,$$

and the partial order

$$\mathbf{l} < \mathbf{k} \equiv \&_{j=1}^m (l_j < k_j), \quad \mathbf{l} = (l_1, l_2, \dots, l_m), \quad \mathbf{k} = (k_1, k_2, \dots, k_m).$$

By  $\mathbf{n} \rightarrow \infty$  we'll mean, that  $\mathbf{n}$  takes the values of some sequence

$$(\mathbf{n}_j), \quad \mathbf{n}_j \in \mathbf{N}, \quad \mathbf{n}_j < \mathbf{n}_{j+1}, \quad j = 1, 2, \dots .$$

Furthermore, in a sake of simplicity we'll write

$$\min(\mathbf{n}) \text{ instead of } \min_{1 \leq j \leq m} \{n_j \mid \mathbf{n} = (n_1, n_2, \dots, n_m) \in \mathbf{N}\}$$

and

$$\max(\mathbf{n}) \text{ instead of } \max_{1 \leq j \leq m} \{n_j \mid \mathbf{n} = (n_1, n_2, \dots, n_m) \in \mathbf{N}\}.$$

For the fixed  $s \in \mathbf{R}$  let  $H^s$  denote  $m$ -dimensional Sobolev space, i.e. the closure of all  $m$ -dimensional smooth  $2\pi$ -periodic by every variable complex-valued functions w.r.t. the norm

$$\|u\|_s = \|u\|_{H^s} = \left( \sum_{\mathbf{l} \in \mathbf{Z}} (1 + \mathbf{l}^2)^s |\widehat{u}(\mathbf{l})|^2 \right)^{1/2},$$

where

$$\widehat{u}(\mathbf{l}) = (2\pi)^{-m} \int_{\mathbf{\Delta}} u(\boldsymbol{\tau}) \bar{e}_1(\boldsymbol{\tau}) d\boldsymbol{\tau}, \quad \mathbf{l} \in \mathbf{Z},$$

are the complex-valued Fourier coefficients of the function  $u \in H^s$  w.r.t the trigonometric monomials

$$e_1(\boldsymbol{\tau}) = \exp(i\mathbf{l} \cdot \boldsymbol{\tau}), \quad \mathbf{l} \in \mathbf{Z}, \quad \boldsymbol{\tau} \in \mathbf{\Delta}, \quad i = \sqrt{-1}.$$

It is known that, being equipped with the inner product

$$\langle u, v \rangle_s = \sum_{\mathbf{l} \in \mathbf{Z}} (1 + \mathbf{l}^2)^s \widehat{u}(\mathbf{l}) \bar{\widehat{v}}(\mathbf{l}), \quad u, v \in H^s,$$

$H^s$  becomes Hilbert space. For the following we'll assume that  $s > m/2$ , providing (see e.g. [2]) the embedding of  $H^s$  in the space of continuous functions.

Let's fix  $\mathbf{n} = (n_1, n_2, \dots, n_m) \in \mathbf{N}$ , denote by

$$\mathbf{I}_{\mathbf{n}} = I_{n_1} \times I_{n_2} \times \dots \times I_{n_m}, \quad I_{n_j} = \{k_j \mid k_j \in Z, |k_j| \leq n_j\}, \quad j = 1, 2, \dots, m,$$

the set of indices and define uniform partition

$$\Delta_{\mathbf{n}} = \{\mathbf{t}_{\mathbf{k}} = (t_{k_1}, t_{k_2}, \dots, t_{k_m}) \mid \mathbf{k} = (k_1, k_2, \dots, k_m) \in \mathbf{I}_{\mathbf{n}},$$

$$t_{k_j} = k_j h_j, \quad h_j = 2\pi/(2n_j + 1), \quad j = 1, 2, \dots, m\},$$

on  $\Delta$ . By  $P_{\mathbf{n}}$  we denote Lagrange interpolation operator that assigns to every function  $u \in H^s$  polynomial

$$(P_{\mathbf{n}}u)(\tau) = \sum_{\mathbf{k} \in \mathbf{I}_{\mathbf{n}}} u(\mathbf{t}_{\mathbf{k}}) \xi_{\mathbf{n}}(\tau, \mathbf{t}_{\mathbf{k}}), \quad \tau = (\tau_1, \tau_2, \dots, \tau_m) \in \Delta,$$

where  $\mathbf{t}_{\mathbf{k}} = (t_{k_1}, t_{k_2}, \dots, t_{k_m}) \in \Delta_{\mathbf{n}}$ , coinciding with  $u$  in the nodes  $\Delta_{\mathbf{n}}$ . Here

