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**THE PSEUDOSPECTRAL METHOD FOR
THERMOTROPIC PRIMITIVE EQUATION AND ITS
ERROR ESTIMATION**

(submitted by A. Lapin)

ABSTRACT. In this paper, a pseudospectral method is proposed for solving the periodic problem of thermotropic primitive equation. The strict error estimation is proved.

1. INTRODUCTION

Thermotropic primitive equation is governed by the following differential equations^[1]:

$$\begin{cases} \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{\partial \phi}{\partial x} - \nu \Delta U - FV = 0, \\ \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{\partial \phi}{\partial y} - \nu \Delta V + FU = 0, \\ \frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial y} + \phi \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) = 0, \end{cases} \quad (1.1)$$

where U, V are the components of the speed in x, y directions respectively, g is the acceleration of gravity, H is the height of the geopotential surface, $\phi = gH$, F is coriolis parameter and ν is the coefficient of friction.

There has been a rapid development in the spectral methods for the last two decades. They have become important tools for numerical solutions of partial differential equations, and have been widely applied to

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numerical simulations in various fields [2-5]. Although the pseudospectral methods are easier to implement for nonlinear partial differential equations, they are not stable as the spectral ones due to 'aliasing'. Therefore some author proposed the filtering technique [10-11] to remedy the deficiency of instability. Some papers have also been devoted to theoretical study and numerical solutions of (1.1) [6-9].

The aim of this paper is to consider the periodic initial boundary-value problem for thermotropic primitive equation. A pseudospectral scheme with restraint operator in combination with first order time differencing technique is considered for thermotropic primitive equation. The stability and rate of convergence for the approximate problem are proved.

2. THE PSEUDOSPECTRAL SCHEME

Let $\Omega = \{(x, y) | -\pi < x, y < \pi\}$ and all functions have the period 2π for the variable x and y . The norm of the space $L^q(\Omega)$ is denoted by $\|\cdot\|_{L^q(\Omega)}$. In particular, the scalar product and the norm of $L^2(\Omega)$ are denoted by (\cdot, \cdot) and $\|\cdot\|_{L^2(\Omega)}$ respectively. Let m_1, m_2 and N be integers and $m = \sqrt{m_1^2 + m_2^2}$. Define

$$V_N = \text{Span} \{e^{i(m_1x+m_2y)} | |m| \leq N\}, \quad N > 0.$$

Let P_N be the orthogonal projection operator, i.e.

$$(P_N \eta, \psi) = (\eta, \psi), \quad \forall \psi \in V_N.$$

For the pseudospectral approximation, we put the nodes

$$(x_{j_1}, y_{j_2}) = \left(\frac{2\pi j_1}{2N+1}, \frac{2\pi j_2}{2N+1} \right), \quad -N \leq j_1, j_2 \leq N,$$

and let \tilde{P}_c be the interpolation operator, i.e. for $\eta(x, y) \in C(\Omega)$

$$\tilde{P}_c \eta(x_{j_1}, y_{j_2}) = \eta(x_{j_1}, y_{j_2}), \quad -N \leq j_1, j_2 \leq N.$$

Define $P_c = P_N \tilde{P}_c$. To weaken the nonlinear instability of computation, we follow the work of [11] to adopt the filtering operator R_γ with $\gamma > 1$, i.e. if

$$\eta(x, y) = \sum_{|m| \leq N} \eta_{m_1, m_2} e^{i(m_1x+m_2y)},$$

then

$$R_\gamma \eta(x, y) = \sum_{|m| \leq N} \left(1 - \left(\frac{|m|}{N} \right)^\gamma \right) \eta_{m_1, m_2} e^{i(m_1x+m_2y)}.$$

