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## STRUCTURE OF FUNCTION ALGEBRAS ON FOLIATED MANIFOLDS

(submitted by M. Malakhaltsev)

ABSTRACT. We consider a manifold  $M$  with a foliation  $F$  given by a locally free action of a commutative Lie group  $H$ . Also we assume that there exists an integrable Ehresmann connection on  $(M, F)$  invariant with respect to the action of the group  $H$ . We get the structure of the restriction of the algebra  $C_0(M)$  to the leaves in three partial cases. Also we consider a classification of the quasiinvariant measures and means on the leaves of  $F$ .

### 1. INTRODUCTION.

In the present paper we consider a manifold  $M$  with a foliation  $F$  given by a locally free action of a commutative Lie group  $H$ . Let us denote this action by  $R : M \times H \rightarrow M$ . Let  $\dim H = n$ , and  $\pi : \mathbb{R}^n \rightarrow H$  be the universal covering. Then  $h \mapsto R_{\pi(h)}$  is the locally free action of  $\mathbb{R}^n$  on  $M$ . Thus, without loss of generality, we can set  $H = \mathbb{R}^n$ .

Also we assume that there exists an integrable Ehresmann connection [1] on  $(M, F)$  which is invariant with respect to the action of  $H$ .

Let us denote by  $C_0(M)$  the algebra consisting of continuous functions on  $M$  vanishing at infinity. The main object of our investigation is the algebra  $C_0(M)|_L$  obtained by restriction of  $C_0(M)$  to a leaf of the foliation.

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Recently the author proved that, under the assumptions given above, there exists an almost everywhere continuous bijection  $\phi : M \rightarrow P \times H/H_P$ , where  $P$  is a connected total transversal tangent to the Ehresmann connection and  $H_P = \{h \in H | hP = P\}$ . The main problem is to describe  $C_0(M)|_L$  for  $L \in F$  under different assumptions. We consider three distinct cases:

- 1)  $P/H_P$  is a Hausdorff space;
- 2) any leaf  $L \in F$  is dense on  $M$ , there exists an  $H_P$ -invariant metric on the transversal  $P$ , and the isotropy groups of all the leaves coincide;
- 3) a base of  $H_P$  is a contraction on  $P$ .

## 2. THE FIRST CASE.

**Definition 1.** For any  $\varepsilon > 0$  we define the set of algebras  $B_{\varepsilon,x}(M) = \{f \in C_0(M) | f \text{ is } H_{\varepsilon,x}\text{-invariant}\}$ , here  $H_{\varepsilon,x} = \{h \in H | hU_{\varepsilon,x} \cap U_{\varepsilon,x} \neq \emptyset\}$ . Set  $B_\varepsilon = \bigcap_{x \in P} B_{\varepsilon,x}$ .

Let us say that  $B_\varepsilon \rightarrow C_0(M)$  as  $\varepsilon \rightarrow 0$  if for any  $f \in C_0(M)$  and a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  ( $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ) there exists a sequence  $(f_n)$ ,  $f_n \in B_{\varepsilon_n}$  such that  $f_n \rightarrow f$  in the usual  $\|\cdot\|_0$  or sup norm.

**Statement 1.** The following assumptions imply that  $B_\varepsilon \rightarrow C_0(M)$  as  $\varepsilon \rightarrow 0$ :

- 1)  $P/H_P$  is a Hausdorff space,
- 2)  $P'_i = \{p \in P | \text{there exists } H_i \subset H_P, H_i p = p\}$  are submanifolds of the global transversal  $P$ .

• The first assumption implies that  $B_{\varepsilon,x} \rightarrow C_0(M)$  as  $\varepsilon \rightarrow 0$  for all  $x \in P$ , otherwise  $P/H_P$  is not a  $T_1$ -space and moreover not a Hausdorff one. From Corollary 27 [4] it follows that there exists a metric on  $P$  invariant with respect to the action of  $H_P$ . (Our assertions imply that  $Q = P/H_P$  is a manifold with a boundary given by  $\bigcup_i P'_i$ . So we take a slice [4]  $\sigma : Q \rightarrow P$ , such that  $\sigma$  is continuous on  $Q \setminus \pi(\bigcup_i P'_i)$ . The last condition of Corollary 27 [4] is met since the set  $P \setminus \bigcup_i P'_i$  ( $P, Q \setminus \pi(\bigcup_i P'_i), \pi$ ) is a cover of  $Q$  with the covering group  $H_P$ . Let us extend  $\sigma$  to  $\bigcup_i P'_i$  by continuity. (In fact we get a multivalued map). Let us define a metric  $d$  on  $P$  as follows: On the factor  $Q$   $d(x, y)$  can be an arbitrary one. Then we must put into consideration the set of continuous sections  $\Sigma = \bigcup_{h \in H_P} h\sigma$ . Let us point out that  $h\sigma \cap h'\sigma \subset \bigcup_i P'_i$ . Then we define metric on  $P$  by gluing metrics on the images  $h\sigma$ , i.e. for any  $h \in H_P$  we

put  $d(hx, hy) = d(x, y)$ . Since the set  $I$  of disjoint continuous sections is at most countable one and  $\bigcup_{i \in I} \sigma_i(Q) = P$  this metric is a correctly defined one on the whole manifold  $P$ . This definition is correct since by construction metrics coincide on the borders of the sets. Thus we apply the construction from [4].)

