Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
Vol. 17, 2005, 147 – 210

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LINEAR ODES AND $\mathcal{D}$-MODULES, SOLVING AND DECOMPOSING EQUATIONS USING SYMMETRY METHODS.

(submitted by M. Malakhaltsev)

Abstract. This text investigates homogeneous systems of linear ODEs with smooth coefficients. Associating to an equation a differential module proves that these equations form a monoidal category with respect to the tensor product of modules, and objects in this category include homomorphisms, symmetric and exterior powers as well as dual equations. Viewing symmetries as endomorphisms of the $\mathcal{D}$-modules enables direct application of results from the theory of representations of Lie algebras. In particular we find decomposition and solution methods of equations with semisimple symmetry algebras, as well as solvable symmetry algebras. Sufficient conditions for equations to be solved by algebraic manipulations and quadrature are given, and unlike most previous results, there is no requirement on the symmetry algebras of having dimension equal to the order of the equations, in some cases even a single symmetry is sufficient to solve an equation.

1. Introduction

This text is devoted to the study of linear ordinary differential equations. Main results are found in Sections 9 and 10, where we obtain methods to decompose and solve equations with both solvable and semisimple

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2000 Mathematical Subject Classification. 34A30, 47E05, 47N20.
Key words and phrases. Linear Ordinary Differential Equations, Symmetry algebras, Representation theory, Symmetry Operators.

Lie algebras of symmetries. We prove that for a number of such equations one can obtain solutions through combining algebraic methods and quadrature. Also, there are no requirements on the dimension of a symmetry algebra of an equation being equal to the order of the equation, the ability to solve the problem rather depends on eigenvalues and weights of the representation of the symmetry algebra into the relevant module of endomorphisms. Given the right conditions it may even be sufficient with a single symmetry to solve an equation through eigenvalue decomposition and quadrature.

The starting point is to connect to systems of linear ODEs algebraic objects, differential modules, or $\mathcal{D}$-modules. The equations considered have coefficients in the differential $\mathbb{R}$-algebra $\mathcal{A} = C^\infty(\mathbb{R})$, with derivation being the usual derivative in the variable x, $\delta_{\mathcal{A}} = \frac{d}{dx}$.

The notion of a differential module appears in differential algebra, see e.g. [14, 15], but differential Galois theory and Picard-Vessiot theory deals with modules over differential *fields*, and mainly the study of differential field extensions by solutions of ODEs. That approach may be used to state whether solutions are algebraic with respect to the base field, study solvability of the extensions and address inverse problems in differential Galois theory etc. The approach in this text has geometrical roots, dating back to Sophus Lie, and points in a different direction with respect to applications.

The correspondence

$$\text{System of linear ODEs} \quad \Leftrightarrow \quad \mathcal{D}\text{-module } (E, \delta) \text{ over } (\mathcal{A}, \delta_{\mathcal{A}}),$$

is given by the isomorphism of vector spaces

$$\text{Solution space of the ODE} \quad \Leftrightarrow \quad \ker \delta \ \subset \ E.$$

A straightforward explanation of this correspondence can be found in Section 2, based purely on the definition of $\mathcal{D}$-modules.

Sophus Lie initiated a geometric approach to differential equations, where one uses symmetries of equations to study their properties and to reduce and solve them. Viewing ODEs as submanifolds of an appropriate jet space provides a geometrical framework widely used to study geometric properties and symmetries of equations. From this framework $\mathcal{D}$-modules emerge in the following way. A linear ODE is a linear subbundle in jet space, with a linear connection determined by the Cartan distribution. The $\mathcal{D}$-module corresponding to an equation can be identified with the $\mathcal{A}$-module of sections in the linear bundle, with derivation determined by the lifting of $\frac{d}{dx}$ by the linear connection. This relation is accounted for in **Section 6**. $\mathcal{D}$-modules form a monoidal category, and

Section 3 describes this category and algebraic constructions within the category. That forms the basic algebraic framework used to produce the results of this text, with the key result being **Theorem 3.3**. It is the main tool allowing us to lift properties and results which apply to the vector space $\ker \delta$ to the whole module E. **Section 4** provides procedures to describe a $\mathcal{D}$-module of a given equation, particularly in terms of so called primitive element bases.