Let τ be the mesh spacing of the variable t and define

$$S_\tau = \{t = k\tau | k = 0, 1, 2, \dots\}.$$

$$\eta_t(t) = \frac{\eta(t + \tau) - \eta(t)}{\tau}.$$

To approximate the nonlinear terms, we define

$$d_\alpha(\eta, u, v) = \alpha d^{(1)}(\eta, u, v) + (1 - \alpha) d^{(2)}(\eta, u, v), \quad 0 \leq \alpha \leq 1,$$

$$d^{(1)}(\eta, u, v) = P_c \left(u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \right),$$

$$d^{(2)}(\eta, u, v) = \frac{\partial}{\partial x} P_c(u\eta) + \frac{\partial}{\partial y} P_c(v\eta).$$

Let u^N, v^N, φ^N be the approximations to U, V and ϕ respectively, where for all $(x, y) \in \Omega$ and $t \in S_\tau$,

$$\eta^N(x, y, t) = \sum_{|m|, |n| \leq N} \eta_{m,n}^N(t) e^{i(mx+ny)}, \quad \eta = u, v, \varphi.$$

The pseudospectral scheme for solving (1.1) is

$$\left\{ \begin{array}{l} u_t^N + R_\gamma d_{1/2}(R_\gamma(u^N + \delta\tau u_t^N), u^N, v^N) + \frac{\partial}{\partial x} \varphi^N - \nu \Delta(u^N + \sigma\tau u_t^N) \\ \quad - F(v^N + \delta\tau v_t^N) = 0, \\ v_t^N + R_\gamma d_{1/2}(R_\gamma(v^N + \delta\tau v_t^N), u^N, v^N) + \frac{\partial}{\partial y} \varphi^N - \nu \Delta(v^N + \sigma\tau v_t^N) \\ \quad + F(u^N + \delta\tau u_t^N) = 0, \\ \varphi_t^N + R_\gamma d_0(R_\gamma(\varphi^N + \delta\tau \varphi_t^N), u^N, v^N) \\ \quad + A(\varphi^N + \delta\tau \varphi_t^N, u^N + \delta\tau u_t^N, v^N + \delta\tau v_t^N) = 0, \end{array} \right. \quad (2.1)$$

where $0 \leq \delta \leq 1$, $0 \leq \sigma \leq 1$ and $A(\eta, \xi, \eta^*) = P_c \left[\eta \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta^*}{\partial x} \right) \right]$.

3. SOME LEMMAS

Lemma 1. [1]. For all $\eta(x, y, t)$

$$2(\eta(t), \eta_t(t)) = (\|\eta(t)\|^2)_t - \tau \|\eta_t(t)\|^2.$$

Lemma 2. [5]. For all $\eta(x, y, t) \in V_N$, then

$$\left\| \frac{\partial \eta}{\partial x} \right\|^2 \leq N^2 \|\eta(t)\|^2, \quad \left\| \frac{\partial \eta}{\partial y} \right\|^2 \leq N^2 \|\eta(t)\|^2.$$

Lemma 3. [10]. For all $\eta(x, y, t) \in V_N$ and $\xi(x, y, t) \in V_N$, then

$$\|\eta(t)\xi(t)\|^2 \leq (2N + 1)^2 \|\eta(t)\|^2 \|\xi(t)\|^2.$$

Lemma 4. [15]. *For all $\eta(x, y, t) \in H^\beta(\Omega)$ and $\xi(x, y, t) \in V_N$, then*

$$\begin{aligned} \|P_N\eta(t) - \eta(t)\|_{H^s(\Omega)} &\leq C_1 N^{S-\beta} \|\eta(t)\|_{H^\beta(\Omega)}, \quad 0 \leq s \leq \beta, \\ \|P_C\eta(t) - \eta(t)\|_{H^s(\Omega)} &\leq C_2 N^{S-\beta} \|\eta(t)\|_{H^\beta(\Omega)}, \quad 0 \leq s \leq \beta, \quad \beta > 1, \\ \|R_\gamma\xi(t) - \xi(t)\|_{H^s(\Omega)} &\leq C_3 N^{S-\beta} \|\xi(t)\|_{H^\beta(\Omega)}, \quad 0 \leq s \leq \beta, \quad \gamma > \beta - s, \\ \|R_\gamma P_N\eta(t) - \eta(t)\|_{H^s(\Omega)} &\leq C_4 N^{S-\beta} \|\eta(t)\|_{H^\beta(\Omega)}, \quad 0 \leq s \leq \beta, \quad \gamma > \beta - s, \end{aligned}$$

where $C_1 - C_4$ are positive constants.