Let us prove now that  $B_\varepsilon \rightarrow C_0(M)$  ( $\varepsilon \rightarrow 0$ ). The proof is by induction on the dimension of the subset of  $P$ . Let us first define the step of the induction as follows:  $f_n(t\gamma) = f(\gamma)$   $\gamma \in P'$ ,  $t \in [0, 1/n]$  from the normal foliation on  $U_{2/n}(\bigcup_i P'_i)$ ,  $f_n(t\gamma) = f(t\gamma)$ ,  $t \geq 2/n$ ,  $f_n(t\gamma) = at + b$ ,  $t \in [1/n, 2/n]$ ,  $a = n(f(2\gamma/n) - f(\gamma))$ ,  $b = 2f(\gamma) - f(2\gamma/n)$ . In the zero-dimensional case (being the base of the induction) we apply the same construction as in the induction step. the first assumption implies that for any point  $y \in P \setminus P'$  one can find such  $\varepsilon(y) > 0$  that  $B_{\varepsilon(y), y} = C_0(M)$ . Then by considering the continuity modulus of the function  $f$  we prove the theorem.  $\triangleright$

**Note 1.** *As a consequence of the proof of the previous statement one can weaken the first assumption. One can assume the existence of such system of  $\varepsilon$ -neighbourhoods  $U_\varepsilon(P'_i)$  of the submanifolds  $P'_i$  ( $\varepsilon_0 > \varepsilon > 0$ ) that for all  $\varepsilon \in (0, \varepsilon_0)$   $H_P(U_\varepsilon(P'_i)) = U_\varepsilon(P'_i)$ , and  $U_\varepsilon(P'_i)/H_P$  is a Hausdorff space.*

**Note 2.** *Let us point out now that  $B_{\varepsilon, x} \rightarrow B_r(G)$  as  $\varepsilon \rightarrow 0$  means that the point  $x$  can be separated from any other point  $y \in P/H_P$ . It is the topological  $T_1$  axiom. Note that the previous statement does not prove the inverse ( $T_1 \Rightarrow B_{\varepsilon, x} \rightarrow B_r(G)$ ) though  $T_2$  axiom almost certainly leads to convergence of  $B_{\varepsilon, x} \rightarrow B_r(G)$  as  $\varepsilon \rightarrow 0$  since it provides us with a slice used in the construction of the invariant metric. Nevertheless in case there exists a structure of a Lie group on  $P$  we have  $B_{\varepsilon, x} \rightarrow B_r(G)$ , hence  $T_1$  is true, and hence  $T_2$  is also true by ([7], Lemma, p. 144 ).*

So later on in this paper we consider only cases in which  $B_{\varepsilon, x} \not\rightarrow B_r(G)$ ,  $\varepsilon \rightarrow 0$ . In general, the situation here is rather complicated so we will consider the partial case when there exists a leaf  $L \in F$  such that  $\overline{L} \cap \overline{P} \setminus L \cap P \neq \emptyset$ .

Then the following cases are possible:

1) Let  $X_h$  stand for the graph of the map  $h : P \rightarrow P$ . We assume that the graphs  $X_h$ ,  $h \in H_P$  do not intersect each other (this is possible in case for example there exists a graph of a number of these maps in any neighbourhood of the diagonal graph  $D = \bigcup_{x \in M} (x, x)$ ).

2) The graphs of the maps generated by elements of  $H_P$  do intersect in a point  $(x, x) \in D$ .

The graphs of the maps are uniformly separated in any other case, that is there exists  $\varepsilon > 0$  such that  $U_\varepsilon(X_i) \cap U_\varepsilon(X_j) = \emptyset$ . Let us mention now that in this case we can apply the first part of Statement 1 and there exists a metric on  $P$  invariant under the action of the group  $H_P$  (note that the converse is not true, e.g. any Kronecker foliation on the torus  $\mathbb{T}^2$ ).

### 3. THE SECOND CASE.

**Lemma 1.** *Let  $P$  be a  $k$ -dimensional manifold endowed with a Riemannian metric  $\rho'$ . Let  $f : P \rightarrow P$  be a diffeomorphism such that*

- 1) *The set  $\bigcup_{n \in \mathbb{Z}} f^n(x)$  is dense for any  $x \in P$ .*
- 2) *There are no fixed points: for any  $p \in P$  and for all  $n \in \mathbb{N}$  ( $f^n(p) \neq p$ ).*
- 3) *The absolute values of the maps  $\rho(f^n(\cdot), f^n(\cdot)) : P \times P \rightarrow \mathbb{R}^+$ ,  $n \in \mathbb{Z}$  are bounded from above and below: there exist  $\delta, \Delta > 0$  such that for any  $n \in \mathbb{Z}$ , and arbitrary  $x, y \in P$ ,  $\delta \rho'(x, y) \leq \rho'(f^n(x), f^n(y)) \leq \Delta \rho'(x, y)$ .*

*Then there exists a metric  $\rho$ ,  $\rho : P \times P \rightarrow \mathbb{R}^+$  such that  $\forall x, y \in P$   $\rho(f(x), f(y)) = \rho(x, y)$ .*

• Let us deform a metric  $\rho'$  on  $P$ , so that for all  $n \in \mathbb{N}$ ,  $\rho(x, f^n(x)) = \text{const}(n)$ . Then there exists  $x \in P$  such that  $\rho(f^{n_k}(x), f^{n_{k+1}}(x)) \rightarrow 0$ , ( $k \rightarrow \infty$ ), this implies that the graphs  $\Gamma(f^n)$  are dense in the space  $P \times P$ . Thus we will consider only the situation in which any leaf is everywhere dense because otherwise we can apply Statement 1.

Then we can consider only the case in which for any  $n \in \mathbb{Z}$ , and for all  $x \in P$ ,  $f^n(x) \neq x$ , otherwise for some fixed  $x \in P$  we can find a natural number  $k = \inf\{n \in \mathbb{N} | f^n(x) = x\}$ . Fix the set of points  $X = \{f^n(x)\}_{n \in \mathbb{Z}}$ . On the one hand  $\overline{X} = P$  but on the other hand  $f^n(x) = f^{lk+j}(x) = f^j(x)$ ,  $0 \leq j \leq k-1$  is a finite set.

