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DYNAMICS OF FINITE-MULTIVALUED TRANSFORMATIONS

(submitted by M. Malakhaltsev)

Abstract. We consider a transformation of a normalized measure space such that the image of any point is a finite set. We call such a transformation an m-transformation. In this case the orbit of any point looks like a tree. In the study of m-transformations we are interested in the properties of the trees. An m-transformation generates a stochastic kernel and a new measure. Using these objects, we introduce analogies of some main concept of ergodic theory: ergodicity, Koopman and Frobenius-Perron operators etc. We prove ergodic theorems and consider examples. We also indicate possible applications to fractal geometry and give a generalization of our construction.

1. Main definitions and examples

Throughout the paper $(X, \mathcal{B}, \mu)$ denotes a normalized measure space. Let m be a positive integer.

Definition 1. *We call a multivalued transformation* $S : X \to X$ *an* **m-transformation** *if* $1 \leq |S(x)| \leq m$ *for any* $x \in X$*, where* $|A|$ *is just a number of elements in* A*.*

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Let
$$S^{-1}_{k;l}(B) \equiv \{x \in X : |S(x)| = k, |S(x) \cap B| = l\},$$
where $B \subset X$ and $k, l \in \mathbb{N}$. Note that sets $S^{-1}_{k;l}(B)$ are pairwise disjoint for the fixed B.

Definition 2. *The m-transformation $S : X \to X$ is* **measurable** *if $S^{-1}_{k;l}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $k, l \in \mathbb{N}$.*

Let $K : X \times \mathcal{B} \to \mathbb{R}^+$ be the function
$$K(x, B) \equiv \frac{1}{|S(x)|} \sum_{y \in S(x)} \chi_B(y) \ .$$
For each $x \in X$, $K(x, \cdot) : \mathcal{B} \to \mathbb{R}^+$ is a normalized measure and for each $B \in \mathcal{B}$, $K(\cdot, B) : X \to \mathbb{R}^+$ is measurable by the Definition 2. Therefore K is a **stochastic kernel** that describes the m-transformation S. We will use K as a tool for proving some results. Fore a more complete study of stochastic kernels the reader is referred to [5].

For any measurable m-transformation S we define a new measure $S\mu$ on $(X, \mathcal{B}, \mu)$
$$S\mu(B) \equiv \int\limits_X K(x, B) \ \mathrm{d}\mu = \sum_{k=1}^{m} \sum_{l=1}^{k} \frac{l}{k} \ \mu(S^{-1}_{k;l}(B)).$$

Definition 3. *We say the measurable m-transformation $S : X \to X$* **preserves measure** *μ or that μ is S-***invariant** *if $S\mu = \mu$.*

Definition 4. *Let the m-transformation $S : X \to X$ preserve measure μ. The quadruple $(X, \mathcal{B}, \mu, S)$ is called an* **m-dynamical system**.

The next proposition gives a number of examples of m-dynamical systems.

Proposition 1. *Let $\{S_i\}_1^k$ be a finite collection of the μ-preserving m_i-transformations of $(X, \mathcal{B}, \mu)$ and let $S(x) = \bigcup_{i=1}^k S_i(x)$ be measurable. Let K, K_i be the stochastic kernels that generates S, S_i, respectively. If for any $B \in \mathcal{B}$*
$$K(x, B) = \frac{1}{k} \sum_{i=1}^{k} K_i(x, B) \tag{1}$$
for almost all $x \in X$, then S is μ-preserving.

► For any measurable B we have
$$S\mu(B) = \int\limits_X K(x, B) \ \mathrm{d}\mu = \frac{1}{k} \sum_{i=1}^{k} \int\limits_X P_i(x, B) \ \mathrm{d}\mu = \mu(B). ◄$$

In the following examples λ denotes the Lebesgue measure on $[0,1]$.

Example 1. *Let* $S:[0,1]\to[0,1]$ *be defined by* $S(x)=\{x,1-x\}$. *Then* S *is* λ*-preserving.*

Example 2. *Let* $S:[0,1]\to[0,1]$ *be defined by*

$$S(x)=\begin{cases}\{2x,1-2x\}, & x\in[0,\frac{1}{2}]\\ \{2x-1\}, & x\in(\frac{1}{2},1].\end{cases}$$

Then S *is* λ*-preserving.*

The following example show that not every λ-preserving m-transformation is union of λ-preserving transformations.