$$\xi_{\mathbf{n}}(\tau, \mathbf{t}_{\mathbf{k}}) = \prod_{j=1}^m \frac{\sin((2n_j + 1)(\tau_j - t_{k_j})/2)}{(2n_j + 1) \sin((\tau_j - t_{k_j})/2)} = [2\mathbf{n} + \mathbf{1}]^{-1} \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} e_{\mathbf{l}}(\tau - \mathbf{t}_{\mathbf{k}}),$$

$$\mathbf{1} = (1, 1, \dots, 1) \in \mathbf{N}, \quad \tau \in \Delta, \quad \mathbf{t}_{\mathbf{k}} \in \Delta_{\mathbf{n}},$$

are fundamental polynomials satisfying

$$\xi_{\mathbf{n}}(\mathbf{t}_{\mathbf{l}}, \mathbf{t}_{\mathbf{k}}) = \begin{cases} 1, & \mathbf{l} = \mathbf{k}, \\ 0, & \mathbf{l} \neq \mathbf{k}, \quad \mathbf{l}, \mathbf{k} \in \mathbf{I}_{\mathbf{n}}. \end{cases}$$

We have to estimate the norm of the operator  $P_{\mathbf{n}} : H^s \rightarrow H^s$ .

## 2. PRELIMINARIES

The results of this section are technical ones. They are gathered in 2 lemmas to exclude less important details from the proof of the main result.

**Lemma 1.** *For every  $m \in \mathbf{N}$ ,  $s \in \mathbf{R}$ ,  $s > m/2$  and  $\mathbf{n} \in \mathbf{N}$*

$$\sum_{\mathbf{j} \in \mathbf{Z}} ((\mathbf{n} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^{-s} \leq 2^m \sum_{\mathbf{l} \in \mathbf{N}} ((\mathbf{n} * (2\mathbf{l} - \mathbf{1}))^2)^{-s}.$$

**Proof.** To change the set of sum indices from  $Z$  to  $N$  let's represent  $Z$  as a merge of two sets:  $\{-l \mid l \in N\}$  and  $\{l - 1 \mid l \in N\}$ . For the  $k$ -th,

$1 \leq k \leq m$ , component of the vector  $\mathbf{n} + \mathbf{j} * (2\mathbf{n} + \mathbf{1})$  we'll obtain

$$\begin{aligned} (n_k + j_k(2n_k + 1))^2 &= (l_k(2n_k + 1) - n_k)^2 \\ &= \left(n_k \left(2l_k + \frac{l_k}{n_k} - 1\right)\right)^2 \geq (n_k(2l_k - 1))^2, \\ &\quad j_k \in \mathbb{Z}, j_k < 0, l_k = -j_k \in \mathbb{N}; \\ (n_k + j_k(2n_k + 1))^2 &= (n_k + (l_k - 1)(2n_k + 1))^2 = (n_k + 2n_k l_k + l_k - 2n_k - 1)^2 \\ &= \left(n_k \left(2l_k + \frac{l_k}{n_k} - 1\right) - 1\right)^2 \geq (n_k(2l_k - 1))^2, \\ &\quad j_k \in \mathbb{Z}, j_k \geq 0, l_k = j_k + 1 \in \mathbb{N}. \end{aligned}$$

As to each summand of index  $\mathbf{l} \in \mathbb{N}$  correspond  $2^m$  summands when adding by  $\mathbb{Z}$  then

$$\sum_{\mathbf{j} \in \mathbb{Z}} ((\mathbf{n} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^{-s} \leq 2^m \sum_{\mathbf{l} \in \mathbb{N}} ((\mathbf{n} * (2\mathbf{l} - \mathbf{1}))^2)^{-s}. \square$$

Let

$$A_p^m = \{\mathbf{k} \mid \mathbf{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}_0, \sum_{j=1}^m k_j = p\}$$

be the set of vectors from  $\mathbb{N}_0$  which component's sum is  $p \in \mathbb{N}_0$ . By  $R(A_p^m)$  we denote the number of elements of  $A_p^m$ .

