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A DOUBLE-SEQUENCE RANDOM ITERATION PROCESS FOR RANDOM FIXED POINTS OF CONTRACTIVE TYPE RANDOM OPERATORS

(submitted by A. Lapin)

Abstract. In this paper, we introduce the concept of a Mann-type double-sequence random iteration scheme and show that if it is strongly convergent then it converges to a random fixed point of continuous contractive type random operators. The iteration is a random version of double-sequence iteration introduced by Moore (Comput. Math. Appl. 43(2002), 1585-1589).

1. INTRODUCTION

Several iteration processes have been established for the constructive approximation of solutions to several classes of (nonlinear) operator equations and many important convergence results have been obtained in terms of these iterative processes(cf. e.g., [1, 3, 5, 6, 9, 13]). Most of these convergence results require that the operator is of the strong (accretive or pseudocontractive) type whereas a few of them do not need the strong type property. Moreover, Mann-type and Ishikawa iteration processes play a key role in most of these convergence results. Most recently, a new Mann-type iteration process called Mann-type double-sequence iteration process was introduced be Moore [8].

On the other hand, random fixed point theory has attracted more and

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more in recent years since the article by Bharucha-Reid [7] come out in 1976. We note some recent works on random fixed points in [2, 10, 11]. In order to construct iterations for finding fixed points of random operators defined on linear spaces, random Ishikawa iteration scheme was introduced in [4].
In this paper, we will introduce the concept of a Mann-type double-sequence random iteration scheme. We will show that if this random iteration scheme converges strongly then it converges to a random fixed point of continuous contractive type random operators defined in the context of a separable Hilbert space.

2. PRELIMINARIES

NOTATIONS: In this paper X is a separable Hilbert space, (Ω, Σ) is measurable space ($i.e$, Σ is a sigma-algebra of subsets of Ω), C is a nonempty subset of X, 2^C is the family of all subsets of C and N_0 is the set of all nonnegative integers.
CONCEPTS: A mapping $\mu : \Omega \to 2^C$ is called *measurable* if for any open subset U of C, $\mu^{-1}(U) = \{w \in \Omega : \mu(w) \cap U \neq \emptyset\} \in \Sigma$. A mapping $T : \Omega \times C \to C$ is called a *random operator* if for any $x \in C$, $T(., x)$ is measurable. A measurable mapping $f : \Omega \to C$ is called a *random fixed point* of random operator $T : \Omega \times C \to C$ if for every $w \in \Omega$, $f(w) = T(w, f(w))$. A random operator $T : \Omega \times C \to C$ is said to be continuous if, for fixed $w \in \Omega$, $T(w, .)$ is continuous.
DOUBLE SEQUENCE.[8] Let E be a normed linear space. By a double sequence in E is meant functions $f_{k,n} : \Omega \to E$ defined by $f_{k,n}(w) := w_{k,n} \in E$, $\forall\, k, n \in N_0$. The double sequence $\{w_{k,n}\}$ is said to converge strongly to w^* if for each $\epsilon > 0$, there exist integers $K, N > 0$, such that $\|w_{k,n} - w^*\| < \epsilon$, $\forall\, k \geq K$, $n \geq N$. If $\forall\, k, r \geq K$, $n, s \geq N$, we have $\|w_{k,r} - w_{n,s}\| < \epsilon$, then the double sequence is said to be Cauchy.
MANN ITERATION SCHEME.[9] Let L be a linear space, $T : L \to L$ be a mapping and $x_0 \in L$. Then the sequence $\{x_n\}$ defined iteratively by:

$$x_{n+1} = (1 - c_n)x_n + c_n T x_n,\ n \in N_0$$

$$\text{where} \quad 0 \leq c_n < 1 \quad \text{and} \quad \sum_{n=0}^{\infty} c_n < \infty$$

is called the Mann iteration scheme.

bf DOUBLE-SEQUENCE RANDOM MANN ITERATION SCHEME. Suppose that C be a nonempty convex subset of a separable Hilbert space X, $T_k : \Omega \times C \to C$ be random operators. The double sequence of functions $\{f_{k,n}\}_{k\geq 0, n\geq 0}$ generated from an arbitrary measurable function $f_{0,0} : \Omega \to C$ defined by

$$\begin{aligned} &f_{k,n+1}(w) = (1 - c_n) f_{k,n}(w) + c_n T_k(w, f_{k,n}(w)), \\ &w \in \Omega, \quad k, n \in N_0 \end{aligned} \tag{2.1}$$

where

$$0 < c_n < 1, \qquad n \in N_0 \tag{2.2}$$

and

$$0 < \lim_{n\to\infty} c_n = h < 1, \tag{2.3}$$

is called double-sequence random mann iteration scheme.
Since C is convex clearly, $f_{k,n}$ is a mapping from $\Omega \to C$ for all $k, n \in N_0$.