Example 3. *Let* $S:[0,1]\to[0,1]$ *be defined by*

$$S(x)=\begin{cases}\{\frac{3}{2}x\}\ , & x\in[0,\frac{1}{3})\\ \\ \{\frac{3}{2}x,\frac{3}{2}x-\frac{1}{2}\}\ , & x\in[\frac{1}{3},\frac{2}{3}]\\ \\ \{\frac{3}{2}x-\frac{1}{2}\}\ , & x\in(\frac{2}{3},1]\ .\end{cases}$$

Then S *is* λ*-preserving, but* S *can not be represented as union of* λ*-preserving transformations.*

▶ Assume $S(x)=\cup_{i=1}^{k}S_i(x)$, where S_i are the λ-preserving transformations. Then there are a measurable set $B\subset[\frac{1}{3},\frac{2}{3}]$ of positive measure and transformation S_i (for instance S_1), such that $S_1(B)\subset[0,\frac{1}{2}]$. We have

$$\lambda(S_1^{-1}(S_1(B)))=\lambda(B\cup(B-\frac{1}{3}))=2\lambda(B)\text{ and }\lambda(S_1(B))=\frac{3}{2}\lambda(B)\ .$$

Since S_1 is the λ-preserving transformation, $\lambda(S_1(B))=\lambda(B)=0$. ◀

Example 4. *Let* $S:[0,1]\to[0,1]$ *be defined by*

$$S(x)=\begin{cases}\{2x,1-2x,x\}\ , & x\in[0,\frac{1}{2}]\\ \\ \{2x-1,x\}\ , & x\in(\frac{1}{2},1]\ .\end{cases}$$

Then S *isn't* λ*-preserving.*

▶ For instance,

$$S\lambda([0,\frac{1}{2}])=\frac{2}{3}\lambda([0,\frac{1}{2}])+\frac{1}{2}\lambda([\frac{1}{2},\frac{3}{4}])=\frac{11}{24}\neq\lambda([0,\frac{1}{2}])\ .$$

Nevertheless, we can represent S as the union of the λ-preserving transformations $S_1(x)=x$ and S_2 from Example 2. Of course, (1) does not hold true. ◀

Let $S^{-1}(B) = \{x \in X : S(x) \cap B \neq \emptyset\}$ denote the full preimage of B.

Definition 5. *A measurable m-transformation $S : X \to X$ is said to be* **nonsingular** *if for any $B \in \mathcal{B}$ such that $\mu(B) = 0$, we have $\mu(S^{-1}(B)) = 0$, i.e., $S\mu \ll \mu$.*

2. Recurrence and ergodic theorems

Let $S : X \to X$ be an m-transformation. The n-th iterate of S is denoted by S^n. The **tree** at $x_0 \in X$ is the set $\{x \in X : x \in S^n(x_0)$ for some $n \geq 0\}$. Any sequence $x_0, x_1, x_2, \ldots$ with $x_{n+1} \in S(x_n)$ for all $n \geq 0$ is called the **orbit** of x_0.

In the study of m-dynamical systems, we are interested in properties of the trees. For example, in the recurrence of trees of S, i.e., the property that if the tree in x starts in a specified set, some orbits of x return to that set infinitely many times.

Proposition 2. *Let S be a nonsingular m-transformation on $(X, \mathcal{B}, \mu)$ and let $\mu(A) \leq \mu(S^{-1}(A))$ for any $A \in \mathcal{B}$. If $\mu(B) > 0$, then for almost all $x \in B$ there is an orbit of x that returns infinitely often to B.*

► Let B be a measurable set with $\mu(B) > 0$, and let us define the set A of points that never return to B, i.e., $A = \{x \in B : S^n(x) \cap B = \emptyset$ for all $n \geq 1\} = B \setminus \cup_{n=1}^{\infty} S^{-n}(B)$. Consider a collection of sets

$$A_1 = A \cup S^{-1}(A),\ A_i = A \cup S^{-1}(A_{i-1}),\ i \geq 2\ .$$

It is clear that $A \cap S^{-1}(A_{i-1}) = \emptyset$. Hence

$$\mu(A_i) = \mu(A) + \mu(S^{-1}(A_{i-1})) \geq \mu(A) + \mu(A_{i-1}) \geq \ldots \geq (i+1)\mu(A)\ .$$

Therefore, $\mu(A) = 0$. Since μ is nonsingular, $\mu(S^{-n}(A)) = 0$ for any $n \geq 0$. This gives $\mu(B \setminus \bigcup_n S^{-n}(A)) = \mu(B)$, and for any $x \in B \setminus \bigcup_n S^{-n}(A)$ there exists an orbit of x that returns infinitely often to B. ◄

If S is measure preserving, then we have an analogue of Poincare's Recurrence Theorem.