In **Section 5** a third view on $\mathcal{D}$-modules is introduced, through linear differential operators. From a practical viewpoint this is an important addition to the theory, giving a generic way to calculate with classes of operators being the elements in the $\mathcal{D}$-modules.

In **Section 7** we investigate equations with Euclidean, symplectic, complex and Hermitian structures. For second order equations we determine classes of equations with such structures. Also, we encounter $\mathcal{D}$-modules with S_2-representations produced by solutions of the Yang - Baxter equation.

Section 8 contains results on symmetries of equations in general. Due to **Theorem 3.3** we establish how to apply results from the theory of representations of Lie algebras into vector spaces to $\mathcal{D}$-modules, and whence to equations with these symmetry algebras. Sections 9 and 10 are based on this observation.

Section 8.2 explains how to incorporate *symmetry operators* in the $\mathcal{D}$-module picture. **Proposition 8.3** determines how a symmetry operator of an equation induces a δ-invariant endomorphism of the corresponding $\mathcal{D}$-module, and **Theorem 8.2** explains how it acts inside $\ker \delta$.

Section 9 deals with equations with solvable symmetry algebras and eigenvalue decompositions of $\mathcal{D}$-modules. We find a sufficient condition for when an equation can be solved by use of a single symmetry, **Theorem 9.1**. **Theorem 9.7** gives a sufficient condition for equations with solvable symmetry algebras to be solvable in terms of quadratures.

In **Section 10** we encounter semisimple symmetry algebras. For a semisimple Lie algebra $\mathfrak{g}$ there is an associated *symmetry ring* $\mathcal{D}(\mathfrak{g})$, an analogue of the the Grothendieck ring of isomorphism classes of finite dimensional vector space representations of $\mathfrak{g}$, and its symmetry ring is generated by a finite number of elements just as its Grothendieck ring is. The generators are isomorphism classes of $\mathcal{D}$-modules with symmetry algebra $\mathfrak{g}$, **Theorem 10.2**.

As a consequence, any $\mathcal{D}$-module with symmetry algebra $\mathfrak{g}$ is polynomial in $\mathcal{D}$-modules isomorphic to the generators, meaning that solutions of the generator differential equations generate all solutions of the original equation.

In particular, any equation with an $\mathfrak{sl}_2$-algebra of symmetries has solution space spanned by powers of solutions of second order model equations of $\mathfrak{sl}_2$, Schrödinger equations, **Theorem 10.6**. Solutions may in many cases be obtained by algebraic methods and quadrature, and an algorithmic approach is outlined.

2. Connecting modules and equations

2.1. **$\mathcal{D}$-modules over a general algebra.** Fix an algebra A over a field K, and a derivation $\delta_A : A \to A$. A pair $(\mathcal{A}, \delta_{\mathcal{A}})$ is called a differential algebra.

Definition 2.1. *A $\mathcal{D}$-module over (A, δ_A) is a pair (E, δ) where E is a module over A and the map*

$$\delta : E \to E$$

is a derivation over δ_A, i. e. it is (i) K-linear, and satisfies a Leibniz rule (ii) with respect to δ_A :

$$(i) \quad \delta(e_1 + e_2) = \delta(e_1) + \delta(e_2) \qquad e_1, e_2 \in E\ , \tag{1}$$

$$(ii) \quad \delta(ae) = \delta_A(a)e + a\delta(e) \qquad a \in A, e \in E\ . \tag{2}$$

Throughout the text we will consider free $\mathcal{D}$-modules, i.e. free modules over an algebra A, that are also $\mathcal{D}$-modules. $\mathcal{D}$-modules (E, δ) over a fixed pair (A, δ_A) constitute the objects of a category which we will denote $\mathcal{C}$, and morphisms are A-homomorphisms of modules that commute with the respective derivations.