Lemma 5. [9]. *Assume that the following conditions are fulfilled:*

- (i) $\xi(t)$ and $\eta(t)$ are non-negative functions defined on S_τ ;
- (ii) ρ, a, M_1, M_2 , and M_3 are nonnegative constants;
- (iii) $A(x)$ is a function such that, if $x \leq M_3$, then $A(x) \leq 0$;
- (iv) $\xi(t) \leq \rho + \tau \sum_{t'=0}^{t-\tau} [M_1\xi(t') + M_2N^a\xi^2(t') + A(\xi(t'))\eta(t')]$;
- (v) $\rho e^{(M_1+M_2)T} \leq \min(M_3, \frac{1}{N^a})$, $\xi(0) \leq \rho$, $t \leq T$.

Then

$$\xi(t) \leq \rho e^{(M_1+M_2)t}.$$

In particular, if $M_2 = 0$ and $A(\xi(t')) = 0$, then for all ρ and n .

$$\xi(t) \leq \rho e^{M_1 t}.$$

4. ERROR ESTIMATION

For simplicity, we take $\delta = 0$, let $U^N = P_N U$, $V^N = P_N V$, and $\phi^N = P_N \phi$, then (1.1) leads to

$$\begin{cases} U_t^N + R_\gamma d_{1/2} (R_\gamma U^N, U^N, V^N) + \frac{\partial}{\partial x} \phi^N - \nu \Delta (U^N + \sigma \tau U_t^N) - F V^N = 0, \\ V_t^N + R_\gamma d_{1/2} (R_\gamma V^N, U^N, V^N) + \frac{\partial}{\partial y} \phi^N - \nu \Delta (V^N + \sigma \tau V_t^N) + F U^N = 0, \\ \phi_t^N + R_\gamma d_0 (R_\gamma \phi^N, U^N, V^N) + A(\phi^N, U^N, V^N) = 0, \end{cases} \quad (4.1)$$

where

$$\begin{aligned}
G_1^N &= U_t^N - \frac{\partial U^N}{\partial t}, \\
G_2^N &= R_\gamma d_{1/2} (R_\gamma U^N, U^N, V^N) - P_N \left[U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right], \\
G_3^N &= -\nu \sigma \tau \Delta U_t^N, \\
G_4^N &= V_t^N - \frac{\partial V^N}{\partial t}, \\
G_5^N &= R_\gamma d_{1/2} (R_\gamma V^N, U^N, V^N) - P_N \left[U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} \right], \\
G_6^N &= -\nu \sigma \tau \Delta V_t^N, \\
G_7^N &= \phi_t^N - \frac{\partial \phi^N}{\partial t}, \\
G_8^N &= R_\gamma d_0 (R_\gamma \phi^N, U^N, V^N) - P_N \left[U \frac{\partial \phi}{\partial x} + V \frac{\partial \phi}{\partial y} \right], \\
G_9^N &= A (\phi^N, U^N, V^N) - P_N \left[\phi \frac{\partial U}{\partial x} + \phi \frac{\partial V}{\partial x} \right].
\end{aligned}$$

Put

$$\tilde{u} = u - U^N, \quad \tilde{v} = v - V^N, \quad \tilde{\varphi} = \varphi - \phi^N.$$