Corollary 1. *Let S be a measure-preserving m-transformation on $(X, \mathcal{B}, \mu)$. If $\mu(B) > 0$, then for almost all $x \in B$ there is an orbit of x that returns infinitely often to B.*

► Note that $S\mu \ll \mu$ and for any measurable A

$$\mu(A) = S\mu(A) = \sum_{k=1}^{m} \sum_{l=1}^{k} \frac{l}{k}\ \mu(S_{k;l}^{-1}(A)) \leq \mu(S^{-1}(A))\ . \blacktriangleleft$$

Example 1 shows there are orbits that do not return to B. If $B = [0, \frac{1}{2})$, then for any $x \in B$ the orbit $\{x, 1-x, 1-x, \ldots\}$ does not return to B.

For any nonsingular m-transformation S and function f on X we define a new function Uf on X by the equality

$$(Uf)(x) \equiv \int_X f \,\mathrm{d}K(x,\cdot) = \frac{1}{|S(x)|} \sum_{y \in S(x)} f(y) \ .$$

Proposition 3. *If S is a nonsingular m-transformation and f is a real-valued measurable function on X, then*

$$\int_X f \,\mathrm{d}S\mu = \int_X Uf \,\mathrm{d}\mu \ ,$$

in the sense that if one of these integrals exists then so does the other integral and the two integrals are equal.

▶ We first show that Uf is measurable. Given any $\alpha \in \mathbb{R}$ consider an increasing sequence of rational numbers $\alpha_1 < \ldots < \alpha_k$, where $k \le m$ and $\sum_{i=1}^{k} \alpha_i < k\alpha$. Then the set

$$B_{\alpha_1,\ldots,\alpha_k} = S^{-1}(f^{-1}(-\infty,\alpha_1]) \cap S^{-1}(f^{-1}(\alpha_1,\alpha_2]) \cap \ldots \cap S^{-1}(f^{-1}(\alpha_{k-1},\alpha_k])$$

is measurable. Taking the union of $B_{\alpha_1,\ldots,\alpha_k}$ for all possible $k \le m$ and $\alpha_1, \ldots, \alpha_k$, we conclude that the set $\{x : (Uf)(x) < \alpha\}$ is measurable.

When $f = \chi_B$ is the characteristic function of $B \in \mathcal{B}$,

$$\int_X \chi_B \,\mathrm{d}S\mu = S\mu(B)$$

and

$$\begin{aligned} \int_X U\chi_B \,\mathrm{d}\mu &= \int_X \left(\int_X \chi_B \,\mathrm{d}K(x,\cdot) \right) \mathrm{d}\mu \\ &= \int_X K(x,B) \,\mathrm{d}\mu = S\mu(B) \ . \end{aligned} \tag{2}$$

Since U is a linear operator, the formula is also true for simple functions. If f is a nonnegative measurable function, then f is the $S\mu$-pointwise limit of an increasing sequence of simple functions f_i, and the result follows from the fact that Uf is the μ-pointwise limit of the increasing sequence of functions Uf_i and the monotone convergence theorem. Finally, any measurable function f can be written as the difference $f = f^+ - f^-$ of two nonnegative measurable functions, so the formula is true in general.◀

Corollary 2. *Let $S : X \to X$ be a measurable m-transformation on $(X, \mathcal{B}, \mu)$. Then S is μ-preserving if and only if*

$$\int_X f \, \mathrm{d}\mu = \int_X Uf \, \mathrm{d}\mu$$

for any $f \in \mathcal{L}^1$.

▶ This follows from the Proposition above and from (2). ◀

Proposition 4. *Let $S : X \to X$ be a μ-preserving m-transformation on $(X, \mathcal{B}, \mu)$. Then the positive linear operator U is a contraction on $\mathcal{L}^p$ for any $1 \le p \le \infty$.*

▶ It is easily seen that U is a contraction on $\mathcal{L}^\infty$. By the Jensen inequality $|Uf|^p \le U|f|^p$ for any $p \ge 1$ and $f \in \mathcal{L}^p$ (see [5], Chapter 1, Lemma 7.4 for a more general statement). Then

$$\|Uf\|_p^p = \int_X |Uf|^p \, \mathrm{d}\mu \le \int_X U|f|^p \, \mathrm{d}\mu = \int_X |f|^p \, \mathrm{d}\mu = \|f\|_p^p \, . \quad ◀$$

For a function f on X and an m-transformation $S : X \to X$, we define the averages

$$A_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} U^k f, \quad n = 1, 2, \dots \, .$$

From the Birkhoff Ergodic Theorem for Markov operators (see [4] for the details) and from the Proposition above we get the following theorem.

Theorem 1. *Suppose $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a measure preserving m-transformation and $f \in \mathcal{L}^1$. Then there exists a function $f^* \in \mathcal{L}^1$ such that*

$$A_n(f) \to f^*, \mu - a.e.$$

Furthermore, $Uf^* = f^*$ μ-a.e. and $\int_X f^* \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$.