Proposition 2.1. *$\mathcal{C}$ is monoidal with respect to the tensor product of modules with the induced derivation δ over δ_A as defined in Definition 3.1.*

Note that (A, δ_A) is a unit object in $\mathcal{C}$.

2.2. **$\mathcal{D}$-modules corresponding to linear ODEs.** Fix $\mathcal{A} = C^\infty(\mathbb{R})$, the $\mathbb{R}$-algebra of smooth functions in one real variable. The pair ($\mathcal{A}$, $\delta_A = \frac{d}{dx}$) is a differential algebra.

Definition 2.2. *By a $\mathcal{D}$-module over $(C^\infty(\mathbb{R}), \frac{d}{dx})$ we mean a pair (E, δ) where E is a free module of rank $= n < \infty$ over $C^\infty(\mathbb{R})$ and the map*

$$\delta : E \to E$$

is a derivation over $\delta_A = \frac{d}{dx}$, i. e. it is (i) $\mathbb{R}$-linear, and satisfies a Leibniz rule (ii) with respect to $\frac{d}{dx}$:

$$\begin{aligned} &(i) \quad \delta(e_1+e_2) \;=\; \delta(e_1)+\delta(e_2)\,, \qquad e_1,e_2\in E\,,\\ &(ii) \qquad \delta(ae) \;=\; \frac{da}{dx}e \,+\, a\delta(e)\;\;, \qquad a\in A, e\in E\,. \end{aligned}$$

From the definition we can immediately deduce a correspondence between a $\mathcal{D}$-module (E,δ) of rank n and a system of linear ordinary differential equations.

Theorem 2.1. *Given a rank $= n$ $\mathcal{D}$-module (E,δ) as in Definition 2.2. Then the $\mathbb{R}$-vector space $E^{\#} = \ker\delta \subset E$ is isomorphic to the solution space of an $n\times n$ system of linear first order differential equations.*

Proof. E is a free module of rank n over $\mathcal{A}$, so there is a basis $\{e_1\,,\;\ldots\;,\,e_n\}$ of E over $\mathcal{A}$. The action of δ on E can be written in matrix form

$$\delta\,(\underline{e}) = A\underline{e}$$

where $A = (a_{ij}(x))$, entries $a_{ij}\in\mathcal{A}$, and $\underline{e} = [e_1,\ldots,e_n]^T$. Considering a general element

$$h = h_1(x)e_1 \,+\, \ldots \,+\, h_n(x)e_n \quad \in E\,, \tag{3}$$

coefficients $h_i(x)\in\mathcal{A}$. Then δ applied to h is

$$\begin{aligned} \delta(h) = \;&\sum_{i=1}^{n}(\,h_i'\,e_i + h_i\,\delta e_i) \;=\; \sum_{i=1}^{n}(\,h_i'\,e_i + h_i\,\sum_{j=1}^{n}a_{ij}e_j)\\ =\;&\sum_{s=1}^{n}(\,h_s' + \sum_{i=1}^{n}a_{is}h_i\,)\,e_s\,. \end{aligned} \tag{4}$$

Thus,

$$\delta\,(h) = 0$$

if and only if the coefficient functions $h_1(x),\ldots,h_n(x)$ satisfy the system

$$\underline{h}' + A^T\underline{h} = 0\,, \tag{5}$$

where $\underline{h} = [h_1(x),\ldots,h_n(x)]^T$.

The map

$$\phi\;:\;\underline{h}\text{ solution of (5)}\;\mapsto\; h \;=\; \sum_{i=1}^{n}h_ie_i \;\in\; \ker\delta$$

is an isomorphism of vector spaces. $\square$

3. The monoidal category of linear ODEs

Proposition 3.1. *$\mathcal{D}$-modules in the sense of Definition 2.2, over the fixed differential algebra*

$$(\mathcal{A} = C^{\infty}(\mathbb{R}) \, , \; \delta_A = \frac{d}{dx})$$

constitute the objects of a category which we will denote $\mathcal{LODE}$, linear ODEs.