Then from (1.1) and (2.1), we obtain

$$\begin{cases} \tilde{u}_t^N + \xi_1^N + \xi_2^N + \frac{\partial}{\partial x} \varphi^N - \nu \Delta (\tilde{u}^N + \sigma \tau \tilde{u}_t^N) - F \tilde{v}^N = \sum_{l=1}^3 G_l^N, \\ \tilde{v}_t^N + \xi_3^N + \xi_4^N + \frac{\partial}{\partial y} \varphi^N - \nu \Delta (\tilde{v}^N + \sigma \tau \tilde{v}_t^N) + F \tilde{u}^N = \sum_{l=4}^6 G_l^N, \\ \tilde{\varphi}_t^N + \sum_{k=5}^9 \xi_k^N = \sum_{l=7}^9 G_l^N, \end{cases} \quad (4.2)$$

where

$$\begin{aligned}
\xi_1^N &= R_\gamma d_{1/2} (R_\gamma \tilde{u}^N, U^N, v^N) + R_\gamma d_{1/2} (R_\gamma U^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_2^N &= R_\gamma d_{1/2} (R_\gamma \tilde{u}^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_3^N &= R_\gamma d_{1/2} (R_\gamma \tilde{v}^N, U^N, V^N) + R_\gamma d_{1/2} (R_\gamma V^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_4^N &= R_\gamma d_{1/2} (R_\gamma \tilde{v}^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_5^N &= R_\gamma d_0 (R_\gamma \tilde{\varphi}^N, U^N, V^N) + R_\gamma d_0 (R_\gamma \phi^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_6^N &= R_\gamma d_0 (R_\gamma \tilde{\varphi}^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_7^N &= -A (\tilde{\varphi}^N, \tilde{u}^N, \tilde{v}^N), \\
\xi_8^N &= -A (\tilde{\varphi}^N, U^N, V^N), \\
\xi_9^N &= -A (\phi^N, \tilde{u}^N, \tilde{v}^N).
\end{aligned}$$

We shall use the following notations

$$E = (U, V, \phi), \quad E^N = (U^N, V^N, \phi^N), \quad \tilde{E}^N = (\tilde{u}^N, \tilde{v}^N, \tilde{\varphi}^N),$$

$$\begin{aligned} \|\tilde{E}^N(t)\|^2 &= \|\tilde{u}^N(t)\|^2 + \|\tilde{v}^N(t)\|^2 + \|\tilde{\varphi}^N(t)\|^2, \\ \|\tilde{E}_t^N(t)\|^2 &= \|\tilde{u}_t^N(t)\|^2 + \|\tilde{v}_t^N(t)\|^2 + \|\tilde{\varphi}_t^N(t)\|^2, \\ \|\tilde{E}^N(t)\|_1^2 &= |\tilde{u}^N(t)|_1^2 + |\tilde{v}^N(t)|_1^2 + |\tilde{\varphi}^N(t)|_1^2. \end{aligned}$$

Let $H^\beta(\Omega)$ be the Sobolev space equipped with the norm $\|\cdot\|_{H^\beta(\Omega)}$. In particular $L^2(\Omega) = H^0(\Omega)$, we define $E_1 = (U, V, \phi)$

$$\|E_1\|_{H^\beta(\Omega)}^2 = \|U(t)\|_{H^\beta(\Omega)}^2 + \|V(t)\|_{H^\beta(\Omega)}^2 + \|\phi(t)\|_{H^\beta(\Omega)}^2,$$

$$\|E_1\|_\beta = \max_{0 \leq t \leq \tau} \|E_1(t)\|_{H^\beta(\Omega)}.$$

Now we suppose

$$\rho(t) = \|\tilde{E}^N(0)\|^2 + \nu\tau(\sigma + \frac{q}{2})|\tilde{E}^N(0)|_1^2 + \tau \sum_{t'=0}^{t-\tau} \|G_l^N(t')\|^2, \quad l = 1, \dots, 9$$

$$\begin{aligned} \tilde{E}(t) &= \|\tilde{E}^N(t)\|^2 + \nu\tau(\sigma + \frac{q}{2})|\tilde{E}^N(0)|_1^2 \\ &\quad + \tau \sum_{t'=0}^{t-\tau} r_1\tau \left[\|\tilde{E}_t^N(t')\|^2 + \nu(2 - 7\varepsilon)|\tilde{E}^N(t')|_1^2 \right]. \end{aligned}$$

