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ON A CLASS OF NON LINEAR DIFFERENTIAL OPERATORS OF FIRST ORDER WITH SINGULAR POINT

(submitted by M. Malakhaltsev)

Abstract. We consider the problem of the existence and uniqueness of solutions for partial differential operator of the form $Lu = D_X u - B(x,u)$ where X is a vector field. The solvability of L may be of some interest since by the Nash-Moser inverse function theorem the equivalence problem in differential geometry can be solved via Lie derivative operator and the later is locally a particular case of L. An application to the equivalence of dynamic systems is given.

1. Introduction.

Let $C(n,s)$ denote the space of germs of C^∞-maps from R^n into R^sendowed with the weak topology. The weak topology is the topology on uniform convergence of derivatives of each order on compact sets. Consider a differential operator $L : C(n,s) \to C(n,s)$ defined by

$$Lu = D_X u - B(x,u),$$

where $X \in C^\infty(n,n)$ is a vector function, B is a C^∞ function with values in $C^\infty(n,s)$ and $D_X u$ stands for the directional derivative of u in direction X. The operators of this type represent a local form (for specific B) of a Lie derivative L_X or covariant derivative ∇_X which are widely used in differential geometry on manifolds. In both these cases $B(x,.)$ depends

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uniquely on a certain jet of X at the point x, and u stands usually for a differentiable section of a fiber bundle over a manifold.

In this paper we are interested in local solvability of the operator L. In other words we ask for the existence and the uniqueness of local solution of the partial differential equation

$$D_X u - B(x,u) = f \tag{1}$$

or in coordinates

$$\sum_{j=1}^{n} X^j(x) D_j u^i - B^i(x,u) = f^i(x) \tag{2}$$

where $i = 1, \ldots, s$.

2. Formal solution of the linearized equation

We are going to present a dynamical method to give an integral formula for a solution of the linearized equation of the equation (1), useful in case where the method works. We shall assume in this section that the coefficients in (2) are of class C^∞ and the vector field X is a half complete in the sense that the flow $\phi(t,x)$ generated by X is defined in a half-cylindrical neighborhood $C_\delta \subset R^n \times R_+$

$$|x| \leq \delta, \quad t \geq 0,$$

where $|.|$ stands for a norm in R^n. The flow of X is the solution of the initial value problem

$$x' = X(x) \quad x(0) = x \tag{3}$$

which means that

$$\phi_t' = X o \phi_t \quad \phi_0(x) = x \tag{4}$$

where $\prime = \frac{d}{dt}$.

Lemma 1 ([1]). *For all s, t, such that $s \geq 0, t \geq 0$ and $s - t \geq 0$ we have*

$$\phi_{s-t}^{-1} = \phi_t o \phi_s^{-1}.$$

*Proof.*Let $f \in C^k(n,s)$, we use Newton method to solve

$$F(x,u) = D_X u - B(x,u) - f(x) = 0.$$

Let u_o be an approximate solution, we try to give a better approximation,

$$u_o + \Delta_o = u_1.$$

The Taylor formula with integral remainder writes

$$F(x,u_1) = F(x,u_o) + (D_X - B_u'(x,u))\Delta_o + o(\Delta_o^2)$$

where

$$o(\Delta_o^2) = \int_0^1 (1-s) D_u^2 F(x, u_o + s\Delta_o) < \Delta_o, \Delta_o > ds \tag{5}$$

and $B'_u(x,u)$ is the Jacobian matrix of B with respect to the variable u. So if Δ_o is solution of

$$(D_X - B'_u(x,u_o))\Delta_o = -F(x,u_o) \tag{6}$$

then

$$F(x,u_1) = o(\Delta_o^2).$$

Before starting iteration we study the linearized equation (6). Consider an another auxiliary differential equation

$$u' = -B'_u(\phi_t^{-1}(x), u_o(\phi_t^{-1}(x))).u \tag{7}$$

x plays here a role of parameter in respect which the right hand side is C^∞. Thus there exists a normalized fundamental solution $R_o(t,x)$ of (7) satisfying

$$R'_o(t,x) = -B'_u(\phi_t^{-1}(x), u_o(\phi_t^{-1}(x))).R_o(t,x) \quad R_o(0,x) = id \tag{8}$$

We shall prove

Lemma 2. [1]*If the integral of the right side hand of the formula*

$$\Delta(x) = -\int_0^\infty R(s,\phi_s(x))G(\phi_s(x))ds \tag{9}$$

is uniformly convergent in a neighborhood of the origin, then Δ is a local solution of (6) where, for a fixed function $v \in C(n,s)$, $G(\phi_s(x)) = F(\phi_s(x), v(\phi_s(x)))$.