For objects (E_1, δ_1), (E_2, δ_2) in $\mathcal{LODE}$, morphisms, $Mor(\,(E_1, \delta_1), (E_2, \delta_2)\,)$, are $\mathcal{A}$-homomorphisms F such that the diagram (6) commutes

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\delta_1"'] & E_2 \arrow[d, "\delta_2"] \\ E_1 \arrow[r, "F"'] & E_2 \end{tikzcd} \tag{6}$$

i.e. $\delta_2 \circ F = F \circ \delta_1$.

Proof. The only category property of composition of $\mathcal{D}$-module morphisms we need to check is that the composition $\psi \circ \phi$ of $\mathcal{A}$-homomorphisms $\phi : E_1 \to E_2$ and $\psi : E_2 \to E_3$ really satisfies the necessary commutator relations. But

$$\begin{aligned} (\psi \circ \phi) \circ \delta_1 &= \psi \circ (\phi \circ \delta_1) = \psi \circ (\delta_2 \circ \phi) \\ &= (\psi \circ \delta_2) \circ \phi = \delta_3 \circ (\psi \circ \phi) \, , \end{aligned} \tag{7}$$

so $\phi \in Mor(\,(E_1, \delta_1), (E_2, \delta_2)\,)$, $\psi \in Mor(\,(E_2, \delta_2), (E_3, \delta_3)\,)$ implies that $\psi \circ \phi \in Mor(\,(E_1, \delta_1), (E_3, \delta_3)\,)$. □

All tensorial constructions of $\mathcal{D}$-modules belong in this category, and each corresponds to an equation. Taking the tensor product of two modules in the category,the resulting $\mathcal{D}$-module with an induced δ is as follows.

Definition 3.1. *The product of two $\mathcal{D}$-modules (E_1, δ_1) and (E_2, δ_2) is the object $(E_1 \otimes_A E_2, \delta)$ where*

$$\delta : E_1 \otimes_A E_2 \longrightarrow E_1 \otimes_A E_2$$

is defined by the requirement that it is a derivation over (A, δ_A) and that

$$\delta(e_1 \otimes e_2) = \delta_1(e_1) \otimes e_2 + e_1 \otimes \delta_2(e_2)$$

on decomposable elements, $e_1 \in E_1, e_2 \in E_2$.

Theorem 3.1. *$\mathcal{LODE}$ is monoidal with respect to the tensor product of modules with the induced δ as defined in Definition 3.1.*
$(\mathcal{A}, \delta_{\mathcal{A}})$ is a unit object in the category.

Proof. We may take basis element $1 \in \mathcal{A}$. For any object (E, δ) in the category we have the following isomorphism $l : \mathcal{A} \otimes E \to E$ defined by

$$1 \otimes e \mapsto e \ , \ \ e \in E \ ,$$

and requiring $\mathcal{A}$-linearity. Note that l is a morphism in the category:

$$\begin{aligned}(\delta_E \circ l)(b \otimes e) &= \delta_E(e) = l\,(b \otimes \delta_E(e)) \\ &= l \circ (\delta_{\mathcal{A}} b \otimes e + b \otimes \delta_E e) \\ &= l \circ \delta_{\mathcal{A} \otimes E}(b \otimes e)\end{aligned} \tag{8}$$

□

Corollary 3.1. *The unit object $(\mathcal{A}, \delta_{\mathcal{A}})$ in the category $\mathcal{LODE}$ corresponds to the first order equation*

$$y' = 0 \tag{9}$$

Proof. Obviously $f \in \ker \delta \subset \mathcal{A}$ if and only if $f' = 0$. □

We also have products of morphisms.

