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CONCAVE SCHLICHT FUNCTIONS WITH BOUNDED OPENING ANGLE AT INFINITY

ABSTRACT. Let D denote the open unit disc. In this article we consider functions $f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$ that map D conformally onto a domain whose complement with respect to $\mathbb{C}$ is convex and that satisfy the normalization $f(1) = \infty$. Furthermore, we impose on these functions the condition that the opening angle of $f(D)$ at infinity is less than or equal to $\pi A, A \in (1, 2]$. We will denote these families of functions by $CO(A)$. Generalizing the results of [1], [3], and [5], where the case $A = 2$ has been considered, we get representation formulas for the functions in $CO(A)$. They enable us to derive the exact domains of variability of $a_2(f)$ and $a_3(f), f \in CO(A)$. It turns out that the boundaries of these domains in both cases are described by the coefficients of the conformal maps of D onto angular domains with opening angle πA.

Let D denote the open unit disc. In this article we consider functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$$

that map D conformally onto a domain whose complement with respect to $\mathbb{C}$ is convex and that satisfy the normalization $f(1) = \infty$. Furthermore, we impose on these functions the condition that the opening angle of $f(D)$ at infinity is less than or equal to $\pi A, A \in (1, 2]$. We will denote these families of functions by $CO(A)$. In [1] and [2], the following

theorem has been proved for the class $CO(2)$, which contains the classes $CO(A), A \in (1, 2]$.

Theorem A. *Let f be holomorphic in D and normalized by $f(0) = f'(0) - 1 = 0$. Let the function Φ be defined by*

$$\Phi(z) = z + \frac{2f'(z)}{f''(z)}, \quad z \in D. \tag{1}$$

Then $f \in CO(2)$ if and only if
(i) Φ is holomorphic in D and $|\Phi(z)| \leq 1$ for $z \in D$.
(ii) Φ has its attractive fixed point at the point $z = 1$ and the angular derivative of Φ at this point satisfies $\Phi'(1) \in [0, 1/3]$.

As a generalization of Theorem A we prove

Theorem 1. *Let f be holomorphic in D and normalized by $f(0) = f'(0) - 1 = 0$. Let the function Φ be defined by*

$$\Phi(z) = z + \frac{2f'(z)}{f''(z)}, \quad z \in D.$$

Then for $A \in (1, 2]$, the function f belongs to the class $CO(A)$ if and only if
(i) Φ is holomorphic in D and $|\Phi(z)| \leq 1$ for $z \in D$.
(ii) Φ has its attractive fixed point at the point $z = 1$ and the angular derivative of Φ at this point satisfies $\Phi'(1) \in [0, (A - 1)/(A + 1)]$.

Proof. If $f(D)$ has an opening angle $\pi\alpha, \alpha \in [1, A]$ at infinity, the boundary C of $f(D)$ may be approximated by concave polygons $C_m, m \in \mathbb{N}$, with opening angle $\pi\alpha, \alpha \in [1, A]$, at infinity and m corners $z_k \in \mathbb{C}, k = 1, ..., m$, such that $\pi\beta_k, k = 1, ..., m, 0 < \beta_k \leq 1$, is the change in the direction of C_m at the corner z_k. In the case $\alpha = 1$, we only have to consider a straight line.
According to the Schwarz-Christoffel formula we get for the maps f that map D onto the concave domain bounded by C_m the existence of m preimages of the corners at the points

$$\exp(-it_k), k = 1, ...m, 0 < t_1 < ... < t_m < 2\pi,$$

such that

$$f'(z) = (1 - z)^{-\alpha - 1} \prod_{k=1}^{m} \left(1 - e^{it_k}z\right)^{\beta_k},$$

where

$$\sum_{k=1}^{m} \beta_k = \alpha - 1.$$

Now, we consider the function

$$g(z) = \frac{1}{f'(z)(1-z)^{A+1}} = \frac{1}{(1-z)^{A-\alpha}\prod_{k=1}^{m}\left(1 - e^{it_k}z\right)^{\beta_k}}.$$