Corollary 3. *Let $1 \le p < \infty$ and let S be a measure preserving m-transformation on $(X, \mathcal{B}, \mu)$. If $f \in \mathcal{L}^p$, then there exists $f^* \in \mathcal{L}^p$ such that $Uf^* = f^*$ μ-a.e. and $\|f^* - A_n(f)\|_p \to 0$ as $n \to \infty$.*

▶ Let us fix $1 \le p \le \infty$ and $f \in \mathcal{L}^p$. Since $\|A_n(f)\|_p \le \|f\|_p$, we have by Fatou's lemma,

$$\int_X |f^*|^p \, \mathrm{d}\mu \le \liminf_{n \to +\infty} \int_X |A_n(f)|^p \, \mathrm{d}\mu \le \int_X |f|^p \, \mathrm{d}\mu \, .$$

Hence, the operator $L : \mathcal{L}^p \to \mathcal{L}^p$ defined by $L(f) = f^*$ is a contraction on $\mathcal{L}^p$. By Theorem 1 $\|f^* - A_n(f)\|_p \to 0$ as $n \to \infty$ for any bounded function $f \in \mathcal{L}^p$. Let $f \in \mathcal{L}^p$ be a function, not necessarily bounded. For any $\varepsilon > 0$ we can find a bounded function $f_B \in \mathcal{L}^p$ such that $\|f - f_B\|_p < \varepsilon$. Then, since L is a contraction on $\mathcal{L}^p$, we have

$$\|f^* - A_n(f)\|_p \leq \|f_B^* - A_n(f_B)\|_p + \|A_n(f - f_B)\|_p + \|(f - f_B)^*\|_p ,$$

which can be made arbitrarily small. ◄

3. Ergodicity

Assume $Uf = f$ for some measurable function f. It is very important to know condition on S under which f is constant.

Definition 6. *We call a nonsingular m-transformation S* **ergodic** *if for any $B \in \mathcal{B}$, such that $B \backslash S^{-1}(B) = B^c \backslash S^{-1}(B^c) = \emptyset$, $\mu(B) = 0$ or $\mu(B^c) = 0$.*

It is obvious that if S is the union of μ-preserving m-transformations (see Proposition 1) one of which is not ergodic, then S is not ergodic.

Theorem 2. *The following three statements are equivalent for any nonsingular m-transformation $S : X \to X$.*

(1) *S is ergodic*
(2) *for any $B \in \mathcal{B}$, such that $\mu(B \backslash S^{-1}(B)) = \mu(B^c \backslash S^{-1}(B^c)) = 0$, $\mu(B) = 0$ or $\mu(B^c) = 0$.*
(3) *for any disjoint sets $B_1, B_2 \in \mathcal{B}$, such that $\mu(B_1 \backslash S^{-1}(B_1)) = \mu(B_2 \backslash S^{-1}(B_2)) = 0$, $\mu(B_1) = 0$ or $\mu(B_2) = 0$.*

► It is evident that (3)⇒(1).

(1)⇒(2) Suppose S is ergodic and $B \in \mathcal{B}$, such that $\mu(B \backslash S^{-1}(B)) = \mu(B^c \backslash S^{-1}(B^c)) = 0$. Let $A_1 = (B \cap S^{-1}(B)) \cup (B^c \backslash S^{-1}(B^c))$, $A_i = A_{i-1} \cap S^{-1}(A_{i-1})$ for $i \geq 2$, and $A = \cap_{i=1}^{\infty} A_i$. We have $A_1 \supset A_2 \supset \dots$ and

$$A_{i-1} \backslash A_i \subset S^{-1}(A_{i-2} \backslash A_{i-1}) \subset \dots \subset S^{-i+2}(A_1 \backslash A_2) \subset S^{-i+1}(B \backslash S^{-1}(B)) .$$

Therefore, $\mu(A \triangle B) = 0$. Let $x \in A$, then there is at least one point in $S(x)$ that belongs to infinite many of A_i. This gives $A \subset S^{-1}(A)$.

Let $C_1 = A^c$, $C_i = C_{i-1} \cap S^{-1}(C_{i-1})$ for $i \geq 2$, and $C = \cap_{i=1}^{\infty} C_i$. We have $C_1 \supset C_2 \supset \dots$ and

$$C_{i-1} \backslash C_i \subset \dots \subset S^{-i+2}(C_1 \backslash C_2) \subset S^{-i+1}(B^c \backslash S^{-1}(B^c)) \cup S^{-i+2}(B \backslash A) .$$

Therefore, $\mu(C \triangle B^c) = 0$. Let $x \in C$, then there is at least one point in $S(x)$ that belongs to infinite many of C_i. This gives $C \subset S^{-1}(C)$. Moreover,

$$C^c = A \cup C_1 \backslash C \subset S^{-1}(A) \cup S^{-1}(C_1 \backslash C) \cup S^{-1}(A) = S^{-1}(C^c) .$$

We conclude from the ergodicity of S that $\mu(B^c) = \mu(C) = 0$ or $\mu(B) = \mu(C^c) = 0$.