*Proof.*While this result is in [1], we include its proof. In order to show that it satisfies (6) we compute first $\Delta(\phi_t(x))$

$$\Delta(\phi_t(x)) = -\int_0^\infty R(s,\phi_{s+t}(x))G(\phi_{s+t}(x))ds$$

$$= -\int_t^\infty R(s-t,\phi_s(x))G(\phi_s(x))ds$$

$$\frac{d}{dt}\Delta(\phi_t(x)) = -G(\phi_t(x)) + \int_t^\infty R'(s-t,\phi_s(x))G(\phi_s(x))ds$$

$$= -G(\phi_t(x)) - B'_u(\phi_t(x), v(\phi_t(x)))\int_0^\infty R(s,\phi_{s+t}(x))G(\phi_{s+t}(x))ds$$

$$= -G(\phi_t(x)) - B'_u(\phi_t(x), v(\phi_t(x))).\Delta(\phi_t(x)).$$

Setting $t = 0$ and knowing $\phi_0(x) = x$, we get finally

$$(D_X - B'_u(x, v)).\Delta = -F(x, v(x)).$$

□

3. Solution of the non linearized equation

We make sufficient conditions to obtain solutions of the non linear equation1

Assume that:

(H_1) the function f is infinitely flat at the origin 0.

(H_2) $B(x, 0) = 0$ for all x in a neighborhood of 0.

(H_3) the origin 0 is a contracting critical point of the vector field X: that means there are positive constants a, c, δ such that

$$|\phi_t(x)| \leq a\,|x|\,e^{-ct} \ \text{ for } x \in B_\delta$$

where B_δ denotes the ball $\{|x| \leq \delta\}$.

We recall some useful facts:

–Contracting critical points are necessary isolated critical points $(cf.$ [3]).

–In [5]it was shown that if ϕ_t has an exponential bound of order e^{-ct}, then so do all the derivative $D^k\phi_t$, $k \geq 1$ that is

$$\left|D_x^k \phi_t(x)\right| \leq a.e^{-ct} \tag{10}$$

where a is a constant depending on k and $x \in B_\delta$.

Let $\alpha(t, x)$ and $\beta(t, x)$ denote respectively the least and the greatest real parts of the eigenvalues occurring in the spectrum of the matrix $A(t, x) = -B'_u(\phi_t^{-1}(x), u_o(\phi_t^{-1}(x)))$ (with fixed t, x and function u). Let $R(t, x)$ be the normalized fundamental solution of the auxiliary equation (7). The following estimates, in version without x are well known $(cf.$ [4]):

$$\exp \int_{t_o}^{t} \alpha(s, x)ds \leq |R(t, t_o, x)| \leq \exp \int_{t_o}^{t} \beta(s, x)ds \tag{11}$$

for $t \geq 0$ and x fixed.

Since we are investigating local solution we can assume without loss of generality that X is of support included in B_δ, hence its flow is defined for all $t \in R$, and the mapping F is bounded in some neighborhood of (x, u), namely since F is continuous in the $C^\infty-$ topology there exist constants $\delta > 0$ and $\epsilon > 0$ such that for every positive integer k, $\left|D^k F(x, u)\right| < \infty$ provided that $|x| < \delta$ and $\|u\|_n^{B_\delta} = \sup_\alpha \sup\{|D^\alpha u(x)| : x \in B_\delta\} < \epsilon$, where α runs over all the derivatives D^α at most equal to the positive integer n.

We define inductively the sequence u_k:

$$u_{k+1} = u_k + \Delta_k \ , \quad u_o = 0$$
$$(D_X - B'_u(x, u_k)).\Delta_k = -F(x, u_k(x)) \tag{12}$$

Theorem 3. *Under the assumptions* $(H_1), (H_2)$ *and* (H_3)*, the function* $u = \sum_{i=1}^{\infty} \Delta_i$ *converges uniformly in a neighborhood of the origin* 0*.*

Proof. Let K be a compact set K. For any $x \in K$ and any $u \in C(n, s)$ sufficiently small (for the norm $\|.\|_n^K$) there is a constant C such that

$$\left|D_u^2 F(x, u(x))\right| \le 2C$$

where $D_u^2 F$ is the second derivative with respect to u. If u_k is a solution of the equation12, then by Taylor's formula with integral remainder we get. So we have the estimate

$$F(x, u_{k+1}(x)) = \int_0^1 (1-t) D_u^2 F(x, u_k + t\Delta_k) < \Delta_k, \Delta_k > dt$$

so

$$|F(x, u_{k+1}(x))| \le C \left|\Delta_k(x)\right|^2 \tag{13}$$

Provided that $x \in K$ and u_{k+1} sufficiently small. Letting $R_k(t, x) = R(t, u_k(x))$ where $R(t, x)$ denotes the fundamental normalized solution of the equation7 and $G_k(x) = -F(x, u_k(x))$. Putting $u_o = 0$, we get

$$G_o(\phi_t(x)) = F(\phi_t(x), 0) = -f(\phi_t(x)).$$

Since f is infinitely flat at the origin 0, for every integer p there exist constants $\delta = \delta_p > 0$ and $M'_p > 0$ such that

$$|F(\phi_t(x)), 0| \le M'_p \delta_p^p \left|\phi_t(x)\right|^p \quad \text{for } \ |x| \le \delta.$$

Taking into account that the vector field X is asymptotically stable, there are constants $a > 0$ and $c > 0$ such that

$$|\phi_t(x)| \le a e^{-ct}$$

provided that x is small (we assume that $x \in B_\delta = \{x : |x| < \delta\}$). Then

$$|F(\phi_t(x), 0)| \le M_p \delta_p^p e^{-pct} \text{ for } \ |x| \le \delta \tag{14}$$

where $M_p = a^p M'_p$ is a constant depending only on p.