Now for a compact manifold  $P$  we take the following form as  $\rho$ :

$$\rho(x, y) = \lim_{n \rightarrow \infty} \frac{\sum_{k=-n}^n \rho'(f^k(x), f^k(y))}{2n}$$

here  $\rho'$  is a nonsingular metric on  $P$ . Let us turn to the proof of metric axioms. Let us prove first that  $\rho(x, y) \geq 0$  for  $x \neq y \in P$ . Assume the contrary. Then there exists a subset  $J$  of density 0 of the set  $\mathbb{N}$  such that the sequence  $\rho(f^n(x), f^n(y)) \rightarrow 0$  as  $n \rightarrow \infty, n \in \mathbb{N} \setminus J$  [2]. But

then the compactness of  $P$  implies that there exists the point  $z \in P$   $f^{n_k}(x), f^{n_k}(y) \rightarrow z$  as  $k \rightarrow \infty$ , this contradicts the third condition of the assumption. The other metric axioms follow from the construction of  $\rho'$ .

Let us prove now that the metric defined above is equivalent to the first one. Assume the contrary, i.e. there exist a  $\varepsilon > 0$  and  $x \in P$  such that for all  $\sigma > 0$  one can find  $y \in P$   $\rho'(x, y) < \sigma$   $\rho(x, y) \geq \varepsilon$ . Then again there exists a sequence  $f^{n_k}(x), f^{n_k}(y)$  such that  $\rho'(f^{n_k}(x), f^{n_k}(y)) \geq \varepsilon$ . In case  $\sigma$  is sufficiently small and  $\varepsilon/\sigma > \delta$  we again arrive to the contradiction with the third assumption of the statement. The same considerations show us that the convergence  $y_n \rightarrow_\rho x$  implies that  $y_n \rightarrow_{\rho'} x$  as  $n \rightarrow \infty$ . Let us again assume the contrary, i.e. there exists a sequence  $(y_n)_{n \in \mathbb{N}}$   $y_n \rightarrow_\rho x$  but  $y_n \not\rightarrow_{\rho'} x$ . We infer from the first convergence that for any  $\varepsilon > 0$  there exist  $n(\varepsilon), l(\varepsilon) \in \mathbb{N}$ ,  $\rho'(f^k(y_n), f^k(x)) < \varepsilon$  for  $\forall n \geq n(\varepsilon)$  and  $k |k| > l(\varepsilon)$ , of the density 1. Then again by the third assumption  $\rho'(y_n, x) = \rho'(f^{-k} \circ f^k(y_n), f^{-k} \circ f^k(x)) \leq D\rho'(f^k(y_n), f^k(x)) + o(\varepsilon) = o(\varepsilon)$ , this contradiction completes the proof.

Now let  $P$  be a noncompact manifold and the set of images  $f^n(p)$  be dense in  $P$ , then any two points from  $A = (P/\sim)$ , (here  $x \sim y$  if and only if there exists  $L \in F$  such that the points  $x, y \in L$ ) can not be separated. Let us construct the invariant metric on  $P$  as follows: Let  $\rho'$  be a fixed metric on  $P$ . Fix a neighbourhood  $U_\varepsilon(x)$  of a point  $x \in P$ . Let us now define the metric inside  $U_\varepsilon(x)$  by putting for  $f^{n+k}(x), f^n(x) \in U_\varepsilon(x)$

$$\begin{aligned} \rho(f^{n+k}(x), f^n(x)) \\ \equiv \sup\{\rho(f^{n+k+l}(x), f^{n+l}(x)) | f^{n+k+l}(x), f^{n+l}(x) \in U_\varepsilon(x)\}. \end{aligned}$$

Now since the set  $f^n(p)$  is dense the sequence  $f^n(U_\varepsilon(x))$  defines a locally finite atlas on  $P$ . So we define the metric on  $P$  as the image of the metric on  $U_\varepsilon(x)$  under the actions of  $f^n$ . Let us then glue metrics on the images of  $U_\varepsilon(x)$  as it was done in the first statement for a Hausdorff factor  $P/H_P$ . Let us then make  $\varepsilon \rightarrow 0$  and consider the limit metric  $\rho_l$ . Note that  $\rho_l \leq \rho_\varepsilon \leq \rho_{\varepsilon'}$  for  $0 < \varepsilon < \varepsilon'$ . This makes the definition correct one.

Equivalence of the constructed metric to the given one holds true due to the third condition and the construction algorithm. The limit length for  $y \rightarrow x$  is bounded from both sides:  $\delta\rho(x, y) < \rho_l(x, y) < \Delta\rho(x, y)$ .  $\triangleright$

**Note 3.** Assume that:

- (a) all points on  $P$  have the same isotropy group,
- (b) there exists an  $H$ -invariant metric  $g$  on  $M$ ,
- (c) the set  $H_P\{x\}$  is dense in  $P$ .

Then for any  $y \in P$ ,  $H_P\{y\}$  is dense in  $P$ .