(2)⇒(3) Suppose (2) holds true and let $B_1, B_2 \in \mathcal{B}$ be the disjoint sets, such that $\mu(B_1 \backslash S^{-1}(B_1)) = \mu(B_2 \backslash S^{-1}(B_2)) = 0$. Let $C_1 = B_1^c$, $C_i = C_{i-1} \cap S^{-1}(C_{i-1})$ for $i \geq 2$, and $C = \cap_{i=1}^{\infty} C_i$. We have $C_1 \supset C_2 \supset \ldots$ and $\mu(B_2 \backslash C_i) = 0$. Therefore $\mu(C) \geq \mu(B_2)$. Let $x \in C$, then there is at least one point in $S(x)$ that belongs to infinite many of C_i. This gives $C \subset S^{-1}(C)$. Moreover $\mu(C^c \backslash S^{-1}(C^c)) = 0$ and $\mu(C^c) \geq \mu(B_1)$. By assumption $\mu(C) = 0$ or $\mu(C^c) = 0$. This finishes the proof. ◀

Example 5. *We will prove the ergodicity of*

$$S(x) = \begin{cases} \{2x, 1-2x\} \ , & x \in [0, \frac{1}{2}] \\ \\ \{2x-1\} \ , & x \in (\frac{1}{2}, 1] \ . \end{cases}$$

▶ Let

$$B \subset S^{-1}(B) \text{ and } B^c \subset S^{-1}(B^c) \ . \tag{3}$$

Set $A_1 = \{x : \{x, 1-x\} \subset B\}$, $A_2 = \{x : \{x, 1-x\} \subset B^c\}$ and $A_3 = (A_1 \cup A_2)^c$.

Let $x \in A_1$. By (3)

$$\frac{1+x}{2} \in B \ , \ \frac{2-x}{2} \in B \ , \ \frac{1-x}{2} \in B \ , \ \frac{x}{2} \in B \ .$$

Therefore $\bar{S}^{-1}(A_1) \subset A_1$, where $\bar{S}$ is the well known ergodic single-valued transformation $\bar{S}(x) = 2x \pmod 1$, $x \in [0,1]$. By ergodicity of $\bar{S}$, $\lambda(A_1) = 0$ or $\lambda(A_1) = 1$. Similarly, $\lambda(A_2) = 0$ or $\lambda(A_2) = 1$.

Since $\lambda(A_1) = 1$ leads to $\lambda(B^c) = 0$ and $\lambda(A_2) = 1$ leads to $\lambda(B) = 0$, we need only consider

$$\lambda(A_3) = 1 \ . \tag{4}$$

Let $x \in B$. By (3) and (4)

$$\frac{1+x}{2} \in B \ , \ \frac{2-x}{2} \in B^c \text{ a.s.}, \ \frac{1-x}{2} \in B^c \text{ a.s.}, \ \frac{x}{2} \in B \text{ a.s.}$$

Therefore $\lambda(\bar{S}^{-1}(B) \backslash B)$. By ergodicity of $\bar{S}$, $\lambda(B) = 0$ or $\lambda(B) = 1$. ◀

Example 6. *The 2-transformation* $S : [0,1] \to [0,1]$

$$S(x) = \begin{cases} \{\frac{3}{2}x\} \ , & x \in [0, \frac{1}{3}) \\ \\ \{\frac{3}{2}x, \frac{3}{2}x - \frac{1}{2}\} \ , & x \in [\frac{1}{3}, \frac{2}{3}] \\ \\ \{\frac{3}{2}x - \frac{1}{2}\} \ , & x \in (\frac{2}{3}, 1] \ . \end{cases}$$

is not ergodic.