Now since the matrix function $B'_u(x, u)$ is continuous in the C^∞-topology with respect to the adjoint variable, it follows that $B'_u(x, u)$ is bounded for $|x| \le \delta$ and u bounded in the space $C(n, s)$ and so do its eigenvalues. By the estimations of the eigenvalues given in 11 we deduce

$$\left|R_o(t, \phi_t(x))\right| . \left|G_o(\phi_t(x))\right| \le M_p \delta^p e^{(-pc+\beta)t} \tag{15}$$

where β denotes the upper bound of the eigenvalues. We choose p large enough so that

$$pc - \beta = \omega \geq 1.$$

By the formula (9), we get that

$$|\Delta_o(x)| \leq \int_0^{+\infty} |R_o(t, \phi_t(x))| \, . \, |G_o(\phi_t(x))| \, dt$$

and by the estimation (15), we obtain for any $|x| \leq \delta$

$$|\Delta_o(x)| \leq M_p \delta^p.$$

Fix $0 < \epsilon < \frac{1}{C}$ and choose δ such that $M_p \delta^p < \frac{\epsilon}{2}$, then for any $|x| \leq \delta$

$$|\Delta_o(x)| < \frac{\epsilon}{2}.$$

Suppose that for any fixed integer $k \geq 1$ and $1 \leq j \leq k$

$$|\Delta_j(x)| < \frac{\epsilon}{2^{j+1}}$$

then

$$|u_k(x)| = \left| \sum_{i=0}^{k} \Delta_j(x) \right| < \epsilon.$$

so for any $|x| \leq \delta$, we get, by the inequality (14) that

$$\begin{aligned}
|\Delta_{k+1}(x)| &= \left| \int_0^{+\infty} R_k(s_o, \phi_{s_o}(x)).G_k(\phi_{s_o}(x)) ds_o \right| \\
&\leq C \int_0^{+\infty} |R_k(s_o, \phi_{s_o}(x))| \, . \, |\Delta_k(x)| \, ds_o \\
&\leq (C)^{2^0} (C)^{2^1} \cdots (C)^{2^k} \int_0^{+\infty} |R_k(s_o, \phi_{s_o}(x))| \cdot \\
&\Big[\int_0^{+\infty} |R_{k-1}(s_1, \phi_{s_o+s_1}(x))| \ldots \\
&\left(\int_0^{+\infty} |R_o(s_{o+\cdots+} s_k, \phi_{s_o+\cdots+s_k}(x))| \, |G_o(\phi_{s_o+\cdots+s_k}(x)) ds_k| \right)^2 \\
&\ldots ds_1 \Big]^2 ds_o. \qquad (16)
\end{aligned}$$

Taking into account of (11) and (15) we obtain

$$
\begin{aligned}
&|R_o(s_{o+.\cdots+}s_k, \phi_{s_o+\cdots+s_k}(x))| \,.\, |G_o(\phi_{s_o+\cdots+s_k}(x))| \\
&\qquad \leq M_p \delta_p^p \exp \int_0^{s_o+\cdots+s_k} (\beta_o(t, \phi_{s_o+\cdots+s_k}(x)) - pc) dt \\
&\qquad\qquad \leq M_p \delta^p \exp \int_0^{s_o+\cdots+s_k} (\beta - pc) dt \\
&\qquad\qquad\qquad \leq M_p \delta^p e^{-\omega(s_o+\cdots+s_k)}. \quad (15)
\end{aligned}
$$

where as it is defined above $\omega = pc - \beta > 1$.

Finally we have

$$|\Delta_{k+1}(x)| \leq (M_p \delta^p)^{2^k} \leq \frac{\epsilon}{2}(\frac{C\epsilon}{2})^{2^k} \quad \text{with} \quad |x| \leq \delta.$$

Since $C\epsilon < 1$, we get

$$|\Delta_{k+1}(x)| \leq \frac{\epsilon}{2^{k+1}} \quad \text{with} \quad |x| \leq \delta.$$

so

$$|u_{k+1}(x)| < \epsilon \quad \text{with} \quad |x| \leq \delta.$$

The series $u(x) = \sum_{i=0}^{\infty} u_i(x)$ converges informally, on the ball $|x| \leq \delta$, and hence it is the solution of the equation $F(x,u) = 0$.

□

Remark 1. *By the same way we have proved that the solution of the linearized equation given by Lemma2 is uniformly convergent in a neighborhood of the origin* 0.