**Lemma 2.** For every  $p, m \in \mathbb{N}$

$$R(A_p^m) \leq mp^{m-1}.$$

**Proof.** We'll show first that

$$R(A_p^m) = C_{m+p-1}^p = \frac{(m+p-1)!}{p!(m-1)!}, \quad p \in \mathbb{N}_0, \quad m \in \mathbb{N}, \quad (1)$$

and then that

$$C_{m+p-1}^p \leq mp^{m-1}, \quad p, m \in \mathbb{N}. \quad (2)$$

Let  $m = 1$ , then for every  $p \in \mathbb{N}_0$  the set  $A_p^1$  contains only one vector, and hence  $R(A_p^1) = C_p^p = p^0 = 1$ . Assume that (1) is valid for some  $m \in \mathbb{N}$ , and prove that it is valid then for  $m + 1$ . We'll construct the set  $A_p^{m+1}$  as a merge of the sets  $A_j^m$ ,  $j = 0, 1, \dots, p$ , adding to each element of the set  $A_j^m$   $m + 1$ -th component equal to  $p - j$ ,  $j = 0, 1, \dots, p$ . Then

$$R(A_p^{m+1}) = \sum_{j=0}^p R(A_j^m) = \sum_{j=0}^p C_{m+j-1}^j = \frac{(m+p)!}{m!p!} = C_{m+1}^p,$$

and hence (1) is valid for all  $p \in \mathbb{N}_0$  and  $m \in \mathbb{N}$ .

Now assume that estimation (2) is valid for some  $m \in N$ , and prove that it is valid then for  $m + 1$ . Indeed,

$$\begin{aligned} C_{m+p}^p &= \frac{(m+p)!}{m!p!} \\ &= C_{m+p-1}^p \frac{m+p}{m} \leq mp^{m-1} \frac{m+p}{m} = p^m \left( \frac{m}{p} + 1 \right) \leq (m+1)p^m. \square \end{aligned} \quad (3)$$

### 3. MAIN RESULTS

**Theorem 1.** *For every  $s \in R$ ,  $m \in N$ ,  $s > m/2$  and  $\mathbf{n} \in \mathbf{N}$  following estimation is valid*

$$\|P_{\mathbf{n}}\|_{H^s \rightarrow H^s} \leq 2^{\frac{m-s}{2}} m^{\frac{s+1}{2}} M(\mathbf{n}, s) \sqrt{1 + \zeta(2s - m + 1)},$$

where

$$M(\mathbf{n}, s) = \left( \frac{\sqrt{\mathbf{n}^2}}{\min(\mathbf{n})} \right)^s,$$

and  $\zeta(t) = \sum_{j=1}^{\infty} j^{-t}$  - is Riemann's  $\zeta$ -function bounded and decreasing for  $t > 1$ .

**Proof.** Let's fix  $m \in N$ ,  $s \in R$ ,  $s > m/2$ ,  $\mathbf{n} \in \mathbf{N}$ , choose an arbitrary function  $u \in H^s$  and write Lagrange interpolative polynomial w.r.t. the nodes  $\Delta_n$  for it

$$(P_{\mathbf{n}}u)(\tau) = \sum_{\mathbf{k} \in \mathbf{I}_{\mathbf{n}}} u(\mathbf{t}_{\mathbf{k}}) \xi_{\mathbf{n}}(\tau, \mathbf{t}_{\mathbf{k}}).$$

It's Fourier coefficients are

$$(\widehat{P_{\mathbf{n}}u})(\mathbf{l}) = \begin{cases} [2\mathbf{n} + \mathbf{1}]^{-1} \sum_{\mathbf{k} \in \mathbf{I}_{\mathbf{n}}} u(\mathbf{t}_{\mathbf{k}}) \bar{e}_{\mathbf{l}}(\mathbf{t}_{\mathbf{k}}), & \mathbf{l} \in \mathbf{I}_{\mathbf{n}}, \\ 0, & \mathbf{l} \notin \mathbf{I}_{\mathbf{n}}. \end{cases}$$

Substituting the values of function  $u$  in the nodes  $\Delta_n$  by its Fourier series expansion we'll obtain

$$\begin{aligned} (\widehat{P_{\mathbf{n}}u})(\mathbf{l}) &= [2\mathbf{n} + \mathbf{1}]^{-1} \sum_{\mathbf{k} \in \mathbf{I}_{\mathbf{n}}} \left( \sum_{\mathbf{j} \in \mathbf{Z}} \hat{u}(\mathbf{j}) e_{\mathbf{j}}(\mathbf{t}_{\mathbf{k}}) \right) \bar{e}_{\mathbf{l}}(\mathbf{t}_{\mathbf{k}}) = \\ &= [2\mathbf{n} + \mathbf{1}]^{-1} \sum_{\mathbf{j} \in \mathbf{Z}} \hat{u}(\mathbf{j}) \sum_{\mathbf{k} \in \mathbf{I}_{\mathbf{n}}} e_{\mathbf{j}}(\mathbf{t}_{\mathbf{k}}) \bar{e}_{\mathbf{l}}(\mathbf{t}_{\mathbf{k}}) = \sum_{\mathbf{j} \in \mathbf{Z}} \hat{u}(\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1})). \end{aligned}$$