CONTRACTIVE INEQUALITY A. Let C be a nonempty convex subset of a Hilbert space X. A mapping $S : C \to C$ is said to satisfy contractive inequality **A** if for all $x, y \in C$,

$$\begin{aligned} \|Sx - Sy\|^2 &\leq a\,\|x - y\|^2 + b\,\|y - Sy\|^2 \,(1 + \|x - Sx\|^2) \\ &+ \tfrac{d}{2}\,\|x - Sy\|^2 \,(1 + \|x - Sx\|^2 + \|y - Sx\|^2), \end{aligned}$$

$$\text{where} \quad a, b, d > 0, \quad k \geq 0, \quad b + \frac{d}{2} < \frac{1}{4}.$$

CONTRACTIVE INEQUALITY B. Let C be a nonempty convex subset of a separable Hilbert space X. The random operator $T : \Omega \times C \to C$ is said to satisfy contractive inequality **B** if for all $x, y \in C$,

$$\begin{aligned} &\|T(w, x) - T(w, y)\|^2 \\ &\leq a\,\|x - y\|^2 + b\,\|y - T(w, y)\|^2 \,(1 + \|x - T(w, x)\|^2) \\ &+ (\tfrac{d}{2})\,\|x - T(w, y)\|^2 \,(1 + \|x - T(w, x)\|^2 + \|y - T(w, x)\|^2), \end{aligned} \tag{2.4}$$

$$\text{where} \quad a, b, d > 0, \quad k \geq 0, \quad b + \frac{d}{2} < \frac{1}{4}. \tag{2.5}$$

3. MAIN RESULTS

Theorem 3.1. Let X be a separable Hilbert space, C be a nonempty closed convex subset of X, $T : \Omega \times C \to C$ be a continuous random operator such that for all $w \in \Omega$, T satisfies contractive inequality **B**. Let $\{b_k\}_{k\geq 0} \subset (0, 1)$ be a sequence such that $lim_{k\to\infty} b_k = 1$.
For an arbitrary but fixed $t \in C$, and for each $k \geq 0$, define $T_k : \Omega \times C \to$

C by $T_k(w,x) = (1-b_k)t + b_kT(w,x)$. Suppose that the double-sequence random Mann iteration scheme satisfying

$$\frac{3}{4[1-(b+d/2)]} < h < 1 \tag{3.1}$$

is strongly convergent. Then it converges to a random fixed point of T.

Proof. Since $0 < b + d/2 < 1/4$, clearly $3/4(1-(b+d/2)) < 1$. So the positive number h satisfying (3.1) exists. Let $\{f_{k,n}(w)\}$ be constructed by (2.1)-(2.3) with h satisfying (3.1) and $\{f_{k,n}(w)\}$ be strongly convergent. Then for all $w \in \Omega$, if for each fixed k, $f_{k,n}(w) \to f_k^*(w)$ as $n \to \infty$ and then $f_k^*(w) \to f(w)$ as $k \to \infty$, then

$$f_{k,n}(w) \to f(w) \quad \text{as} \quad k,n \to \infty. \tag{3.2}$$

Since C is closed, it follows that f is a mapping from $\Omega \to C$. Since C is a subset of a separable Hilbert space X, for any continuous random operator F and any measurable function g from $\Omega \to C$, $G(w) = F(w, g(w))$ is also a measurable function [12]. It thus follows from (2.1)-(2.3) that $\{f_{k,n}\}$ is a sequence of measurable functions. Hence, $f : \Omega \to C$, being the limit of a sequence of measurable functions, is also measurable. For $w \in \Omega$, from (2.1) and parallelogram law we have

$$\begin{aligned}
&\|f(w) - T_k(w, f(w))\|^2 \\
&= \|f(w) - f_{k,n+1}(w) + f_{k,n+1}(w) - T_k(w, f(w))\|^2 \\
&\le 2\|f(w) - f_{k,n+1}(w)\|^2 + 2\|f_{k,n+1}(w) - T_k(w, f(w))\|^2 \\
&= 2\|f(w) - f_{k,n+1}(w)\|^2 \\
&+2\|(1-c_n)f_{k,n}(w) + c_nT_k(w, f_{k,n}(w)) - T_k(w, f(w))\|^2 \\
&\le 2\|f(w) - f_{k,n+1}(w)\|^2 + 4(1-c_n)^2\|f_{k,n}(w) - T_k(w, f(w))\|^2 + \\
&\quad 4c_n^2\|T_k(w, f_{k,n}(w)) - T_k(w, f(w))\|^2.
\end{aligned}$$