• Let  $x \in P$  be such a point that  $(f^n(x))_{n \in \mathbb{Z}}$  is dense. Let there exist  $y \in P$  such that the set  $\bigcup_{n \in \mathbb{Z}} f^n(y)$  is not everywhere dense. Let  $f^{n_k}(x) \rightarrow y$  be a subsequence of  $f^n(x)$  then for arbitrary  $l \in \mathbb{Z}$   $f^l \circ f^{n_k}(x) \rightarrow f^l(y)$  ( $n_k \rightarrow \infty$ ), moreover the union of all shifted subsequences coincides with the sequence  $f^n(x)$ . Then the limit point set of  $f^n(x)$  must coincide with the same set of the sequence  $f^n(y)$ . This contradiction completes the proof.  $\triangleright$

**Example 1.** *Let us show that the third condition of the previous statement is necessary. For the irrational flow on torus  $\mathbb{T}^2$  as the foliation  $F$  and take a parallel of torus as a transversal submanifold  $P$ . Thus leaves of the foliation define a rotation of the transversal. The standard Kronecker foliation satisfies all conditions of the lemma and the standard angle metric on the  $\mathbb{S}^1$  is invariant under the action of the rotation group.*

*There is an example constructed by Arnold which shows us that this homeomorphism is not necessary a diffeomorphism on a dense set of points [5].*

In order to make condition 3) more evident we consider the following example:

**Example 2.** [12] *Fix a sequence of points  $x_n = 1/n \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . Let us consider a 1-dimensional foliation on the cylinder  $\mathbb{S}^1 \times \mathbb{R}$ . Let the Ehresmann connection be given by the distribution orthogonal to each  $\mathbb{S}^1 \times \{t\} \subset \mathbb{S}^1 \times \mathbb{R}$ . Let the set  $(x_n)_{n \in \mathbb{N}}$  be the set of fixed points of the first return map on the transversal  $P \cong \mathbb{R}$ . Assume also that this first return map is strictly monotone on the intervals  $(\frac{1}{n}, \frac{1}{n+1})$ . This is the so-called  $III_\lambda$  case. Then there is no invariant metric on the transversal  $P$ .*

**Statement 2.** *For complete geodesic manifold  $P$  the nonsingular metric from the previous statement is unique (up to multiplication by a constant) under the same restrictions on the map  $f : P \rightarrow P$ .*

• First let us prove this for a 1-dimensional manifold  $P$  with a Riemannian metric. Let us consider a groupoid of geodesics on  $P$ .  $\gamma : [0, 1] \rightarrow P$ ,  $\gamma(0) = x$ ,  $\gamma(1) = y$ ,  $\gamma \circ \gamma' \equiv \gamma''$ , for  $\gamma'(0) = \gamma(1)$ ,  $\gamma''$  being the geodesic from  $\gamma(0)$  to  $\gamma'(1)$ . In this case these rays are uniquely defined by source and range, so we get a groupoid structure on  $P \times P$ .

Let us point out now that a metric  $\rho$  on  $P$  gives rise to a cocycle  $D : \tilde{P} \times \tilde{P} \rightarrow \mathbb{R}$  [9, 11] defined on the product of the universal covering

spaces of  $P$ . Since there exists a direction on the universal covering space of  $P$  we can correctly define  $D(x, y) = \rho(x, y)$  in case  $x < y$  and  $D(x, y) = -\rho(x, y)$  otherwise. Now, since  $\rho$  is not a singular metric, the cocycle  $D$  gives rise to an invariant measure on  $\tilde{P}$  [9]. Since this measure is unique  $D$  must be equal to  $K$  for a  $K \in \mathbb{R}$ . This proves the statement in this case.

Now we turn to the general case. Fix the set of complete geodesic lines for any  $x \in P$  and any direction  $v \in T_x X$ ,  $\gamma : \mathbb{R} \rightarrow P$ ,  $\gamma(0) = x$ ,  $\frac{d\gamma}{dt} = v$ . Next we can construct for  $f^k : P \rightarrow P$ ,  $k \in \mathbb{Z}$  a pair of maps  $\gamma_{\pm}^{-1} \circ f \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$  for any  $\gamma'$  with  $\gamma'_{\pm}(\pm t(k)) = f(x)$ , where  $x = \gamma(0)$ , and  $t(k) = \rho(x, f^k(x))$ . This is true since the set of directions is compact. The set  $\bigcup_{k \in \mathbb{Z}} f^k(x)$  is everywhere dense on  $P$ , hence  $\bigcup_{k \in \mathbb{Z}} t(k)$  is dense on  $\mathbb{R}$ . Thus we reduce the general case to the 1-dimensional one. From this follows the statement for the manifolds of dimension greater than 1.  $\triangleright$

Now the statements above provide us with the set of almost periodic transformations [3].

**Note 4.** Assume that a function  $f : P \rightarrow P$  satisfies the following conditions:

- 1) For all  $n \in \mathbb{Z}$  and  $p \in P$   $f^n(p) \neq p$ .
- 2) There exists a point  $x \in P$ ,  $f^{n_k}(x) \rightarrow y$ , ( $n_k \in K \subset \mathbb{Z}$ ).

Then there may exist a homeomorphism  $\phi : P \rightarrow P$  such that the set  $f^n(x)$  is dense on  $P' \subset P$  for some not discrete  $P'$ .

This homeomorphism can be constructed as follows. Let us take the set of graphs  $\Gamma f^n(P) \subset P \times P$ . Let us deform all these graphs simultaneously. Let  $\phi(f^{n_k - n_{k-1}}(x)) \equiv y(n_k - n_{k-1}) \in P$  such that  $\rho(y, x) = \rho(f^{n_k}(x), y)$  (there is a wide range of choice of points that satisfy this property). Then we define  $\phi(f^{n + n_k - n_{k-1}}(x)) := z(n + n_k - n_{k-1})$  such that  $\rho(z, x) = \rho(f^{n + n_k - n_{k-1}}(x), f^n(x))$ . The open problem here is whether there exists a map  $\phi$ , which at the same time is a homeomorphism.

With the help of this homeomorphism one can improve the situation from the first example. Nevertheless the second example can not be improved in the same way.

So we consider only the possibilities given below:

- 1) There exists a metric  $\rho : P \times P \rightarrow \mathbb{R}^+$  and  $\varepsilon > 0$  such that for all  $x \in P$  and  $k, l \in \mathbb{Z}$ ,  $k \neq l$ ,  $\rho(f^l(x), f^k(x)) \geq \varepsilon$ . This situation was explored in [4] where  $P/H_P$  is a Hausdorff space. One can apply Statement 1.