Theorem 1. Suppose the following conditions are fulfilled

- (i) $\delta = 0, \quad \tau N^2 < \infty,$
- (ii) $\sigma > 1/2$ or $\tau N^2 < \frac{2}{v(1-2\sigma)},$
- (iii) for suitably small positive constant M_1 and all $t \leq T$, such that $\rho(T) \leq \frac{M_1}{N^2}.$

Then there exists a positive constant M_2 such that for all $t \in S_\tau, t \leq T$, we have

$$\tilde{E}(t) \leq \rho(t)e^{M_2 t}.$$

Theorem 2. Assume that the conditions (i), (ii) of Theorem 1 are satisfied. In addition $E \in C^2(0, T; m_0^0(\Omega)), E_1 \in C(0, T; H_{\frac{5}{2}+r}(\Omega) \cap m_0^{B+1}(\Omega)), r > 0, \quad \beta \geq 1$, then

$$\tilde{E}(t) \leq M_3(\tau^2 + N^{-2\beta})e^{M_4 t}.$$

M_ℓ being positive constants depending only on $\|E_1\|_{\frac{5}{2}+r}$ and ν .

Now we define

$$\|\eta(t)\|_{m_r^\beta} = \max_{0 \leq s \leq q} \left(\frac{1}{4\pi^2} \sum_{a+b=0}^{\beta} \iint_{\Omega} \left(\frac{\partial^{a+b}\eta(x,y,t)}{\partial x^a \partial y^b} \right)^2 dx dy \right)^{1/2},$$

$$\|\eta\|_{C^q(0,T;m_r^\beta(\Omega))} = \max_{0 \leq s \leq q} \max_{0 \leq t \leq T} \left\| \frac{\partial^s \eta(t)}{\partial t^s} \right\|_{m_r^\beta(\Omega)},$$

$$C^q(0,T;m_r^\beta(\Omega)) = \left(\eta(x,y,t) \mid \|\eta\|_{C^q(0,T;m_r^\beta(\Omega))} < \infty \right).$$

5. THE PROOF OF THEOREM 1

Let c be a positive constant which may be different in different cases, q denote an undetermined positive constant and $\varepsilon > 0$. Taking the scalar product (4.2) with $2\tilde{u}^N + q\tau\tilde{u}_t^N$, we have

$$\begin{aligned} & (\|\tilde{u}^N(t)\|^2)_t + \tau(q-1-\varepsilon)\|\tilde{u}_t^N(t)\|^2 \\ & + (2\tilde{u}^N(t) + q\tau\tilde{u}_t^N(t), \tilde{\xi}_1^N(t) + \tilde{\xi}_2^N(t) \\ & + \frac{\partial\tilde{\phi}^N}{\partial x}(t) + F\tilde{u}^N(t)) + 2\nu|\tilde{u}^N(t)|_1^2 \\ & + \nu\tau(\sigma + \frac{q}{2})(|\tilde{u}(t)|_1^2)_t + \nu\tau^2(\sigma q - \sigma - \frac{q}{2})|\tilde{u}_t^N(t)|_1^2 \\ & \leq c\|\tilde{u}^N(t)\|^2 + c(1 + \frac{q^2\tau}{4\varepsilon}) \sum_{l=1}^3 \|G_l^N(t)\|^2. \end{aligned} \tag{5.1}$$