3.1. **Smoothness of solutions.** Now state the following

Theorem 4. *Under the assumptions* $(H_1), (H_2)$ *and* (H_3)*, the function* $u = \sum_{i=1}^{\infty} \Delta_i$ *converges in the* $C^\infty-$*topology in a neighborhood of the origin* 0.

We establish some estimates from which the proof of Theorem4 follows.

3.1.1. *Estimation of* $D_x^k F(\phi_s(x), u_m(\phi_s(x)))$.

Lemma 5. *For any integers* $k, m \geq 1$*, there exist constants* K *and* A *depending on* k *and* $\delta > 0$ *such that*

$$\left| D_x^k F(\phi_s(x), u_m(\phi_s(x))) \right| \leq K e^{-cs} 2^{2k-1} k^{4k+3} A^k \left(\sum_{j=0}^{k} \left| D_y^j \Delta_{m-1}(\phi_s(x)) \right| \right)^2$$

for all $s \geq 0$, $|x| \leq \delta$ *and* $y = \phi_s(x)$.

*Proof.*Since for any $m \geq 1$, $u_m = u_{m-1} + \Delta_{m-1}$, where Δ_{m-1} is assumed to be a solution of the linear equation $D_X - B'_u(x,u)\Delta_{m-1} = -F(x, u_{m-1})$, then $F(x, u_m(x))$ is the integral remainder that is to say

$$\begin{aligned} &F(x, u_m(x)) \\ &= \int_0^1 (1-t) D_u^2 F(x, u_{m-1}(x) + t\Delta_{m-1}(x)) < \Delta_{m-1}(x), \Delta_{m-1}(x) > dt \end{aligned}$$

so we have by taking derivative with respect to x that

$$\begin{aligned} &D_x^k F(x, u_m(x)) \\ &\qquad = \int_0^1 (1-t) \sum_{r=0}^{k} C_k^r D_x^r \left(D_u^2 F(x, u_{m-1}(x) + t\Delta_{m-1}(x)) \right) \\ &\qquad\qquad D_x^{k-r} \langle \Delta_{m-1}(x), \Delta_{m-1}(x) \rangle dt. \quad (16) \end{aligned}$$

Setting $v_{m-1}(x,t) = u_{m-1}(x) + t\Delta_{m-1}(x)$, we get

$$\begin{aligned} D_x^r \left(D_u^2 F(x,\ v_{m-1}(x,t)) \right) &= \sum_{L+N=r} \sum_{Q=1}^{N} D_y^L D_u^{Q+2} F(x, v_m(x,t)) \\ &\sum_{j_1+\dots+j_Q=N} C_{N,J_1\cdots J_Q} D_y^{j_1} v_{m-1}(x,t) \dots D_y^{j_Q} v_{m-1}(x,t). \quad (17) \end{aligned}$$

Now interchanging x by $y = \phi_s(x)$, and differentiating with respect to x we obtain

$$\begin{aligned} &D_x^r \left(D_u^2 F(\phi_s(x),\ v_{m-1}(\phi_s(x), t)) \right) \\ &\qquad = \sum_{L+N=r} \sum_{P=1}^{L} \sum_{Q=1}^{N} D_y^P D_u^{Q+2} F(\phi_s(x), v_{m-1}(\phi_s(x), t)) \\ &\qquad\quad \times \sum_{i_1+\dots+i_P=L} C_{L,i1\cdots i_P} D^{j_1}\phi_s(x) \dots D^{j_Q}\phi_s(x) \\ &\qquad\qquad \times \sum_{j_1+\dots+j_Q=N} C_{N,j_1\cdots j_Q} \\ &\times \left(\sum_{r_1=1}^{j_1} D_y^{r_1} v_{m-1}(\phi_s(x), t) \sum_{l_1+\dots+l_{r_1}=j_1} C_{j_1,l_1\dots l_{r_1}} D^{l_1}\phi_s(x) \dots D^{l_{r_1}}\phi_s(x) \right) \\ &\dots \left(\sum_{r_p=1}^{j_Q} D_y^{r_Q} v_{m-1}(\phi_s(x), t) \sum_{l_1+\dots+l_{r_Q}=j_Q} C_{j_Q,l_1\dots l_{r_Q}} D^{l_1}\phi_s(x) \dots D^{l_{r_Q}}\phi_s(x) \right) \end{aligned}$$

where

$$C_{L,j_1\cdots J_Q} = \frac{L!}{j_1!...j_Q!}.$$

Since ϕ_t has an exponential bound of order e^{-ct} then, by ([5]), so do all its derivatives $D^i\phi_t$, $k \geq 1$, that is

$$\left|D_x^i \phi_t(x)\right| \leq \eta(i).e^{-ct} \tag{18}$$

where $\eta(i)$ is a constant depending on i and x small enough.

For any integer $k \geq 1$, let $A = \sup\{1, \eta(i) : i = 1, ..., k\}$ and

$$C = \sup\Big\{\left|D_y^P D_u^{Q+2} F(x, v(x,\tau))\right| : P + Q = k,\ \|v\|_j^{B_\delta} < \rho, \\ j = 0, 1, 2, \cdots,\ x \in B_\delta \tau \in [0,1]\}$$

where B_δ is the closed ball of center 0 and radius δ.