- 2) There exists at least one point  $x \in P$  such that  $(f^l(x))_{l \in \mathbb{Z}}$  is not closed. Then we can apply Note 4 and — in partial cases — Statement

2 (namely the case of almost periodic transformations [3]). Here nevertheless we can find ourselves in the situation similar to one described in Example 2.

3) For any  $\varepsilon > 0$  there exist  $p \in P$  and  $n, l \in \mathbb{Z}$  such that

$$\rho(f^l(p), f^n(p)) < \varepsilon.$$

So we must find out what can be said about  $B_r(G)$  under the specified restrictions. As was already mentioned the only interesting situations are 2) and 3).

**Statement 3.** *Under the assumptions of Statement 2, if  $P$  be a compact manifold, then the mean  $M_x(f(x))$  on each leaf  $L$ ,  $M_x$  being the limit of  $M_{mn}(f(x)) = \frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m f(a_i \gamma_j)$  ( $a_i \in H_x$ ,  $\gamma_j \in U_j$ ) as  $m, n \rightarrow \infty$ , is finite, not equal to 0 and coincides with the mean defined in [10] as  $1/n \sum_{i=1}^n f(x_i)$ . Here  $H_x$  is the isotropy subgroup of the leaf passing through  $x$  and  $U_j$  is a  $\varepsilon$ -scattering of  $L$  given by  $f$  [10].*

• This is a consequence of the almost periodicity of the considered set of functions on the leaves and the existence of the integrable Ehresmann connection on  $(M, F)$ .

We must point out first that for any  $x \in P$ ,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(a_i x)/n \neq 0$  for  $f \in B_r(M)$ ,  $\mu(\text{supp}(f)) \neq 0$  by Poincaré theorem [2]. Thus we obtain the limit function  $f_l(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a_i x)/n$  and note that if  $f_l(x) \neq 0$ ,  $x \in \text{sat}_{F'}(\text{supp}(f)) = H_P \cdot \text{supp}(f)$  then the mean of the function  $M(f_l) = \lim_{k, l \rightarrow \infty, x_l \in \delta(S \cap \text{supp}(f))} \frac{1}{(kl)} \sum_{j=1}^{j=k, l=m} f(a_j x_l) \neq 0$ .  $\triangleright$

Hence we get the map  $M : P \rightarrow (B_r(G)|_L)'$  which does not depend on the point  $p \in P$ . Moreover, almost periodicity provides us with the following

**Statement 4.** *For any  $L \in F$  there exists the mean  $M' : B_r(G)|_L \rightarrow \mathbb{C}$ ,*

$$M'(g) = \lim_{k \rightarrow \infty} \frac{1}{2k} \sum_{i=-k}^k g(f^{n_k}(x))$$

such that  $\ker(M') = 0$ ,  $(n_k)_{k \in \mathbb{N}} \subset \mathbb{Z}$  has density 0.

• By general assumption there exists a sequence  $(f^{n_k}(x)) \subset P$ ,  $f^{n_k}(x) \rightarrow x$ ,  $(k \rightarrow \infty)$ . Then, since  $g \in C_0(M)$  is a continuous function,  $g(f^{n_k}(x)) \rightarrow$

$g(x)$ ,  $k \rightarrow \infty$ . The only thing left to prove is that these sequences coincide for all  $y \in P$ .

Since any leaf is dense and there exists an invariant metric, these sequences coincide for the everywhere dense set of points. Let  $f^{n_k}(y) \not\rightarrow y$ ,  $k \rightarrow \infty$ . Then there exists  $f^{n_k}(y) \rightarrow y$  ( $k \rightarrow \infty$ ) because otherwise  $D_v f^{n_k}$  are not bounded from above.  $\triangleright$

Let us describe the algebra  $C_0(M)|_L$  in this case.

**Statement 5.** *Assume that all the assumptions of Note 3 are satisfied. Then the algebra  $C_0(M)|_L$  consists of uniform almost periodic functions if  $M$  is compact.*

- Let  $x \in P$  then for all  $\varepsilon > 0$  there exists  $y(\varepsilon) \in U_\varepsilon(x) \setminus \{x\}$ . For  $h_m(\varepsilon) = \min\{h \in H_P | y = hx\}$  we can cover a manifold  $M$  by varieties  $\{hU_\varepsilon(x) | h \in H, h \geq h_m\}$ . It is clear that  $h_m(\varepsilon) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . Consider the continuity modulus  $\Delta_f$  of the function  $f$  of  $C_0(M)|_L$ . Then by general assumption for all  $x \in M$ ,  $|f(hx) - f(x)| < \Delta_f(\varepsilon)$  if  $d(x, hx) < \varepsilon$ . For any  $\Delta > 0$ , there exist  $\varepsilon > 0$  and  $h \in H_P$ .  $\triangleright$

The last case is then the worst to explore: there is no leaf with the property similar to one described above, since here we have only sequence of leaves such that there are leaves with points infinitely close one to another.

#### 4. THE THIRD CASE.

Let there exist a point  $x \in P$  such that  $B_{\varepsilon,x} \not\rightarrow B_r$ , and the leaf passing through  $x$  is compact. Assume also the following: for any  $p \in P$ ,  $p \neq x$  there exists a neighbourhood  $U(p) \subset P$  of the point  $p$  such that  $H_{U(p)} = H_{L(p)}$ . Thus we assume that the equivalence relation induced on  $P_i$  by  $H_{P_i}$  is topological [13]. Let us suppose also that for any point  $p \in P$  there exists a basis  $(a_1, \dots, a_n)$  of  $H_P$  such that  $a_i^k(p) \rightarrow x$  as  $k \rightarrow +\infty$ . Thus the set  $H_P$  is a contraction [8] (each  $a_i$ ,  $i = \overline{1, n}$  is a contraction of  $P$  with the common accumulation point  $x$ ).