► For instance, $[0, \frac{1}{2}) \subset S^{-1}([0, \frac{1}{2}))$ and $[\frac{1}{2}, 1] \subset S^{-1}([\frac{1}{2}, 1])$. ◄

Proposition 5. *Let S be ergodic. If f is measurable and $(Uf)(x) = f(x)$ a.e., then f is constant a.e.*

► For each $r \in \mathbb{R}$, $E_r = \{x \in X : (Uf)(x) = f(x) > r\}$ is measurable. Then $E_r \subset S^{-1}(E_r)$ and $E_r^c \subset S^{-1}(E_r^c)$, hence E_r has measure 0 or 1. But if f is not constant a.e., there exists an $r \in \mathbb{R}$ such that $0 < \mu(E_r) < 1$. Therefore f must be constant a.e. ◄

Corollary 4. *If a measure preserving m-transformation S is ergodic and $f \in \mathcal{L}^1$, then the limit of the averages $f^* = \int_X f \, \mathrm{d}\mu$ is constant a.e. Thus, if $\mu(B) > 0$, then for almost all $x \in X$ there is a orbit of x that returns infinitely often to B.*

► We conclude from Theorem 1 and from Proposition 5, that $f^* = \int_X f \, \mathrm{d}\mu$. To prove the second statement we consider $f = \chi_B$ and apply Corollary 1. ◄

Corollary 5. *Let measure preserving m-transformation S be ergodic and $\mu(S_{11}^{-1}(X)) < 1$, i.e., the set $\{x \in X : |S(x)| \geq 2\}$ has positive measure. If $\mu(B) > 0$, then for almost all $x \in X$ there are uncountable many orbits of x that return infinitely often to B.*

► We just apply the Corollary above to the sets B and $S_{11}^{-1}(X)^c$. ◄

Corollary 6. *Let S be a measure preserving ergodic m-transformation and $f \in \mathcal{L}^1$ such that $f(x) \geq f(y)(f(x) \leq f(y))$, for any $y \in S(x)$. Then f is constant a.e.*

► We have $Uf \leq f$, hence the limit of averages $f^* \leq f$. By Corollary 4 $f = f^*$ is constant a.e. ◄

4. The Frobenius-Perron operator

Assume that a nonsingular m-transformation $S : X \to X$ on a normalized measure space is given. We define an operator $P : \mathcal{L}^1 \to \mathcal{L}^1$ in two steps.

1. Let $f \in \mathcal{L}^1$ and $f \geq 0$. Write

$$\nu(B) = \int_X f(x) K(x, B) \, \mathrm{d}\mu \ .$$

Then, by the Radon-Nikodym Theorem, there exists a unique element in $\mathcal{L}^1$, which we denoted by Pf, such that

$$\nu(B) = \int_B Pf \, \mathrm{d}\mu \ .$$

2. Now let $f \in \mathcal{L}^1$ be arbitrary, not necessarily nonnegative. Write $f = f^+ - f^-$ and define $Pf = Pf^+ - Pf^-$. From this definition we have

$$\int\limits_B Pf \, \mathrm{d}\mu = \int\limits_X f^+(x)K(x,B) \, \mathrm{d}\mu - \int\limits_X f^-(x)K(x,B) \, \mathrm{d}\mu$$

or, more completely,

$$\int\limits_B Pf \, \mathrm{d}\mu = \int\limits_X f(x)K(x,B) \, \mathrm{d}\mu \, . \tag{5}$$

Definition 7. *If $S : X \to X$ is a nonsingular m-transformation the unique operator $P : \mathcal{L}^1 \to \mathcal{L}^1$ defined by equation (5) is called the* **Frobenius-Perron operator** *corresponding to S.*

It is straightforward to show that P is a positive linear operator and

$$\int\limits_X Pf \, \mathrm{d}\mu = \int\limits_X f \, \mathrm{d}\mu \, .$$

Proposition 6. *If $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$, then $\langle Pf, g\rangle = \langle f, Ug\rangle$, i.e.,*

$$\int\limits_X (Pf) \cdot g \, \mathrm{d}\mu = \int\limits_X f \cdot (Ug) \, \mathrm{d}\mu \, . \tag{6}$$

► Let B be a measurable subset of X and $g = \chi_B$. Then the left hand side of (6) is

$$\int\limits_B Pf \, \mathrm{d}\mu = \int\limits_X f(x)K(x,B) \, \mathrm{d}\mu$$

and the right hand side is

$$\int\limits_X f \cdot (U\chi_B) \, \mathrm{d}\mu = \int\limits_X f \cdot \left(\int\limits_X \chi_B \, \mathrm{d}K(x,\cdot) \right) \, \mathrm{d}\mu \; = \int\limits_X f(x)K(x,B) \, \mathrm{d}\mu \, .$$

Hence (6) is verified for characteristic functions. Since the linear combinations of characteristic functions are dense in $\mathcal{L}^\infty$, (6) holds for all $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$. ◄

The following proposition says that a density f_* is a fixed point of P if and only if it is a density of an S-invariant measure ν, absolutely continuous with respect to a measure μ.