Similarly from the second and third formulas of (4.2), we have

$$\begin{aligned} & (\|\tilde{v}^N(t)\|^2)_t + \tau(q-1-\varepsilon)\|\tilde{v}_t^N(t)\|^2 \\ & + (2\tilde{v}^N(t) + q\tau\tilde{v}_t^N(t), \tilde{\xi}_3^N(t) + \tilde{\xi}_4^N(t) \\ & + \frac{\partial\tilde{\phi}^N}{\partial y}(t) + F\tilde{u}^N(t)) + 2\nu|\tilde{v}^N(t)|_1^2 \\ & + \nu\tau(\sigma + \frac{q}{2})(|\tilde{v}(t)|_1^2)_t + \nu\tau^2(\sigma q - \sigma - \frac{q}{2})|\tilde{v}_t^N(t)|_1^2 \\ & \leq c\|\tilde{v}^N(t)\|^2 + c(1 + \frac{q^2\tau}{4\varepsilon}) \sum_{l=4}^6 \|G_l^N(t)\|^2 \end{aligned} \tag{5.2}$$

$$\begin{aligned}
& (\|\tilde{\varphi}^N(t)\|^2)_t + \tau(q-1-\varepsilon)\|\tilde{\varphi}_t^N(t)\|^2 \\
& + (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \sum_{l=5}^9 \tilde{\xi}_l^N(t)) \\
& \leq c\|\tilde{\varphi}^N(t)\|^2 + c(1 + \frac{q^2\tau}{4\varepsilon}) \sum_{l=7}^9 \|G_l^N(t)\|^2.
\end{aligned} \tag{5.3}$$

Putting (5.1)-(5.3) together, we get

$$\begin{aligned}
& (\|\tilde{E}^N(t)\|^2)_t + \tau(q-1-\varepsilon)\|\tilde{E}_t^N(t)\|^2 \\
& + 2\nu|\tilde{E}^N(t)|_1^2 + \nu\tau(\sigma + \frac{q}{2})(|\tilde{E}(t)|_1^2)_t \\
& + \nu\tau^2(\sigma q - \sigma - \frac{q}{2})|\tilde{E}_t^N(t)|_1^2 + \sum_{l=1}^6 M_l^N(t) \\
& \leq c\|\tilde{E}^N(t)\|^2 + c(1 + \frac{q^2\tau}{4\varepsilon}) \sum_{l=1}^9 \|G_l^N(t)\|^2,
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
M_1^N(t) &= (2\tilde{u}^N(t) + q\tau\tilde{u}_t^N(t), \xi_1^N(t)) + (2\tilde{v}^N(t) + q\tau\tilde{v}_t^N(t), \xi_3^N(t)) \\
&+ (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \xi_5^N(t)) - q\tau(\tilde{u}_t^N(t), F\tilde{v}^N(t)) \\
&+ q\tau(\tilde{v}_t^N(t), F\tilde{u}^N(t)), \\
M_2^N(t) &= (2\tilde{u}^N(t) + q\tau\tilde{u}_t^N(t), \xi_2^N(t)) + (2\tilde{v}^N(t) + q\tau\tilde{v}_t^N(t), \xi_4^N(t)) \\
&+ (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \xi_6^N(t)), \\
M_3^N(t) &= (2\tilde{u}^N(t) + q\tau\tilde{u}_t^N(t), \frac{\partial}{\partial x}\varphi^N(t)) + (2\tilde{v}^N(t) + q\tau\tilde{v}_t^N(t), \frac{\partial}{\partial y}\tilde{\varphi}^N(t)), \\
M_4^N(t) &= (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \xi_7^N(t)), \\
M_5^N(t) &= (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \xi_8^N(t)), \\
M_4^N(t) &= (2\tilde{\varphi}^N(t) + q\tau\tilde{\varphi}_t^N(t), \xi_9^N(t)).
\end{aligned}$$