To these products we apply Lemma 1 and Theorem 1 of [4]. They imply that products of the form

$$g(z) = \prod_{k=1}^{m+1}\left(1 - e^{it_k}z\right)^{-\delta_k}, \quad \sum_{k=1}^{m+1}\delta_k = A - 1,$$

have a representation

$$g(z) = \int_{\partial D}\frac{d\mu(x)}{(1-xz)^{A-1}},$$

where μ is a probability measure on ∂D. Hence, we get for the derivatives of our polygonal mappings f the representation formula

$$\frac{1}{f'(z)} = (1-z)^{A+1}\int_{\partial D}\frac{d\mu(x)}{(1-xz)^{A-1}}, \quad z \in D,$$

and obviously this formula is valid for all $f \in CO(A)$.

Consideration of the Taylor expansion of both sides reveals that the inequality

$$|2a_2(f) - (A+1)| \leq A - 1 \tag{2}$$

is valid. Now, we proceed exactly as in the proof of Theorem A (ii) in [1] to get the assertion of Theorem 1.

Now we use Theorem 1 and the following Theorem B that was shown in [5] to prove a representation formula for $f \in CO(A)$.

Theorem B. *Let $\lambda > 1$.*
1) *For any function Φ holomorphic in $D, \Phi(D) \subset D$, with an attractive boundary fixed point at the point 1 and an angular derivative $\Phi'(1) \in [0, 1/\lambda]$ the function φ defined by*

$$\varphi(z) = \frac{\lambda - 1 + z - \lambda\Phi(z)}{\lambda z - ((\lambda-1)z+1)\Phi(z)}, \quad z \in D,$$

is holomorphic in D and satisfies $\varphi(D) \subset \overline{D}$.
2) *For any function φ holomorphic in $D, \varphi(D) \subset \overline{D}$, the function Φ defined by*

$$\Phi(z) = \frac{\lambda - 1 + z - \lambda z\varphi(z)}{\lambda - ((\lambda-1)z+1)\varphi(z)}, \quad z \in D, \tag{3}$$

is holomorphic in D. Furthermore, there are two possibilities. The first one occurs if $\varphi \equiv 1$, then $\Phi \equiv 1$. In all other cases $\Phi(D) \subset D$. In the

latter cases Φ *has an attractive boundary fixed point at the point* 1 *and an angular derivative* $\Phi'(1) \in [0, 1/\lambda]$.

The said representation theorem is as follows.

Theorem 2. *Let* $A \in (1, 2]$. *A function* f *holomorphic in* D *and satisfying* $f(0) = 0$ *belongs to the class* $CO(A)$ *if and only if there exists a function* $\varphi : D \to \overline{D}$, *holomorphic in* D, *such that*

$$f'(z) = (1-z)^{-(A+1)} \exp\left(-(A-1)\int_0^z \frac{\varphi(t)\, dt}{1-t\varphi(t)}\right), \quad z \in D. \quad (4)$$

Proof. We apply Theorem 1 and Theorem B in the case $\lambda = (A + 1)/(A-1)$. From the formulas (1) and (3) we get by a little computation that $f \in CO(A)$ if and only if there exists a function φ holomorphic in D such that $\varphi(D) \subset \overline{D}$ and

$$\frac{d}{dz}\left(\log\left(f'(z)(1-z)^{(A+1)}\right)\right) = \frac{-(A-1)\varphi(z)}{1-z\varphi(z)}, \quad z \in D.$$

Integration using the initial condition $f'(0) = 1$ immediately yields the assertion of Theorem 2.

Now, we want to present some corollaries to Theorem 2. Firstly, we make (2) more precise.