By (18) we get

$$\begin{aligned} &\left|D_y^r(D_u^2 F(\phi_s(x), v_{m-1}(\phi_s(x),\tau)))\right| \\ &\quad\leq \sum_{L+N=r} \sum_{P=1}^{L} \sum_{Q=1}^{N} \left|D_y^P D_u^{Q+2} F(\phi_s(x), v_{m-1}(\phi_s(x),\tau))\right| \\ &\quad \sum_{i_1+\cdots+i_P=L} C_{L,i1\cdots i_P} \eta(i_1)\ldots\eta(i_P) e^{-Pcs} \sum_{j_1+\cdots+j_Q=N} C_{N,j1\cdots j_Q} \\ &\sum_{r_1=1}^{j_1} \left|D_y^{r_1} v_{m-1}(\phi_s(x),\tau)\right| \sum_{l_1+\cdots+l_{lr_1}=j_1} C_{j_1,l_1\ldots l_{r_1}} \eta(l_1)\ldots\eta(l_{j_1}) e^{-j_1 cs} \ldots \\ &\sum_{r_Q=1}^{j_Q} \left|D_y^{r_Q} v_{m-1}(\phi_s(x),\tau)\right| \sum_{l_1+\cdots+l_{r_Q}=j_Q} C_{j_Q,l_1\ldots l_Q} \eta(l_1)\ldots\eta(l_{j_Q}) e^{-j_Q cs}. \end{aligned} \tag{19}$$

Using

$$\sum_{j_1+\cdots+j_Q=N} C_{N,j1\cdots j_Q} = Q^N$$

we obtain

$$\left|D_y^r\left(D_u^2F(\phi_s(x),\ v_{m-1}(\phi_s(x),\tau))\right)\right| \leq$$
$$Ce^{-cs}\sum_{L+N=r}\sum_{P=1}^{L}\sum_{Q=1}^{N}A^PP^L$$
$$\varepsilon^Q\sum_{j_1+\cdots+j_Q=N}C_{N,j_1\cdots j_Q}A^{j_1+\cdots+j_Q}(j_1)^{J_1+1}\ldots(j_Q)^{J_Q+1}$$
$$\leq \rho' CA^r r^{4r}e^{-cs}. \quad (20)$$

where $\rho' = \sup(1,\rho)$.

By the chain rule we get

$$D_y^\gamma\langle\Delta_{m-1}(y),\Delta_{m-1}(y)\rangle = \sum_{\alpha=0}^{\gamma}C_\gamma^\alpha\langle D_y^\alpha\Delta_{m-1}(y), D_y^{\gamma-\alpha}\Delta_{m-1}(y)\rangle,$$

setting $y=\phi_s(x)$, the above equality writes

$$D_x^l\langle\Delta_{m-1}(\phi_s(x)),\Delta_{m-1}(\phi_s(x))\rangle$$
$$= 2 < \Delta_{m-1}(\phi_s(x)), \sum_{i=1}^{l}D_y^i\Delta_{m-1}(\phi_s(x))\sum_{i_1+\cdots+i_i=l}C_{l,i_1,\ldots,i_i}\,D_x^{i_1}\phi_s(x)\cdots D_x^{i_i}\phi_s(x) >$$
$$+\sum_{q=1}^{l-1}C_l^q\left\langle\sum_{i=1}^{q}D_y^i\Delta_{m-1}(\phi_s(x))\sum_{i_1+\cdots+i_i=q}C_q\sum_{i_1\cdots i}D_x^{i_1}\phi_s(x)\cdots D_x^{i_i}\phi_s(x),\right.$$
$$\left.\sum_{j=1}^{l-q}D_y^j\Delta_{m-1}(\phi_s(x)).\sum_{j_1+\cdots+j_j=l-q}C_{l-q,j_1\ldots j_j}\,D_x^{j_1}\phi_s(x)\ldots D_x^{j_j}\phi_s(x)\right\rangle$$

So, by (18), we get

$$\left|D_y^l\langle\Delta_{m-1}(\phi_s(x)),\Delta_{m-1}(\phi_s(x))\rangle\right| \leq 2\left|\Delta_{m-1}(\phi_s(x))\right|.$$

$$\sum_{i=1}^{l}\left|D_y^i\Delta_{m-1}(\phi_s(x))\right| \sum_{i_1+\cdots+i_i=l} C_{l,i_1\cdots i}\eta(i_1)\ldots\eta(i_i)e^{-ics}+$$

$$\sum_{q=1}^{l-1} C_l^q \sum_{j=1}^{q}\left|D_y^j\Delta_{m-1}(\phi_s(x))\right| \sum_{j_1+\cdots+j_j=q} C_{q,j_1\ldots j_j}\eta(j_1)\ldots\eta(j_j)e^{-jcs}$$

