Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
Vol. 17, 2005, 43 – 46

R.N. Gumerov

ON THE EXISTENCE OF MEANS ON SOLENOIDS

(submitted by V. Lychagin)

Abstract. A mean on a topological space is a continuous idempotent and symmetric operation on it. A proof of a criterion for the existence of means on solenoids is given.

An n-mean, $n \geq 2$, on a topological space X is a continuous mapping μ from the Cartesian product of n copies of X into X such that $\mu(x, x, \ldots, x) = x$ and $\mu(x_1, x_2, \ldots, x_n) = \mu(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for all $x, x_1, x_2, \ldots, x_n \in X$ and any permutation σ of the set $\{1, 2, \ldots, n\}$.

There is a large literature concerning the problem on the existence of means on topological spaces (see, e.g., [1]—[5] and the references cited there). G. Aumann [2] showed that the circle does not admit an n-mean for any n. J. Keesling [4] gave necessary and sufficient conditions for the existence of n-means on compact connected Abelian topological groups. In particular, a compact connected Abelian group G admits an n-mean if and only if the one-dimensional Čech cohomology group $H^1(G, \mathbb{Z})$ with the integers $\mathbb{Z}$ as the coefficient group, or equivalently the Pontryagin dual of G, is n-divisible (see [4, Theorem 1.1]). We recall that an additive Abelian group H is said to be *n-divisible* provided that, for each element $h \in H$, there exists an element $g \in H$ such that $h = ng$.

Let $N = (n_1, n_2, \ldots)$ be a sequence of integers that are greater than 1. *The solenoid* Σ_N is defined as the inverse limit of the inverse sequence

---

2000 Mathematical Subject Classification. 54H11, 54F15.

Key words and phrases. n-mean, solenoid.

$\{X_k, f_k^{k+1}, \mathbb{N}\}$, where $\mathbb{N}$ is the set of all positive integers and for each $k \in \mathbb{N}$ the factor space X_k is the unit circle $\mathbb{S}^1$ in the complex plane and the bonding mapping f_k^{k+1} is the n_k-fold covering mapping $\mathbb{S}^1 \longrightarrow \mathbb{S}^1 : z \mapsto z^{n_k}$. The solenoid is a compact connected Abelian group under the coordinatewise multiplication with the identity $e = (1, 1, \ldots)$. In case $N = (2, 2, \ldots)$ the solenoid Σ_N is said to be *dyadic*.

The exponential covering mapping from the reals $\mathbb{R}$ onto $\mathbb{S}^1$ induces a one-to-one continuous homomorphism $\theta : \mathbb{R} \to \nprec_N$ defined by

$$\theta(\alpha) = (\exp(i2\pi\alpha), \exp(i\frac{2\pi\alpha}{n_1}), \exp(i\frac{2\pi\alpha}{n_1 n_2}), \ldots), \quad \alpha \in \mathbb{R}, i^2 = -1,$$

which is not a topological embedding. The image of the homomorphism θ is the arc component of Σ_N containing the identity e (see [6, § 5]).

The Pontryagin dual of Σ_N is isomorphic (see [7, (25.3)]) to the discrete additive group of rationals $\mathbb{Q}_N$ generated by the set

$$\left\{\frac{1}{n_1}, \frac{1}{n_1 n_2}, \cdots, \frac{1}{n_1 n_2 \ldots n_k}, \cdots \mid k \in \mathbb{N}\right\}.$$

It is easy to see that the group $\mathbb{Q}_N$ is n-divisible if and only if each prime factor of n divides infinitely many terms of the sequence N.

Thus, by the above-mentioned facts, one has the criterion for the existence of n-means on solenoids:

Theorem 1. *The solenoid Σ_N admits an n-mean if and only if each prime factor of n divides infinitely many terms of the sequence N.*

A simple proof of Theorem 1 for 2-means was given by P. Krupski in [5]. It is based on a method of B. Eckmann [3] and the following theorem of W. Scheffer ([8, Corollary 2]).

Theorem 2. *Let G be a compact connected topological group, and H be a locally compact Abelian topological group. Then every continuous mapping from G into H that preserves the unit element is homotopic to exactly one continuous homomorphism from G into H, and the homotopy can be chosen to preserve the identity.*

In this note a similar proof of Theorem 1 is given for arbitrary n-means.

Proof of Theorem 1. Necessity. Suppose that the solenoid Σ_N admits an n-mean and $p \geq 2$ is a prime factor of the integer n. It follows immediately from the definition of an n-mean that the solenoid Σ_N admits a p-mean as well. We denote by μ a p-mean on Σ_N.

According to Theorem 2, there exists a continuous homomorphism

$$\phi : \Sigma_N \times \Sigma_N \times \ldots \times \Sigma_N \to \Sigma_N,$$

which is homotopic to μ.

Choose any $g \in \Sigma_N$. The points g and $\phi(g, g, \ldots, g)$ lie in the same arc component Γ of Σ_N. It follows then from the equality

$$\phi(g, g, \ldots, g) = \phi(g, e, \ldots, e)\phi(e, g, e, \ldots, e)\phi(e, \ldots, e, g)$$

that $(\mu(g, e, \ldots, e))^p \in \Gamma$. Consequently, we have $\phi(g^p, e, \ldots, e) \in \Gamma$. This implies that $\mu(g^p, e, \ldots, e) \in \Gamma$. In other words, for each $g \in \Sigma_N$, both points g and $\mu(g^p, e, \ldots, e)$ are contained in the same arc component of the space Σ_N.