Therefore by (2.4) we obtain

$$\|f(w) - T_k(w, f(w))\|^2 \le \beta + 4c_n^2(\gamma + \alpha), \tag{3.3}$$

where

$$\alpha = \delta\left(\frac{d}{2}\right)\|f_{k,n}(w) - T_k(w, f(w))\|^2,$$

$$\delta = 1 + \|f_{k,n}(w) - T_k(w, f_{k,n}(w))\|^2 + \|f(w) - T_k(w, f_{k,n}(w))\|^2,$$

$$\beta = 2\|f(w) - f_{k,n+1}(w)\|^2 + 4(1-c_n)^2\|f_{k,n}(w) - T_k(w, f(w))\|^2,$$

$$\gamma = a \left\| f_{k,n}(w) - f(w) \right\|^2 \\ + b \left\| f(w) - T_k(w, f(w)) \right\|^2 \left(1 + \left\| f_{k,n}(w) - T_k(w, f_{k,n}(w)) \right\|^2\right).$$

Since

$$\left\| f_{k,n}(w) - T_k(w, f_{k,n}(w)) \right\|^2 = \frac{\left\| f_{k,n}(w) - f_{k,n+1}(w) \right\|^2}{c_n^2}, \tag{3.4}$$

It follows that

$$\left\| f(w) - T_k(w, f_{k,n}(w)) \right\|^2 \leq 2 \left\| f(w) - f_{k,n}(w) \right\|^2 \tag{3.5}$$
$$+2 \left\| f_{k,n}(w) - T_k(w, f_{k,n}(w)) \right\|^2$$
$$= 2 \left\| f(w) - f_{k,n}(w) \right\|^2 + \left(\frac{2}{c_n^2}\right) \left\| f_{k,n}(w) - f_{k,n+1}(w) \right\|^2. \tag{3.6}$$

Using (3.4) and (3.5) in (3.3), we have, for all $w \in \Omega$,

$$\left\| f(w) - T_k(w, f(w)) \right\|^2 \leq \beta^* + 4c_n^2(\gamma^* + \alpha^*),$$

where

$$\alpha^* = \delta^* \left(\frac{d}{2}\right) \left\| f_{k,n}(w) - T_k(w, f(w)) \right\|^2,$$

$$\delta^* = 1 + 2 \left\| f(w) - f_{k,n}(w) \right\|^2 + \left(\frac{2}{c_n^2} + \frac{1}{c_n^2}\right) \left\| f_{k,n}(w) - f_{k,n+1}(w) \right\|^2,$$

$$\beta^* = 2 \left\| f(w) - f_{k,n+1}(w) \right\|^2 + 4(1 - c_n)^2 \left\| f_{k,n}(w) - T_k(w, f(w)) \right\|^2,$$

$$\gamma^* = a \left\| f_{k,n}(w) - f(w) \right\|^2$$
$$+b \left\| f(w) - T_k(w, f(w)) \right\|^2 \left(1 + \frac{\left\| f_{k,n}(w) - f_{k,n+1}(w) \right\|^2}{c_n^2}\right).$$

Letting $k, n \to \infty$, using (3.2), (2.3) and the fact that T_k are continuous random operators, we obtain, for $w \in \Omega$,

$$\left\| f(w) - T(w, f(w)) \right\|^2 \leq 4(1 - h)^2 \left\| f(w) - T(w, f(w)) \right\|^2$$
$$+4h^2 \left\{ b \left\| f(w) - T(w, f(w)) \right\|^2 + \left(\frac{d}{2}\right) \left\| f(w) - T(w, f(w)) \right\|^2 \right\}$$
$$= \left\{ 4(1 - h)^2 + 4h^2 \left(b + \frac{d}{2}\right) \right\} \left\| f(w) - (w, f(w)) \right\|^2.$$

From (2.5) and (3.1) we have

$$(1 - h)^2 + h^2(b + \frac{d}{2}) < \frac{1}{4}.$$

Therefore, for all $w \in \Omega$ and $k \geq 0$, we have $f(w) = T(w, f(w))$. This completes the proof.

Following is the immediate consequence of theorem 3.1.

Corollary 3.1. Let X be a Hilbert space, C be a nonempty closed convex subset of X, $S : C \to C$ be a function satisfying contractive inequality **A**. Suppose that the Mann iteration scheme satisfying

$$\frac{3}{4[1-(b+d/2)]} < h < 1$$

is convergent. Then it converges to a fixed point of T.

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