Corollary 1. *Let* $A \in (1, 2]$. *Then the domain of variability of* $a_2(f), f \in CO(A)$ *is determined by the inequality*

$$\left| a_2(f) - \frac{A+1}{2} \right| \leq \frac{A-1}{2}. \quad (5)$$

Equality in (5) *is attained if and only if*

$$f(z) = f_\theta(z) = \frac{1}{A\,(1+e^{i\theta})}\left(\left(\frac{1+e^{i\theta}z}{1-z}\right)^A - 1\right), \quad \theta \in [0, 2\pi] \setminus \{\pi\}. \quad (6)$$

These functions map the unit disc onto an angular domain with opening angle πA. *The boundary point* $a_2(f) = 1$ *of* (5) *is attained if and only if* $f \in CO(2)$ *maps* D *onto a half plane.*

Proof. We insert the Taylor series

$$\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$$

into (4) from which we get the representation

$$2\,a_2(f) = A + 1 - (A-1)c_0.$$

The proof of (5) is a consequence of the fact that under our assumptions on the function φ the inequality $|c_0| \leq 1$ is valid. In this inequality

equality is attained if and only if $\varphi \equiv \exp(i\theta), \theta \in [0, 2\pi]$. This together with the integration of the corresponding differential equation (4) proves the rest of the assertion.

The central role of the mappings f_θ defined by (6) in the family $CO(A)$ may be recognized in a more formal way from the next corollary.

Corollary 2. *Let $A \in (1,2]$ and $f \in CO(A)$. Then there exists a function $\omega : D \to \overline{D}$, holomorphic in D, such that*

$$f'(z) = \frac{(1 + z\omega(z))^{A-1}}{(1 - z)^{A+1}}, \quad z \in D. \tag{7}$$

Proof. Theorem 2 implies that $f \in CO(A)$ if and only if there exists a function $g \in CO(2)$ such that

$$f'(z) = (g'(z))^{A-1}(1 - z)^{2A-4}.$$

The representation (7) follows from this equation and the fact that for the derivative of any function $g \in CO(2)$ there exists a function $\omega : D \to \overline{D}$, holomorphic in D, such that

$$f'(z) = \frac{1 + z\omega(z)}{(1 - z)^3}, \quad z \in D. \tag{8}$$

(8) has been proved in [3] and [5] in two different ways.

It is easily seen that a computation of the domain of variability of $a_2(f), f \in CO(A)$ with the help of Corollary 2 delivers Corollary 1 again. A detailed comparison of these two possibilities in the investigation of $a_3(f)$ shows that in this case the analogous fact is not longer true for all $A \in (1, 2]$. We shall prove here only the results of the determination of $a_3(f), f \in CO(A)$.

Corollary 3. *Let $A \in (1, 2]$ and the function h be defined by*

$$h(\zeta) = \zeta + \frac{A - 2}{2(A + 1)}\zeta^2.$$

Then the equation

$$\{a_3(f) \mid f \in CO(A)\} = \left\{ \frac{(A + 1)(A + 2)}{6} + (A^2 - 1)\frac{\tau}{3} \;\middle|\; \tau \in \overline{h(D)} \right\} \tag{9}$$

is valid. A point on the boundary of the set of variability of $a_3(f)$ given in (9) is attained if and only if f is one of the functions defined in (5) or the mapping of D onto a halfplane belonging to $CO(2)$.

Proof. To prove (9), we consider a variation of a function $f \in CO(A)$. To this end, let for $\epsilon \in D$ and the holomorphic function $\varphi : D \to \overline{D}$ the function $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(z, \epsilon) = \frac{\varphi(z) - \epsilon}{1 - \overline{\epsilon}\varphi(z)}, \quad z \in D.$$

Obviously, $\tilde{\varphi}(\cdot, \epsilon)$ is holomorphic in D and $\tilde{\varphi}(D, \epsilon) \subset \overline{D}$. Hence, for any $f \in CO(A)$ given by (4) the function $f(\cdot, \epsilon)$ defined by

$$f'(z, \epsilon) = (1 - z)^{-(A+1)} \exp\left(-(A - 1) \int_0^z \frac{\tilde{\varphi}(t, \epsilon)\, dt}{1 - t\tilde{\varphi}(t, \epsilon)}\right), \quad z \in D, \quad (10)$$

and $f(0, \epsilon) = 0$ belongs to $CO(A)$ for any $\epsilon \in D$ as well. Now, we consider the Taylor expansion

$$f'(z, \epsilon) = 1 + \sum_{k=2}^{\infty} k\, a_k(f, \epsilon) z^{k-1}. \quad (11)$$