Thus we consider the situation in which the accumulation point belongs to the diagonal  $D = \{(x, x) | x \in X\} \subset X \times X$  and is separated from the other similar points.

**Statement 6.** *The algebra  $C_{0,F}(M) = \{f \in C_0(M) | \text{for all } L \in Ff|_L \in C_0(M)\}$  coincides with the algebra  $C_{L_x} = \{f \in C_0(M) | f|_{L_x} = 0\}$ . Moreover this subalgebra is an ideal of  $C_0(M)$ .*

- The inclusion  $C_{0,F}(M) \subset C_{L_x}$  is evident. The converse follows from the consideration of the modulus of continuity.

The ideal property follows from the definition of the  $C_{0,F}(M)$ .  $\triangleright$

Thus we get  $C_0(M)/C_{0,F}(M) \cong C(L_x)$ . Let us take into consideration the following subspace of the almost periodic functions in Bohr sense:  $V_\tau = \{f \in C(X) \mid |f(n\tau + x) - f((n+1)\tau + x)| \rightarrow 0 (n \rightarrow +\infty)\}$ . We must

span the following set of vectors  $f_n = \begin{cases} 0, & \text{if } x < 0; \\ xe^{in\tau x}, & \text{if } 0 \leq x < 1; \ n \in \mathbb{N} \\ e^{in\tau x}, & \text{if } x \geq 1 \end{cases}$

as a subspace of  $V_\tau$ . For the mean we take the limit  $M_{mn}(f(x)) = \sum_{i=1}^n \sum_{j=1}^m f(a_i \gamma_j) / mn$  ( $a_i \in H_x$ ,  $\gamma_j \in U_j$ ). One can induce the scalar product on  $V_\tau$  with the help of this mean. This scalar product will be correctly defined on  $V' = \text{span}(\bigcup_{n \in \mathbb{N}} f_n)$ . Then the following statement holds true on any leaf of the foliation:

**Statement 7.** *Let all the assumptions from the first part of this chapter be true. Then for all  $f \in V_\tau$ ,  $f - f_0 \in C_0(X)$ , here  $f_0$  is the element which provides the best approximation of  $f$  with respect to  $V'$ .*

• For any  $\varepsilon > 0$  let us find an element  $g \in V'$  such that  $M(|f(x) - g(x)|) < \varepsilon$ . To do this, first one must find  $h(x) = \lim_{n \rightarrow \infty} f(x + n\tau)$  which exists by the general assumption on the structure of  $(M, F)$ . Let us then construct the function  $g(x + k\tau) = h(x)$ . Since  $g$  is a periodic function, it is enough to approximate it on one period by elements from  $V'|_{[0, \tau]}$ , but this is possible since  $V'$  is dense in  $C([0, \tau])$ .  $\triangleright$

**Statement 8.** *The sequence  $C_{0,F}(M) \hookrightarrow C_0(M) \rightarrow C_0(M)/C_{0,F}(M)$  can be split.*

• To construct the homomorphism  $\phi : C_0(M)/C_{0,F}(M) \rightarrow C_0(M)$  one can take the following function:

$$f(t, \theta p) = \begin{cases} 0, & \text{if } t < -1; \\ (t+1)\theta f_0(t), & \text{if } -1 \leq t < 0; \\ (t+\theta(1-t))f_0(t), & \text{if } 0 \leq t < 1; \\ f_0(t), & \text{if } t \geq 0. \end{cases}$$

Here  $t = \sum_{i=1}^{\dim(L)} t_i$ ,  $t_i$  are the natural coordinates on  $H \simeq \mathbb{R}^n$ ,  $\theta$  is the radius coordinate on  $\Delta = P \setminus \{x\}/H_P \simeq \mathbb{S}^{\dim(P)-1} \times [0, 1]$ , and  $p$  is the coordinate on  $\mathbb{S}^{\dim(P)-1}$ . The last diffeomorphism exists due to the assumptions on the equivalence relation on  $P$ .  $\triangleright$

Thus  $V_\tau = C_0(X) \oplus C(S)$ .

Fix any  $f \in B_r \setminus B_0$ .

**Definition 2.** Let us define mean on the leaf  $L$  of the foliation  $F$  as the limit for  $m$  and  $n \rightarrow \infty$  of  $M_{mn}(f(x)) = \sum_{i=1}^n \sum_{j=1}^m f(a_i \gamma_j) / mn$  ( $a_i \in H_x$ ,  $\gamma_j \in U_j$ )

Let  $H_x$  be an isotropy subgroup of the leaf passing through the point  $x$  and  $\{U_j\}$  be a  $\varepsilon$ -scattering of  $L$  defined by  $f$  [10].

**Statement 9.** The mean  $M_x(f(x))$  on any leaf  $L$  in the third situation, is finite and strictly less than the mean defined in [10] as  $\lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n f(x_i)$ .

- The proof follows from the fact that in this case there exists at least one element  $h \in H_P$  that gives rise to a contraction in a neighbourhood of the singular point  $x \in P$ . Moreover, the assumptions of this statement imply that there are no periodic or almost periodic functions in the considered class, this gives us the second part of the proposition. So any continuous function must satisfy: for all  $y \in P$   $\lim_{n \rightarrow \infty} f(h^n y) = f(x)$  implies that for all  $g \in S = H/H_P$ ,  $\lim_{n \rightarrow \infty} f(h^n g y) = f(g x)$ .