Proposition 7. *Let $S : X \to X$ be nonsingular and let $f_* \in \mathcal{L}^1$ be a density function on $(X, \mathcal{B}, \mu)$. Then $Pf_* = f_*$ a.e., if and only if the measure $\nu = f_* \cdot \mu$, defined by $\nu(B) = \int_B f_* \, \mathrm{d}\mu$, is S-invariant.*

▶ Let $B \subset X$ be measurable. Then

$$S\nu(B) = \int_X K(x, B)\, \mathrm{d}\nu = \int_X f_*(x) K(x, B)\, \mathrm{d}\mu = \int_B P f_*\, \mathrm{d}\mu\ .$$

On the other hand

$$\nu(B) = \int_B f_*\, \mathrm{d}\mu\ . \ ◀$$

Proposition 8. *Let $S : X \to X$ be a nonsingular m-transformation and P the associated Frobenius-Perron operator. Assume that an $f \geq 0,\ f \in \mathcal{L}^1$ is given. Then*

$$\operatorname{supp} f \subset S^{-1}(\operatorname{supp} Pf) \ \textit{a.s.}$$

▶ By the definition of the Frobenius-Perron operator, we have $Pf(x) = 0$ a.e. on B implies that $f(x) = 0$ for a.a. $x \in S^{-1}(B)$. Now setting $B = (\operatorname{supp} f)^c$, we have $Pf(x) = 0$ for a.a. $x \in B$ and, consequently, $f(x) = 0$ for a.a. $x \in S^{-1}(B)$, which means that $\operatorname{supp} f \subset (S^{-1}(B))^c$. Since $(S^{-1}(B))^c \subset S^{-1}(B^c)$ a.s., this completes the proof. ◀

Proposition 9. *Let $S : X \to X$ be a nonsingular m-transformation and P the associated Frobenius-Perron operator. If S is ergodic, then there is at most one stationary density f_* of P.*

▶ Assume that S is ergodic and that f_1 and f_2 are different stationary densities of P. Set $g = f_1 - f_2$, so that $Pg = g$. Since P is a Markov operator, g^+ and g^- are both stationary densities of P. By assumption, f_1 and f_2 are not only different but are also densities we have $g^+ \not\equiv 0$ and $g^- \not\equiv 0$. Set

$$B_1 = \operatorname{supp} g^+ \quad \text{and} \quad B_2 = \operatorname{supp} g^-\ .$$

It is evident that B_1 and B_2 are disjoint sets and both have positive measure. By Proposition 8, we have

$$B_1 \subset S^{-1}(B_1) \text{ a.s.} \quad \text{and} \quad B_2 \subset S^{-1}(B_2) \text{ a.s.}$$

But, from Theorem 2 it follows that $\mu(B_1) = 0$ or $\mu(B_2) = 0$. ◀

5. Applications and generalization

We now apply the method of m-transformation to the intersection of two middle-β Cantor sets (see [8] and the references given there).

Let $\alpha \in [\frac{1}{3}, \frac{2}{3}]$ and $\psi_1(x) = \alpha x$, $\psi_1(x) = \alpha x + 1 - \alpha$ be contracting similarity maps on $I = [0, 1]$ endowed with Lebesgue measure λ. There is a unique compact set $C_\alpha \subset I$ which satisfies the set equation

$$C_\alpha = \psi_1(C_\alpha) \cup \psi_2(C_\alpha) \ .$$

It is easily checked that C_α is the middle-β Cantor set for $\beta = 1 - 2\alpha$. Let $x \in I$ and $f(x) = \dim_{\mathrm{H}}(C_\alpha \cap (C_\alpha + x))$ denotes the Hausdorff dimension of the set $C_\alpha \cap (C_\alpha + x)$. Let $B_{ij} = \psi_i(C_\alpha) \cap \psi_j(C_\alpha + x)$, $i, j = 1, 2$. From the construction of C_α it follows that $B_{12} = \emptyset$,

$$\dim_{\mathrm{H}} B_{11} = \dim_{\mathrm{H}} B_{22} = \begin{cases} f(\frac{x}{\alpha}) \ , & 0 \le x \le \alpha \\ \\ 0 \ , & \alpha < x \le 1 \end{cases}$$

and

$$\dim_{\mathrm{H}} B_{21} = \begin{cases} 0 \ , & 0 \le x < 1 - 2\alpha \\ \\ f(-\frac{x}{\alpha} + \frac{1}{\alpha} - 1) \ , & 1 - 2\alpha \le x < 1 - \alpha \\ \\ f(\frac{x}{\alpha} - \frac{1}{\alpha} + 1) \ , & 1 - \alpha \le x \le 1 \ . \end{cases}$$