Further, according to the proof of Lemma 2 [1], we get

$$\begin{aligned}
\|P_{\mathbf{n}}u\|_s^2 &= \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} (1+\mathbf{l}^2)^s |(\widehat{P_{\mathbf{n}}u})(\mathbf{l})|^2 = \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} (1+\mathbf{l}^2)^s \left| \sum_{\mathbf{j} \in \mathbf{Z}} \widehat{u}(\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1})) \right|^2 \\
&= \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \left| \sum_{\mathbf{j} \in \mathbf{Z}} (1 + \mathbf{l}^2)^{\frac{s}{2}} \widehat{u}(\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1})) \right|^2 \\
&= \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \left| \sum_{\mathbf{j} \in \mathbf{Z}} (1 + \mathbf{l}^2)^{\frac{s}{2}} (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^{-\frac{s}{2}} \widehat{u}(\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1})) (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^{\frac{s}{2}} \right|^2 \\
&\leq \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \left( \sum_{\mathbf{j} \in \mathbf{Z}} ((1 + \mathbf{l}^2) / (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2))^s \right) \\
&\sum_{\mathbf{j} \in \mathbf{Z}} |\widehat{u}(\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))|^2 (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^s \leq \\
&\leq \max_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \left( \sum_{\mathbf{j} \in \mathbf{Z}} ((1 + \mathbf{l}^2) / (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2))^s \right) \|u\|_s^2.
\end{aligned}$$

It is easy to check that sum

$$\sum_{\mathbf{j} \in \mathbf{Z}} ((1 + \mathbf{l}^2) / (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2))^s$$

reaches maximum when  $\mathbf{l} = \mathbf{n}$ , so using Lemma 1 we have

$$\begin{aligned}
&\max_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \left( \sum_{\mathbf{j} \in \mathbf{Z}} ((1 + \mathbf{l}^2) / (1 + (\mathbf{l} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2))^s \right) \\
&= \sum_{\mathbf{j} \in \mathbf{Z}} ((1 + \mathbf{n}^2) / (1 + (\mathbf{n} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2))^s \\
&\leq 2^s (\mathbf{n}^2)^s \sum_{\mathbf{j} \in \mathbf{Z}} ((\mathbf{n} + \mathbf{j} * (2\mathbf{n} + \mathbf{1}))^2)^{-s} \leq 2^{s+m} (\mathbf{n}^2)^s \sum_{\mathbf{j} \in \mathbf{Z}} ((\mathbf{n} * (2\mathbf{j} - 1))^2)^{-s} \\
&\leq 2^{s+m} M^2(\mathbf{n}, s) \sum_{\mathbf{j} \in \mathbf{N}} ((2\mathbf{j} - 1)^2)^{-s}.
\end{aligned}$$

Summands could be estimated as

$$((2\mathbf{j} - 1)^2)^{-s} = \left( \sum_{k=1}^m (2j_k - 1)^2 \right)^{-s} \leq \left( \frac{m}{\left( \sum_{k=1}^m (2j_k - 1)^2 \right)} \right)^s,$$

and using Lemma 2 we obtain

$$\begin{aligned}
& 2^{s+m} M^2(\mathbf{n}, s) \sum_{\mathbf{j} \in \mathbf{N}} ((2\mathbf{j} - \mathbf{1})^2)^{-s} \\
& \leq 2^{s+m} m^s M^2(\mathbf{n}, s) \sum_{\mathbf{j} \in \mathbf{N}} \left( \sum_{k=1}^m (2j_k - 1) \right)^{-2s} \\
& = 2^{s+m} m^s M^2(\mathbf{n}, s) \sum_{\mathbf{j} \in \mathbf{N}} \left( 2 \sum_{k=1}^m j_k - m \right)^{-2s} = 2^{s+m} m^s M^2(\mathbf{n}, s) \sum_{j \in \mathbf{N}_0} \frac{R(A_j^m)}{(m+2j)^{2s}} \\
& \leq 2^{s+m} m^s M^2(\mathbf{n}, s) \left( m^{-2s} + \sum_{\mathbf{j} \in \mathbf{n}} \frac{mj^{m-1}}{(m+2j)^{2s}} \right) \\
& \leq 2^{m-s} m^{s+1} M^2(\mathbf{n}, s) \left( 1 + \sum_{j \in \mathbf{N}} j^{-(2s-m+1)} \right).
\end{aligned}$$

Theorem is proved.  $\square$

Denote polynomial of the best approximation to  $u \in H^s$  of degree not higher than  $\mathbf{n} \in \mathbf{N}_0$  and the corresponding best approximation

$$(S_{\mathbf{n}}u)(\boldsymbol{\tau}) = \sum_{\mathbf{l} \in \mathbf{I}_{\mathbf{n}}} \hat{u}(\mathbf{l}) e(i\boldsymbol{\tau} \cdot \mathbf{l}), \quad E_{\mathbf{n}}(u)_s = \|u - S_{\mathbf{n}}u\|_s,$$

where  $(S_{\mathbf{n}}u)(\boldsymbol{\tau})$  is the  $\mathbf{n}$ -th partial sum of Fourier series of  $u$ .