We now estimate $|M_l^N(t)|$. Because of the Schwarz inequality and embedding theorem, we have

$$\begin{aligned}
|M_1^N(t)| &\leq \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \varepsilon \tau \|\tilde{E}_t^N(t)\|^2 + \frac{c}{\varepsilon \nu} (1 + \tau q^2 N^2) \|E_1^N\|_{\frac{5}{2}+r} \|\tilde{E}^N(t)\|^2, \\
|M_2^N(t)| &\leq \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \varepsilon \tau \|\tilde{E}_t^N(t)\|^2 + \frac{cN^2}{\varepsilon \nu} (\|\tilde{E}^N(t)\|^2 \\
&\quad + |\tilde{E}^N(t)|_1^2 + \|E_1^N\|_{\frac{5}{2}+r}) \|\tilde{E}^N(t)\|^2, \\
|M_3^N(t)| &\leq \varepsilon \tau |\tilde{E}_t^N(t)|_1^2 + \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \frac{c}{\varepsilon \nu} (1 + \tau q^2 N^2) \|\tilde{E}^N(t)\|^2, \\
|M_4^N(t)| &\leq \varepsilon \tau \|\tilde{E}_t^N(t)\|_1^2 + \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \frac{c}{\varepsilon \nu} (1 + \tau q^2 N^2) \|\tilde{E}^N(t)\|^2, \\
|M_5^N(t)| &\leq \varepsilon \tau \|\tilde{E}_t^N(t)\|_1^2 + \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \frac{cN^2}{\varepsilon \nu} (1 + \tau q^2 N^2) \|\tilde{E}^N(t)\|^4, \\
|M_6^N(t)| &\leq \varepsilon \tau \|\tilde{E}_t^N(t)\|_1^2 + \varepsilon \nu |\tilde{E}^N(t)|_1^2 + \frac{c}{\varepsilon \nu} \|E_1\|_{\frac{5}{2}+r} \|\tilde{E}^N(t)\|^2.
\end{aligned}$$

By substituting the above estimates into (5.4), we get

$$\begin{aligned}
&(\|\tilde{E}^N(t)\|^2)_t + \tau(q - 1 - 7\varepsilon) \|\tilde{E}^N(t)\|^2 + \nu(2 - 6\varepsilon) |\tilde{E}^N(t)|_1^2 \\
&\quad + \nu\tau(\sigma + \frac{q}{2}) (|\tilde{E}^N(t)|_1^2)_t + \nu\tau^2(\sigma q - \sigma - \frac{q}{2}) |\tilde{E}^N(t)|_1^2 \quad (5.5) \\
&\leq H^N(t) + A_1 \|\tilde{E}^N(t)\|^2 + B_1 \|\tilde{E}^N(t)\|^4 + B_2 |\tilde{E}^N(t)|_1^2,
\end{aligned}$$

where

$$A_1 = c \left(1 + \frac{c}{\varepsilon \nu} + \frac{c}{\varepsilon \nu} (1 + \tau q^2 N^2) \right) \|E_1\|_{\frac{5}{2}+r},$$

$$B_1 = \frac{cN^2}{\varepsilon \nu} (1 + \tau q^2 N^2) + \frac{cN^2}{\varepsilon \nu},$$

$$B_2 = \frac{cN^2}{\varepsilon \nu} \|\tilde{E}^N(t)\|^2,$$

$$H^N(t) = c \left(1 + \frac{q^2 \tau}{4\varepsilon} \right) \sum_{l=1}^9 \|G_l^N\|^2.$$

Now let ε be suitably small, $r_1 > 0$, and

$$q_1 = \max \left[1 + r_1 + 7\varepsilon, \frac{2\sigma}{2\sigma - 1} \right],$$

$$q_2 = r_1 + 1 + 7\varepsilon + \frac{\nu\tau N^2}{2},$$

$$q_3 = (2r_1 + 2 + 14\varepsilon + 2\sigma\nu\tau N^2) [2 - \nu\tau N^2(1 - 2\sigma)]^{-1}.$$