Let us point out that the mean defined above coincides with the mean defined in [10] for the set of continuous uniform almost periodic functions. This is a consequence of

$$\lim_{\varepsilon \rightarrow 0} 1/n \sum_{i=1}^n f(a_i) = \lim_{\varepsilon \rightarrow 0} 1/(nk) \sum_{i=1}^n \sum_{j=1}^k f(a_{ij})$$

for  $a_i, a_{ij}, i = \overline{1, n}, j = \overline{1, k}$ , being elements from an  $\varepsilon$ -scattering of the leaf  $L$ , i.e. for all  $j \in \{1, \dots, k\}, i \in \{1, \dots, n\} |f_i - f(a_{ij})| \leq \varepsilon, |f_i - f(a_i)| \leq \varepsilon$  [10]. Then we must pass to the limit as  $k, n \rightarrow \infty$  to obtain the result.  $\triangleright$

### 5. MEASURES AND MEANS.

Let us point out the natural connection between our problem and the so-called Radon-Nikodym problem on the leaves of the foliation. We can classify the equivalent, in general non-Borel invariant probability measures. Assume that the equivalence relation defined on  $M$  is generalized ( $\gamma \equiv a' \gamma$ ) and extended ( $\gamma \equiv \gamma' \leftrightarrow s(\gamma) = h s(\gamma'), r(\gamma) = h' r(\gamma')$ ) to the trivial groupoid  $G = L \times L$  of the leaf  $L$ . Let us then assume that the Radon-Nikodym derivative  $D(x, y) = D(x, y) + D(y, z)$  of the given measure belongs to the set of functions on the trivial groupoids of the leaves and the corresponding measure is quasiinvariant. So in the first case  $D(x, y) \rightarrow 0$  as  $x, y \rightarrow \infty$ .  $D(a^{n_k} x, a^{n_k} y) \rightarrow D(x, y)$  ( $n_k \rightarrow \infty$ ) in

the second case, and in the third case there exists  $c \in \mathbb{R}$ ,  $D(a^n x, a^n y) \rightarrow c$  ( $n \rightarrow \infty$ ). The measure can be a Borel probability measure only in the first case and in the third case when  $c = 0$ . In the first case we can naturally induce a measure  $\mu(E)$  on the manifold  $M$  as an integral of the measures  $\nu_p(E \cap L_p)$  over  $P/H_P$ .

Since the first case is trivial let us turn to the other cases.

The second case. Fix the cocycle  $D$  such that

$$D(x, a^l x) = \prod_{i=1}^l D(a^{i-1} x, a^i x).$$

As  $D(x, a^{n_k} x) \rightarrow D(x, x) = 1$ , this  $D$  is a uniform almost periodic function with respect to each variable.

**Statement 10.** *If there exists a cocycle  $D(x, y) : R \subset H \times H \rightarrow \mathbb{R}$  which is almost periodic with respect to both variables then*

1) *there exists a potential  $p : H \rightarrow \mathbb{R}$  such that  $D(x, y) = p(x) - p(y)$  (i.e.  $D$  is a coboundary cocycle);*

2) *the mean  $M_p(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)| p(x) dx$  is equivalent to the mean*

$$M(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)| dx \text{ in the following sense: } M(f) = 0 \Leftrightarrow M_p(f) =$$

0 *in case the potential from the first part is a uniform almost continuous function and for all  $x \in H$   $p(x) > 0$ .*

• 1) Fix  $y \in H$  and take  $M_y(D(x, y))$  as  $p(x)$ . The correctness of this definition follows from the properties of  $D$ :

$$p(x) - p(y) = \lim_{T \rightarrow \infty} 1/2T \int_{-T}^T D(x, t) - D(y, t) dt = D(x, y).$$

2) By assumption, the potential constructed in the first part is bounded from above and below by constants, from this follows the result.  $\triangleright$

So it seems possible to define a measure on the manifold  $\mu(E)$  as an integral over  $P$  of the mean  $E \cap L_p$ , here  $L_p$  is a leaf passing through  $p \in P$ . Later on we will show that it suffices in some cases to take only one leaf. Conversely, any measure on  $M$  invariant under  $H_P$  gives rise to a quasiinvariant measure on any leaf  $L \in F$ : Let us consider  $\lambda(E_1) = \mu(E)/\nu(E_2)$  for an  $H_P$ -invariant transverse measure  $\nu$  on  $P$  and  $E = E_1 \times E_2$ ,  $E_1 \subset S$ ,  $E_2 \subset S$  in the decomposition  $\phi : M \leftrightarrow S \times P$  [14].

**Statement 11.** *Assume that  $\dim H = 1$ . Let  $P$  be a compact submanifold of  $M$ . For any closed  $E \subset M$  we can take  $\mu(E) = M(I_E|L)$  for any leaf  $L \in F$ .*

• First let us prove this fact for the sets  $E_1 \times E_2$ , where  $E_1 \subset P$ ,  $E_2 \subset \sigma(H/H_P) \subset H$ . The main problem here is to show that  $M(E) > 0$  in case  $\mu(E) > 0$ . Let us consider  $\lim_{N \rightarrow \infty} 1/(2N) \int_{-N}^N I_E d\lambda = \lim_{N \rightarrow \infty} \frac{K_N}{2N} I_E$ , where  $K_N$  is the number of returns from a point  $x \in E_1$  to  $E_1$  on the set  $[-N, N]$  and  $I_E$  is the restriction of the characteristic function of  $E$  to the arbitrary leaf  $L$ . Since  $H_P$  generates an ergodic transformation on  $P$ , the Kats theorem [2] implies that  $\lim_{N \rightarrow \infty} \frac{K_N}{2N} \rightarrow \frac{\mu(E_1)}{\mu(P)}$ . This completes the proof since we get a measure, which is correctly defined for simple functions on  $M$ .  $\triangleright$

To get a measure of any measurable subset  $E$  of  $M$  one can't consider an arbitrary leaf as a support of the restriction of the characteristic function  $I_E$ , nevertheless, since  $M \simeq P \times S$  almost every leaf will serve as the target one.