Since $C_\alpha \cap (C_\alpha + x) = B_{11} \cup B_{21} \cup B_{22}$, we have

$$f(x) = \max\{\dim_{\mathrm{H}} B_{ij} : i, j = 1, 2\} = \max\{f(y) : y \in S(x)\} \ , \qquad (7)$$

where

$$S(x) = \begin{cases} \{\frac{x}{\alpha}\} \ , & 0 \le x < 1 - 2\alpha \\ \\ \{\frac{x}{\alpha}, -\frac{x}{\alpha} + \frac{1}{\alpha} - 1\} \ , & 1 - 2\alpha \le x \le \alpha \\ \\ \{-\frac{x}{\alpha} + \frac{1}{\alpha} - 1\} \ , & \alpha < x \le 1 - \alpha \\ \\ \{\frac{x}{\alpha} - \frac{1}{\alpha} + 1\} \ , & 1 - \alpha < x \le 1 \end{cases}$$

(compare with Examples 2 and 5 under $\alpha = \frac{1}{2}$).

Using Leibniz's rule, we find the Frobenius-Perron operator corresponding to S:

$$(Pf)(x) = \begin{cases} \alpha(f(1 - \alpha + \alpha x) + f(1 - \alpha - \alpha x) + f(\alpha x)), & 0 \le x < \frac{1}{\alpha} - 2 \\ \\ \alpha(f(1 - \alpha + \alpha x) + \frac{1}{2} f(1 - \alpha - \alpha x) + \frac{1}{2} f(\alpha x)), & \frac{1}{\alpha} - 2 \le x \le 1. \end{cases}$$

Assume there exist a stable point f_* of P. Then by Proposition 7 the measure $\mu = f_* \cdot \lambda$ is S-invariant. If in addition $S : (I, \mathcal{B}, \mu) \to (I, \mathcal{B}, \mu)$ is ergodic, then by (7) and Corollary 6 f is constant μ-a.e. The same method works in case of the intersection of two arbitrary self-similar sets.

Using m-transformations, we can develop a new approach to the self-similar sets with overlaps (see [2], [7]). Let $\psi_1, \ldots, \psi_m$ be contracting similarity maps on $\mathbb{R}^n$, and let $X = \cup_{i=1}^m \psi_i(X)$ be an attractor of the iterated function system. Given normalized measure μ on X we consider m-transformation of X

$$S(x) = \bigcup_{\{i : x \in \psi_i(X)\}} \psi_i^{-1}(x) \ .$$

Assume, using the Frobenius-Perron operator corresponding S, we have found S-invariant ergodic measure on X. This measure gives us an interesting information about X. For instance, if the conditions of Corollary 5 hold true, we see that a.a. points of X have uncountable many of addresses (see [3] for details).

From these examples we see that the main problem of the investigation is to find an S-invariant ergodic measure. To decide this problem we propose a following generalization of an m-transformation.

Given m-transformation S on a normalized measure space $(X, \mathcal{B}, \mu)$ we consider a collection of pairs $\{S_i, \alpha_i\}_{i=1}^m$, where $S_i : X \to X$ are the single-valued measurable transformations such that $S(x) = \cup_{i=1}^m S_i(x)$ for any $x \in X$, and $\alpha_i : X \to [0,1]$ are the measurable functions such that $\sum_{i=1}^m \alpha_i(x) = 1$ for any $x \in X$. Let us consider the stochastic kernel

$$K(x, B) = \sum_{i=1}^m \alpha_i(x) \chi_B(S_i(x))$$

and a new measure on X

$$S\mu(B) \equiv \int_X K(x, B) \ \mathrm{d}\mu \ .$$

If we choose S_i and α_i such that $S\mu = \mu$, we can employ the results of this paper to the measure preserving transformation S.

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References

[1] Boyarsky, A. and Gora, P. (1997), *Laws of Chaos. Invariant Measures and Dynamical Systems in one Dimension*, Birkhauser, Boston.

[2] Broomhead, D., Montaldi, J. and Sidorov, N. (2004), *Golden gaskets: variations on the Sierpinski sieve*, Nonlinearity, **17**, 1455 – 1480.

[3] Falconer, K.J. (1990), *Fractal Geometry*, Wiley.

[4] Foguel, S. (1980), *Selected Topics in the Study of Markov Operators*, Carolina Lecture Series, Department of Mathematics, University of North Carolina.

[5] Krengel, V. (1985), *Ergodic Theorems*, Walter de Gruyter, N. York.

[6] Lasota, A. and Mackey, M. (1994), *Chaos, Fractals, and Noise*, Appl. Math. Sci. **97**, Springer-Verlag, New York.

[7] Ngai, S.M. and Wang, Y. (2001), *Hausdorff dimension of self-similar sets with overlaps*, J. Lond. Math. Soc. **63**, 655 – 672.

[8] Peres, Y. and Solomyak, B. (1998), *Self-similar measures and intersections of Cantor sets*, Trans. Amer. Math. Soc. **350**, 4065 – 4087.

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