The point $a_3(f)$ is a inner point of the set $\{a_3(f)|f \in CO(A)\}$ if $a_3(f, \cdot)$ maps a neighbourhood of $\epsilon = 0$ onto a neighbourhood of $a_3(f)$. These points are characterized by

$$\left.\left(\left|\frac{\partial a_3(f, \epsilon)}{\partial \overline{\epsilon}}\right|^2 - \left|\frac{\partial a_3(f, \epsilon)}{\partial \epsilon}\right|^2\right)\right|_{\epsilon=0} \neq 0. \quad (12)$$

Using (10), (11), and the above Taylor series for φ we get

$$3 \left.\frac{\partial a_3(f, \epsilon)}{\partial \overline{\epsilon}}\right|_{\epsilon=0} = -(A - 1)\left(c_0^2(A + 1 + (2 - A)c_0) + c_0 c_1\right)$$

and

$$3 \left.\frac{\partial a_3(f, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = (A - 1)\left(A + 1 + (2 - A)c_0\right).$$

To verify that (12) is valid for $|c_0| < 1$ we use that $|c_1| \leq 1 - |c_0|^2$. Hence, in this case it is sufficient to prove that

$$|A + 1 + (2 - A)c_0| > |A + 1 + (2 - A)c_0||c_0|^2 + |c_0|\left(1 - |c_0|^2\right).$$

The inequality

$$|A + 1 + (2 - A)c_0| > |c_0|$$

is equivalent to the inequality mentioned before and it is easy to see using the triangle inequality that it holds as a consequence of $|c_0| < 1$ and $A > 1$.

Now, we have proved that the boundary points of the set $\{a_3(f)|f \in CO(A)\}$ can stem only from $|c_0| = 1$. Since $|c_0| = 1$ if and only if $\varphi \equiv \exp(i\theta), \theta \in [0, 2\pi]$, the only possibilities for extremal functions of the set $\{a_3(f)|f \in CO(A)\}$ are the extremal functions of Corollary

1. The computation of their third Taylor coefficients has as result the boundary of the set on the right hand side of equation (9). Here, we have to recognize that $h(\partial D)$ is a (convex) Jordan curve. The inequalities

$$0 \le \frac{2 - A}{2(A + 1)} < \frac{1}{4}$$

imply that h belongs to the family of univalent convex functions. This completes the proof of Corollary 3.

We conclude with a little geometric observation closely related to the mappings f_θ defined in (6).

Theorem 3. *Let $A \in (1, 2]$. Then the Koebe domain of $CO(A)$ is determined by*

$$\bigcap_{f \in CO(A)} f(D) = \left\{ w \,\middle|\, \operatorname{Re} w > -\frac{1}{2A} \right\}.$$

Proof. For $A \in (1, 2)$ let us fix $\alpha \in [1, A]$ and consider the functions $f \in CO(A)$ with opening angle $\pi\alpha$ at infinity. For $\alpha = 1$, there exists only one such mapping, namely $f(z) = z/(1 - z), z \in D$. In all other cases, we may proceed as follows. Since $\mathbb{C} \setminus f(D)$ is convex, this set is contained in one of the sets $\mathbb{C} \setminus f_\theta(D), \theta \in [0, 2\pi] \setminus \{\pi\}$. The fact that the union of the sets $\mathbb{C} \setminus f_\theta(D), \theta \in [0, 2\pi] \setminus \{\pi\}$, forms the closed half plane

$$\left\{ w \,\middle|\, \operatorname{Re} w \le -\frac{1}{2\alpha} \right\}$$

proves the assertion of Theorem 3 in the cases $A \in (1, 2)$.
For the proof in the case $A = 2$, it is sufficient to recognize that

$$\bigcup_{A \in (1,2)} CO(A) \subset CO(2)$$

and that the end points of the half lines $\mathbb{C} \setminus f_\theta(D), \theta \in [0, 2\pi] \setminus \{\pi\}$, in this case form the line

$$\left\{ w \,\middle|\, \operatorname{Re} w = -\frac{1}{4} \right\}.$$

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