To obtain a contradiction we suppose now that there exists an integer $k \in \mathbb{N}$ such that n_j is not a multiple of p for each $j \geq k$. Since Σ_N and $\Sigma_{(n_k, n_{k+1}, \ldots)}$ are homeomorphic we can assume that for every $j \in \mathbb{N}$ the prime p is not a divisor of the integer n_j.

Now we shall construct an element $g \in \Sigma_N$ such that $g^p = e$ (cf. [9, the proof of Proposition 4]). Let $\sqrt[p]{1}$ denote the multiplicative cyclic group of all values of the p-th root of 1 generated by $\xi = \exp(i\frac{2\pi}{p})$. For each term n_j of the sequence N we consider the homomorphism $\psi_{n_j} : \sqrt[p]{1} \to \sqrt[p]{1}$ defined by $\psi_{n_j}(z) = z^{n_j}, z \in \sqrt[p]{1}$. Since the integers p and n_j are relatively prime the mapping ψ_{n_j} is a bijection. Denote by $\phi_{n_j} : \sqrt[p]{1} \to \sqrt[p]{1}$ the inverse of ψ_{n_j}. We have $(\phi_{n_j}(z))^{n_j} = z$ for each $z \in \sqrt[p]{1}$. So that the sequence

$$g = (\xi, \phi_{n_1}(\xi), \phi_{n_2} \circ \phi_{n_1}(\xi), \ldots)$$

is an element of Σ_N and $g^p = e$. Therefore we get $\mu(g^p, e, \ldots, e) \in \theta(\mathbb{R})$.

On the other hand, it is obvious that the point g does not lie in $\theta(\mathbb{R})$ (see also Remark below). Thus the points g and $\mu(g^p, e, \ldots, e)$ belong to distinct arc components of the space Σ_N. This contradicts the observation made above.

Sufficiency. If each prime factor of an integer n divides infinitely many terms of the sequence N, then one can readily show that the solenoid Σ_N is homeomorphic to Σ_M, where $M = (m_1, m_2, \ldots)$ with $m_j = nk_j, k_j \in \mathbb{N}$, for all $j \in \mathbb{N}$. It is straightforward to check that the mapping

$$\mu : \Sigma_M \times \Sigma_M \times \ldots \times \Sigma_M \to \Sigma_M$$

determined by the formula

$$\mu((z_{11}, z_{12}, \ldots), (z_{21}, z_{22}, \ldots), \ldots, (z_{n1}, z_{n2}, \ldots)) =$$

$$= ((z_{12}z_{22} \ldots z_{n2})^{k_1}, (z_{13}z_{23} \ldots z_{n3})^{k_2}, \ldots)$$

is an n-mean on the solenoid Σ_M. This completes the proof of Theorem 1.

Remark. It is interesting to note that any non-trivial continuous self-homomorphism of the solenoid Σ_N bijectively maps arc components onto arc components (see [10, Proposition 3]).

It is known that the problem on the existence of n-means on compact connected Abelian groups is closely related to the question of existence or nonexistence of finite-sheeted connected coverings (see, e.g., [11]). By *a connected covering* of a topological group we mean a covering mapping from a connected Hausdorff topological space onto a group. Using Theorem 1 and the conditions for the existence and nonexistence of finite-sheeted connected coverings of solenoids (see, e.g., [9, Theorem 2]), one can easily obtain the following theorem.

Theorem 3. *The solenoid Σ_N admits an n-mean if and only if for each prime factor p of n there is no p-fold connected covering of Σ_N.*

References

[1] Kolmogoroff A. N., *Sur la notion de la moyenne,* Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat., (6) 12 (1930), 388–391.

[2] Aumann G., *Über Räume mit Mittelbildungen,* Math. Ann., 119 (1944), 210–215.

[3] Eckmann B., *Räume mit Mittelbildungen,* Comment. Math. Helv., 28 (1954), 329–340.

[4] Keesling J., *The group of homeomorphisms of a solenoid,* Trans. Amer. Math. Soc., 172 (1972), 119–131.

[5] Krupski P., *Means on solenoids,* Proc. Amer. Math. Soc., 131 (2002), 1931–1933.

[6] McCord M. C., *Inverse limit sequences with covering maps,* Trans. Amer. Math. Soc., 114 (1965), 197–209.

[7] Hewitt E., Ross K. A., *Abstract harmonic analysis, Vol. I,* Springer–Verlag, Berlin, 1963.

[8] Scheffer W. A., *Maps between topological groups that are homotopic to homomorphisms,* Proc. Amer. Math. Soc., 33 (1972), 562–567.

[9] Gumerov R. N., *On finite-sheeted covering mappings onto solenoids,* Proc. Amer. Math. Soc. (to appear).

[10] Aarts J. M., Fokkink R. J., *Mappings on the dyadic solenoid,* Comment. Math. Univ. Carolinae, 2003, 44 (4), 697–699.

[11] Gumerov R. N., *Means on compact groups,* Trudy Lobachevskii Centre Math., Vol. 25, 97–98, International Conference, Kazan Math. Soc., 2004 (in Russian).

Department of Mechanics and Mathematics, Kazan State University, Kremlevskaya 18, Kazan, 420008, Russian Federation

E-mail address: renat.gumerov@ksu.ru