$$\sum_{p=1}^{l-q}\left|D_y^p\Delta_{m-1}(\phi_s(x))\right| \sum_{p_1+\cdots+p_p=l-q} C_{l-q,p_1\ldots p_l}\eta(p_1)\ldots\eta(p_p)e^{-pcs}$$

$$\leq 2A^l l^l e^{-cs}\left|\Delta_{m-1}(\phi_s(x))\right|\sum_{i=1}^{l}\left|D_y^i\Delta_{m-1}(\phi_s(x))\right|+$$

$$A^l e^{-cs}\sum_{q=1}^{l-1} C_l^q q^q(l-q)^{l-q}\sum_{j=1}^{q}\left|D_y^j\Delta_{m-1}(\phi_s(x))\right|\sum_{p=1}^{l-q}\left|D_y^p\Delta_{m-1}(\phi_s(x))\right|$$

$$\leq e^{-cs}(2lA)^l\left(\sum_{j=0}^{l}\left|D_y^j\Delta_{m-1}(\phi_s(x))\right|\right)^2. \quad (21)$$

Now combining the inequalities (20) and (21), we get easily that

$$\left|D_x^k F(\phi_s(x),u_m(\phi_s(x)))\right|$$

$$\leq Ce^{-cs}2^{2k}k^{4k}A^k\left(\sum_{j=0}^{l}\left|D_y^j\Delta_{m-1}(\phi_s(x))\right|\right)^2. \quad (22)$$

□

3.1.2. *Estimation of $D_x^k R_m(t,x)$.*

Lemma 6. *With the same notations as in Lemma 5, we have*

$$\left|D_x^k R_m(t,\phi_t(x))\right|$$

$$\leq C^k(2A)^{k+1}\prod_{i=1}^{k} i^{3i+2}\sum_{j=0}^{k-1} C_{k-1}^j\frac{1}{c^{j+1}}\exp\int_0^t \beta_m(\tau,\phi_t(x))d\tau.$$

*Proof.*From the equation (8) we obtain by derivation

$$D_x^k R'_m(t,x)$$

$$= A_m(t,x).D_x^k R_m(t,x)+\sum_{l=1}^{k} C_k^l D_x^l A_m(t,x)D_x^{k-l}R_m(t,x), \quad (23)$$

where

$$A_m(t,x) = -D_u B(\phi_t^{-1}(x), u_m(\phi_t^{-1}(x))).$$

So $D_x^k R_m(t,x)$ appears as the solution of a non homogeneous matrix linear equation with initial value $D_x^k R_m(0,x)$ =0 and, as a particular solution, is given by

$$D_x^k R_m(t,x) = \int_0^t S(t,s,x) \sum_{l=1}^{k} C_k^l D_x^l (A_m(s,x)) D_x^{k-l} R_m(s,x) ds,$$

where $S(t,s,x)$ stands for the normalized fundamental solution of the differential equation (23).

Following the same calculations as above and taking into account of (11), we get

$$\begin{aligned} &\left| D_x^k R_m(t,\phi_t(x)) \right| \\ &\qquad \le K\left(2A\right)^k k^{3k+2} \int_0^t e^{-c(t-s)} \exp\left(\int_s^t \beta_m(\tau,\phi_t(x)) d\tau \right) \\ &\qquad\qquad \sum_{l=1}^{k} \left| D_x^{k-l} R_m(s,\phi_t(x)) \right| ds. \end{aligned}$$

And by induction we obtain

$$\begin{aligned} &\left| D_x^k R_m(t,\phi_t(x)) \right| \\ &\qquad \le (K)^k \left(2A\right)^{k+1} \prod_{i=1}^{k} i^{3i+2} \sum_{j=0}^{k-1} C_{k-1}^j \frac{1}{c^{j+1}} \exp \int_0^t \beta_m(\tau,\phi_t(x)) d\tau. \end{aligned} \tag{24}$$

□

3.1.3. *Estimate of* $D_x^k \Delta_m(x)$.

Lemma 7. *For any integers $k, m \ge 1$, and large positive integer p, there exist positive constants $M = M_{k,p}$, $\delta = \delta_{k,p}$ (depending on p and k), W depending on k such that*

$$\left| D_x^k \Delta_m(x) \right| \le \left(2^k k^3 M \delta^p W^{1-\frac{1}{2^m}} \right)^{2^m} \simeq \left(2^k k^3 M_{k,p} \delta^p W \right)^{2^m}.$$

*Proof.*Let $0 < \epsilon < 0$. For $m = 0$, we have

$$G_o(\phi_t(x)) = -B(\phi_t(x), 0) + f(\phi_t(x)) = f(\phi_t(x))$$

so

$$\left| D_x^k G_o(\phi_t(x)) \right| \le \sum_{i=1}^{k} \left| D^i f o \phi_t(x) \right| \sum_{j_1+\cdots+j_i=k} C_{k,j_1\ldots j_i \eta(j_1)\ldots\eta(j_i)} e^{-ict} \tag{25}$$

$$\leq A^k k^k \sum_{i=1}^{k} \left| D^i f o\phi_t(x) \right| e^{-ct}.$$

Since f is infinitely flat at origin 0, for any positive integer p there exist constants $M_{i,p} > 0$ and $\delta_{i,p} > 0$ (depending on i and p) such that

$$\left| D^i f o\phi_t(x) \right| \leq M_{i,p} \delta_{i,p}^p e^{-pct}$$

provided that $|x| < \delta_{i,p}$.