**Corollary 1.** *For every  $s \in \mathbf{R}$ ,  $m \in \mathbf{N}$ ,  $s > m/2$ ,  $\mathbf{n} \in \mathbf{N}$  and arbitrary function  $u \in H^s$*

$$\|u - P_{\mathbf{n}}\|_s \leq (1 + 2^{\frac{m-s}{2}} m^{\frac{s+1}{2}} M(\mathbf{n}, s) \sqrt{1 + \zeta(2s - m + 1)}) E_{\mathbf{n}}(u)_s. \quad (4)$$

The proof is obvious.  $\square$

**Corollary 2.** *For every  $s \in \mathbf{R}$ ,  $m \in \mathbf{N}$ ,  $s > m/2$ ,  $\mathbf{n} \in \mathbf{N}$ , arbitrary function  $u \in H^s$  and sequence of indices  $(\mathbf{n}_j)_{j \in \mathbf{N}}$  satisfying*

$$\lim_{\mathbf{n} \rightarrow \infty} M(\mathbf{n}, s) < \infty,$$

*sequence of polynomials  $(P_{\mathbf{n}}u)$  converges to function  $u$  with the error estimate*

$$\|u - P_{\mathbf{n}}u\|_s = O(E_{\mathbf{n}}(u)_s).$$

Proof follows directly from Corollary 1.  $\square$

**Corollary 3.** *For any  $p, s \in \mathbf{R}$ ,  $m \in \mathbf{N}$ ,  $p \geq s > m/2$ ,  $\mathbf{n} \in \mathbf{N}$  and arbitrary function  $u \in H^p$  the following estimation is valid*

$$E_{\mathbf{n}}(u)_s \leq (1 + \mathbf{n}^2)^{\frac{s-p}{2}} E_{\mathbf{n}}(u)_p.$$

Proof follows from properties of the best approximation and definitions of norms in  $H^s$  and  $H^p$ .  $\square$

Corollary 3 allows to generalize Corollary 2.

**Corollary 4.** *For any  $p, s \in \mathbb{R}$ ,  $m \in \mathbb{N}$ ,  $p \geq s > m/2$ ,  $\mathbf{n} \in \mathbf{N}$ , arbitrary function  $u \in H^p$  and sequence of indices  $(\mathbf{n}_j)_{j \in \mathbb{N}}$  satisfying*

$$\lim_{\mathbf{n} \rightarrow \infty} M(\mathbf{n}, s)(\mathbf{n}^2)^{\frac{s-p}{2}} < \infty,$$

*sequence of polynomials  $(P_{\mathbf{n}}u)$  converges to function  $u$  with error estimate*

$$\|u - P_{\mathbf{n}}u\|_s = O(E_{\mathbf{n}}(u)_p).$$

Proof follows from Corollaries 1 - 3.

**Remark 1.** *For any constant  $C$ ,  $C \geq \sqrt{m^s}$ , the set  $\{\mathbf{n} \mid M(\mathbf{n}, s) \leq C\}$  is a cone in  $\mathbf{N}$ . Choosing indices from this cone we'll obtain sequence of interpolation polynomials converging with estimation (4) where  $M(\mathbf{n}, s)$  is substituted by  $C$ . The minimal possible value of  $M(\mathbf{n}, s) = \sqrt{m^s}$  will be on the set*

$$\{\mathbf{n} \mid \mathbf{n} \in \mathbf{N}, \mathbf{n} = (n_1, n_2, \dots, n_m), n_k = n_l, 1 \leq k, l \leq m\}$$

*of indices with equal components.*

#### REFERENCES

- [1] Fedotov A.I. On the asymptotic convergence of the polynomial collocation method for singular integral equations and periodic pseudodifferential equations// Archivum mathematicum. 2002. V.1. P.1-13.
- [2] Taylor, M.E. Pseudodifferential operators, Princeton University Press, Princeton 1981.

RESEARCH INSTITUTE OF MATHEMATICS AND MECHANICS, KAZAN STATE UNIVERSITY, UNIVERSITETSKAYA STR. 17, KAZAN:420008, RUSSIA

*E-mail address:* fedotov@mi.ru

Received December 5, 2003