It seems that this statement is true in general case. To prove it one should find a generalization of Kats theorem.

So, to get a measure on the closed subsets of  $M$  one can take only an  $H_P$ -invariant measure on the leaf and consider the standard mean with respect to this measure.

The third case. Any  $f \in V_\tau$  defines two measures: the periodic measure on the summand  $C_p(S)$  and the ordinary measure on the algebra  $C_0(L)$  for all  $L \in F$ .

Both of them are rather well investigated. The classification problem for each of them was solved by the Radon-Nikodym theorem. The problem here is to classify the sum of these objects. We must turn again to the set of functions on the groupoid of the foliation. Since we must consider ordinary measures with the multiplicative law of substantiation the Radon-Nikodym derivatives of the quasiinvariant measures on the equivalence relation [11] must satisfy the condition  $\lim_{x,y \rightarrow +\infty} D(x,y) = 1$ .

Then as in the previous statement we can prove the existence of a potential  $\rho$  of the probability measure (sum of the objects defined above) by passing to the mean  $p(x) = M'_y(D(x,y)) = \lim_{T \rightarrow +\infty} \int_0^T D(x,t) dt$ . Note that

this potential satisfies the following condition:  $\lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n \rho(a^i x) = 1$ ,

$a \in H_P$ . Let us define the conditional expectation  $E_D : C(L/H_P) \rightarrow \mathbb{R}$  corresponding to  $\rho$  as  $E_D(f)(\omega) = \lim_{n \rightarrow \infty} 1/n \sum_{i=1}^n \rho(a^i \sigma(\omega)) f(a^i \sigma(\omega))$ .

**Statement 12.** *Let  $\mu'$  be a measure on  $X$  equivalent to the fixed measure  $\mu$ . Assume  $\mu(K) = \mu(a^i K)$ . Let the cocycle  $D_{\mu'}$  being a R-N derivative of  $\mu'$  meet the condition  $\lim_{x, y \rightarrow \infty} D(x, y) = 1$ . Then  $\mu'$  is a measure of the type  $\mu \circ E_D$ , for  $E_D$  defined as before.*

- We have

$$\begin{aligned} \int \lim_{n \rightarrow \infty} 1/n \sum_{\pi(x)=\pi(y)} f(x, y) d\mu(x) \\ = \int \lim_{n \rightarrow \infty} 1/n \sum_{\pi(x)=\pi(y)} f(x, y) D(x, y) d\mu(y). \end{aligned}$$

Again as in [9] one can put  $f(x, y) = \rho(x)f(y)$ .  $\triangleright$

Now let us turn to the measures generated by the summand from  $C_0(L)$ .

**Statement 13.** *Let the equivalence relation on  $\mathbb{R}$  be given by  $x \sim x + k$ ,  $k \in N$ . Let  $\mu$  be a measure on  $\mathbb{R}$  such that the R-N derivative (A)  $D(x, y) \rightarrow 0$  as  $x$  or  $y \rightarrow \infty$ . Then*

- 1) *there exists a potential  $p : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $D(x, y) = p(x)/p(y)$ ;*
- 2) *if (B) for all  $y \in \mathbb{R}^+ \sum_{x \in R(y)} D(x, y) \leq \infty$  then there exists the unique potential  $p' : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $\sum_{x \in R(y)} p(x) = 1$ .*

- The proof repeats that of Renault [9]. Let us consider the natural projection  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ , a locally finite open cover  $\{V_j\}$  of  $\mathbb{S}^1$  and continuous sections  $\sigma_j : V_j \rightarrow \mathbb{R}^+$ .

Let us take  $p_j(x) = D(x, \sigma_j(\pi(x)))$ . Then the desired potential  $p$  is given by the product  $\prod_j h_j(\pi(x)) b_j$ . Hence the first part of the statement holds true.

The second part. First we note that if  $b(x)/b(y) = b'(x)/b'(y)$  then for any  $x \in \mathbb{S}^1$   $b/b'|_{R_x} = \text{const}$ , then  $b/b' = f \circ \pi$  for  $f \in C(\mathbb{S}^1)$ . Then we must take  $p'(x) = p(x) / \sum_{k \in N} p(k + x)$  as the desired potential of the second part of the statement.  $\triangleright$

**Note 5.** *The second part of the previous statement implies that the quasi-invariant measure  $\mu$  is a finite one. And vice versa, if the measure  $\mu$  is finite then the assumption of the second part of the statement holds true.*

- The statement follows from the representation of the measure

$$\mu(\mathbb{R}) = \lim_{n \rightarrow \infty} 1/n \sum_{j=1}^n \sum_{k \in \mathbb{Z}} p(j/n + k).$$

▷

Then the quasiinvariant measure whose R-N derivative satisfies the assumptions of the previous statement can be classified as in the second case. Naturally  $L_2(\mathbb{R}, \mu_1) = L_2(\mathbb{R}, \mu_2)$  if  $D(\mu_1)$  and  $D(\mu_2)$  both satisfy conditions (A) and (B).

It is clear that the measure on the foliated manifold  $M$  can be defined as follows:  $\mu(K)$  equals the integral of the values of the compactly supported summand on the characteristic function of  $K$  over the set  $P \setminus \{p\}$  with measure whose R-N derivative satisfies (A) and (B). It holds true due to the fact that the summand defined by the periodic part of the function lies on the 0-measure set. Thus, in the second and the third cases one can exclude subsets on the transversal  $P$  in order to get measure on the foliated manifold  $M$ .

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