Now (25) becomes

$$\left| D_x^k G_o(\phi_t(x)) \right| \leq A^k k^{k+1} M_{k,p} \delta_{k,p}^p e^{-(p+1)ct} \tag{26}$$

with $|x| \leq \delta_{k,p}$. For simplicity, we put $\delta = \delta_{k,p}$ and $M = A^k k^{k+1} M_{k,p}$, we have

$$\left| D^k \Delta_o(x) \right| \leq M\delta^p \int_0^{+\infty} \exp \int_0^s (-pc + \beta_o(\phi_t(x), 0)) dt$$

and we choose p large enough so that

$$pc - \beta_o = \omega > 1.$$

where $\beta_o = \sup\left\{ |\beta_o(\phi_t(x), 0)| :\ |x| \leq \delta_o,\ t \geq 0 \right\}$. Hence

$$\left| D^k \Delta_o(x) \right| \leq M\delta^p$$

with $|x| \leq \delta$. Taking δ such that $M\delta_o^p < \frac{\epsilon}{2}$, we obtain

$$\left| D^k \Delta_o(x) \right| < \frac{\epsilon}{2} \tag{27}$$

Provided that $|x| \leq \delta$.

Suppose that for any integer $m \geq 1$,

$$\left| D^k \Delta_{m-1}(x) \right| < \frac{\epsilon}{2^m}$$

for $|x| < \delta$.

Denote by

$$U(k, A, C) = 2^{2k-1} k^{4k+3} A^k C$$

and

$$W(k, A, c, C) = C^k (2A)^{k+1} \prod_{i=1}^{k} i^{3i+2} \sum_{j=0}^{k-1} C_{k-1}^j \frac{1}{c^{j+1}}.$$

From (23) and (25) we obtain

$$\begin{aligned}&\left|D_x^k\Delta_m(x)\right|\\&\quad\leq\int_0^\infty\sum_{l=1}^k C_k^l U(l,K,A)U(k-l,K,A)\left(\sum_{j=0}^n\left|D_x^j\Delta_{m-1}(\phi_t(x))\right|\right)^2\\&\qquad\qquad\qquad\qquad\qquad\qquad\exp\int_0^t\beta_m(\tau,\phi_t(x))d\tau.\end{aligned}$$

and by induction, we get

$$\begin{aligned}\left|D_x^k\Delta_m(x)\right|&\leq 2^{(2^m-1)k}k^{3(2^m-1)}W^{2^m-1}\left(M\delta^p\right)^{2^m}\\&\leq\left(2^kk^3W'M\delta^p\right)^{2^m}\end{aligned}$$

with $W'=\max(1,W)$.

We choose δ small enough so that

$$2^kk^3W'M\delta^p<\frac{\epsilon}{2}$$

then

$$\left|D_x^k\Delta_m(x)\right|<\frac{\epsilon}{2^m}$$

and

$$\sum_{m=0}^{\infty}\left|D_x^k\Delta_m(x)\right|<\varepsilon.$$

□

4. Uniqueness theorems

First, we give a uniqueness theorem for the linearized equation 6.

Let $A(x,u)=D_uB(x,u)$ the solutions of the equation (6) are unique if the equation

$$(D_X-A(x,u))\Delta=0,\Delta(0)=0\tag{28}$$

has only trivial solution $\Delta(x)=0$ in neighborhood of the origin. Let $S(t,x)$ be the normalized fundamental matrix of the auxiliary equation

$$\Delta'=A(\phi_t(x),u(\phi_t(x)))\Delta\tag{29}$$

Setting $\phi_t(x)$ as argument, (29) yields to

$$(\Delta(\phi_t(x)))'=A(\phi_t(x),u(\phi_t(x)))\Delta(\phi_t(x))\tag{30}$$

So if a solution $\Delta(x)$ satisfies 29 then $\Delta(\phi_t(x))$ satisfies 13 and we get

$$\Delta(\phi_t(x))=S(t,x)\Delta(x),\quad t\geq 0$$

Theorem 8. *Suppose that the flow $\phi_t(x)$ generated by X is quasi-asymptotically stable and either integral*

$$\int_0^\infty \alpha(s,x)ds$$

or

$$int_0^\infty |A(\phi_t(x), u(\phi_t(x)))| \, dt$$

converges for x from a neighborhood of the origin and fixed function u. Then every solution of (13) is locally trivial.