If $\sigma > 1/2$, we put $q = q_1$, and it follows from (5.5) that

$$\begin{aligned} & (\|\tilde{E}^N(t)\|^2)_t + r_1\tau\|\tilde{E}^N(t)\|^2 + \nu(2 - 6\varepsilon)|\tilde{E}^N(t)|_1^2 \\ & \quad + \nu\tau(\sigma + \frac{q}{2})(|\tilde{E}^N(t)|_1^2)_t \\ & \leq H^N(t) + A_1\|\tilde{E}^N(t)\|^2 + B_1\|\tilde{E}^N(t)\|^4 + B_2|\tilde{E}^N(t)|_1^2. \end{aligned} \quad (5.6)$$

If $\sigma = 1/2$, we put $q = q_2$, and so

$$\tau(q - 1 - 7\varepsilon)\|\tilde{E}_t^N\|^2 + \nu\tau^2(\sigma q - \sigma - \frac{q}{2})|\tilde{E}_t^N(t)|_1^2 \geq r_1\tau|\tilde{E}_t^N(t)|^2.$$

Therefor (5.6) is still holds.

If $\sigma < 1/2$, $\tau N^2 < \frac{2}{\nu(1-2\sigma)}$, we put $q = q_3$, and thus (5.12) holds.

By summing up (5.6) for $t \in S_\tau$, we get

$$\begin{aligned} & (\|\tilde{E}^N(t)\|^2)_t + \nu\tau(\sigma + \frac{q}{2})(|\tilde{E}^N(t)|_1^2)_t + \tau \sum_{t'=0}^{t-\tau} r_1\tau\|\tilde{E}_t^N(t')\|^2 \\ & \quad + \nu(2 - 7\varepsilon)|\tilde{E}^N(t')|_1^2 \leq \rho(t) \\ & \quad + \tau \sum_{t'=0}^{t-\tau} A_1\tilde{E}_1^N(t') + B_1\tilde{E}_1^2(t') \\ & \quad + B_2|\tilde{E}_2^N(t')|_1^2, \end{aligned} \quad (5.7)$$

where

$$\rho(t) = \|\tilde{E}^N(0)\|^2 + \nu\tau(\sigma + \frac{q}{2})|\tilde{E}^N(0)|_1^2 + \tau \sum_{t'=0}^{t-\tau} H^N(t')$$

from which and Lemma 5, the proof is completed.

6. THE PROOF OF THEOREM 2

We first have

$$\|G_l^N(t)\| \leq c\tau\|E\|_{C^2(0,T;m_0^0(\Omega))}, \quad l = 1, 4, 7.$$

From Lemma 4 and the embedding theorem, we get

$$\|G_l^N(t)\| \leq c\|E_1\|_{\frac{5}{2}+r}N^{-\beta}\|E_1\|_{m_0^{\beta+1}(\Omega)}, \quad l = 2, 5, 8.$$

It is easy to show that

$$\|G_l^N(t)\| \leq c\tau \left(\left\| \frac{\partial E}{\partial t}(t) \right\|_{m_2^0(\Omega)} + \left\| \frac{\partial E}{\partial t}(t) \right\|_{m_2^0(\Omega)} \right), \quad l = 3, 6.$$

We have also

$$\|G_9^N(t)\| \leq cN^{-\beta}\|E_1\|_{\frac{5}{2}+r}\|E(t)\|_{m_0^{\beta+1}(\Omega)}, \quad \|\tilde{E}^N(0)\| \leq cN^{-\beta}\|E(0)\|_{m_0^{\beta}(\Omega)}.$$

Therefore if the conditions of Theorem 1 are fulfilled, then

$$\rho(t) \leq c (\tau^2 + N^{-2\beta}).$$

By combining the above estimations with Theorem 1, we complete the proof of Theorem 2.

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