*Proof.*There exists $\eta > 0$ such that if $|x| \leq \eta$ then $u(\phi_t(x)) \to 0$ as $t \to \infty$. From (11) we have

$$|u(\phi_t(x))| \geq |u(x)| \exp \int_0^t \alpha(s,x)ds$$

Letting $t \to \infty$ we get $u(x) = 0$ for sufficiently small x.□

Theorem 9. *Suppose that $\phi_t(x)$ is bounded as $t \to \infty$ and x is small. If for such x*

$$\sup_{t>0} \int_0^t \alpha(s,x)ds = +\infty$$

then $\Delta(x) = 0$ for every solution of 13.

Now we are in position to establish a uniqueness theorem for the original equation, let , for any subset compact subset $K \subset R^n$, $\|.\|_o^K$ stand for the semi-norm on the space $C(n,s)$, the space of germs of C^∞-maps from R^n into R^s, given by $\|u\|_o^K = \sup_K |u(x)|$.

Lemma 10. *Let $u_1, u_2 \in C(n,s)$; there exist constant $\delta > 0$ and $C > 0$ such that if $\|u_2\|_o^B \leq \delta$ and $\|u_1\|_o^B \leq \delta$, then*

$$\|u_2 - u_1\|_o^B \leq C \, \|F(.,u_1) - F(.,u_2)\|_o^B$$

where $F(.,u)(x) = F(x,u(x))$ and B is a small closed ball centered at the origin $0 \in R^n$.

Proof. We use Taylor's formula with integral remainder

$$F(x,u_2) = F(x,u_1) + D_u F(x,u_1)(u_2 - u_1) \\ + \int_0^1 (1-s) D_u^2 F(u_1 + s(u_2 - u_1))(u_2 - u_1)^2 ds$$

By the Theorems 8 and 9, $D_uF(x,u_1) = D_X - A(x,u_1)$ is invertible on a sufficiently small ball neighborhood of the origin 0 in $C(n,s)$; let $VF(x,u_1)$ be its inverse. So

$$u_2 - u_1 =$$
$$VF(x,u_1)\left\{F(x,u_2) - F(x,u_1) - \int_0^1 (1-s)D_u^2F(u_1 + s(u_2-u_1))^2 ds\right\}$$

and

$$\begin{aligned}\|u_2 - u_1\|_o^B & \\ &\leq \|VF(x,u_1)\|_o^B \{\|F(.,u_2) - F(.,u_1)\| + \\ &\quad \sup_{s\in[0,1]} \left\|D_u^2F(u_1 + s(u_2-u_1))\right\|_o^B .(\|u_2-u_1\|_o^B)^2\right\}.\end{aligned}$$

On the other hand, $VF(x,u_1)$ is bounded on a sufficiently small neighborhood W of the origin $(0,0)$ in $R^n \times C(n,s)$; consequently, if the diameter of W is less than δ, there exists a constant C' such that

$$\|u_2-u_1\|_o^B (1-2\delta C') \leq C' \|F(.,u_2) - F(.,u_1)\|_o^B$$

provided that $(x,u_1),(x,u_2) \in W$.

We choose $\delta < \frac{1}{2C'}$ and take the constant $C = \frac{C'}{1-2\delta C'}$.$\square$

As a consequence of Lemma10, we have the following uniqueness theorem

Theorem 11. *Under the assumptions (H_1), (H_2) and (H_3) the nonlinear differential equation $D_Xu - B(x,u) = f$ has a unique local solution.*

5. Application to dynamic

Theorem 12. *Let $X = \sum_{i=1}^n(\lambda_i x_i + f_i(x)\frac{\partial}{\partial x_i})$ be a vector field where the functions f_i are infinitely flat at the origin 0 and $\lambda_i < 0$. Then there exists a local diffeomorphism tangential to the identity $x_i = y_i + \varphi_i(y)$ which transforms X in its linear part $\sum_{i=1}^n \lambda_i y_i$.*

*Proof.*This result is not new, it is a special case of the Sternberg linearization. If $\phi(y) = y + \varphi(y)$, ϕ satisfy

$$X = \phi_* X_o,$$

where

$$\phi_* X_o = D\phi o X_o o \phi^{-1}$$

which writes in coordinates

$$\sum_{i=1}^{n}(\lambda_i y_i \frac{\partial \varphi^j}{\partial y_i} - \lambda_j \varphi_j(y)) = f_j(y + \varphi(y)), \quad j = 1, \cdots, n. \tag{13}$$

Putting

$$B_j(y, \varphi) = \lambda_j \varphi_j(y) + f_j(y + \varphi(y)) - f_j(y) \qquad j = 1, \cdots, n$$

we get from (32) that

$$\sum_{i=1}^{n} \lambda_i y_i \frac{\partial \varphi^j}{\partial y_i} - B_j(y, \varphi) = f_j(y) \qquad j = 1, \cdots, n$$

or in a short form

$$\sum_{i=1}^{n} \lambda_i y_i \frac{\partial \varphi}{\partial y_i} - B(y, \varphi) = f(y)$$

where $B(y, \varphi) = (B_1(y, \varphi), \cdots, B_n(y, \varphi)) f(y) = (f_1(y), \cdots, f_n(y))$. The above equation fulfils manifestly the assumption of Theorem4 and we obtain Theorem 12. $\square$

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