Given $\mathcal{D}$-modules (E_1, δ_{E_1}), (E_2, δ_{E_2}), (F_1, δ_{F_1}) and (F_2, δ_{F_2}) and two module homomorphisms $\phi : E_1 \to F_1$, $\psi : E_2 \to F_2$ we may consider their usual tensor product

$$\phi \otimes \psi \ : \ E_1 \otimes E_2 \ \to \ F_1 \otimes F_2 \, , \tag{10}$$

which on decomposable elements is

$$\phi \otimes \psi \ : \ (e_1 \otimes e_2) \ \mapsto \ \phi(e_1) \otimes \psi(e_2) \, .$$

Proposition 3.2. *Given morphisms $\phi \in Mor(\,(E_1, \delta_{E_1})\,,\,(F_1, \delta_{F_1})\,)$ and $\psi \in Mor(\,(E_2, \delta_{E_2})\,,\,(F_2, \delta_{F_2})\,)$. Then their product is again a morphism, $\phi \otimes \psi \in Mor(\,(E, \delta_E)\,,\,(F, \delta_F)\,)$, where $E = E_1 \otimes_{\mathcal{A}} E_2$, $F = F_1 \otimes_{\mathcal{A}} F_2$ and δ_E, δ_F the induced derivations on the products E, F.*

Proof. We need only check that $\phi \otimes \psi$ satisfies the necessary composition property $(\phi \otimes \psi) \circ \delta_E = \delta_F \circ (\phi \otimes \psi)$. Writing $\delta_E = \delta_{E_1} \otimes I_{E_2} + I_{E_1} \otimes \delta_{E_2}$ and $\delta_F = \delta_{F_1} \otimes I_{F_2} + I_{F_1} \otimes \delta_{F_2}$ in product notation we see that

$$\begin{aligned}(\phi \otimes \psi) \circ \delta_E &= (\phi \otimes \psi) \circ (\delta_{E_1} \otimes I_{E_2} + I_{E_1} \otimes \delta_{E_2}) \\ &= (\phi \circ \delta_{E_1}) \otimes \psi + \phi \otimes (\psi \circ \delta_{E_2}) \\ &= (\delta_{F_1} \circ \phi) \otimes \psi + \phi \otimes (\delta_{F_2} \circ \psi) \\ &= (\delta_{F_1} \otimes I_{F_2} + I_{F_1} \otimes \delta_{F_2}) \circ (\phi \otimes \psi) \\ &= \delta_F \circ (\phi \otimes \psi) \, .\end{aligned} \tag{11}$$

Thus, $(\phi \otimes \psi) \in Mor(\,(E, \delta_E)\,,\,(F, \delta_F)\,)$. □

Some $\mathcal{D}$-modules come with a bit of extra structure, we will encounter both algebras and Lie algebras, the $\mathcal{D}$-module versions are as follows.

Definition 3.2. *A $\mathcal{D}$-algebra (E, δ) is a $\mathcal{D}$-module with a product*

$$m : E \otimes_{\mathcal{A}} E \longrightarrow E$$

such that

$$\begin{array}{ccc} E \otimes E & \xrightarrow{m} & E \\ \delta \downarrow & & \downarrow \delta \\ E \otimes E & \xrightarrow[m]{} & E \end{array} \tag{12}$$

commutes, which satisfies the associativity condition

$$\begin{array}{ccc} E \otimes E \otimes E & \xrightarrow{1 \otimes m} & E \otimes E \\ m\otimes 1 \downarrow & & \downarrow m \\ E \otimes E & \xrightarrow[m]{} & E \end{array} \tag{13}$$

on E.

Note: If (E, δ) is a $\mathcal{D}$-algebra, then $ker \delta \subset E$ is an $\mathbb{R}$-algebra.

Definition 3.3. *A $\mathcal{D}$-Lie-algebra (E, δ) is a $\mathcal{D}$-module with a bracket*

$$[\,\cdot\,,\,\cdot\,] : E \times E \longrightarrow E$$

which is
(1) *$\mathcal{A}$-linear in both arguments, skew-symmetric and satisfies the Jacobi identity, i.e.*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \; = \; 0 \, ,$$

and,
(2) *the bracket operation is δ-invariant, i.e.*

$$\delta[X, Y] \; = \; [\delta X, Y] + [X, \delta Y] \, .$$

Note: If (E, δ) is a $\mathcal{D}$-Lie algebra, then the solution space $\ker \delta$ is a Lie algebra over $\mathbb{R}$ in the usual sense, with respect to the restriction of the bracket to $\ker \delta$.

A natural construction to consider in the category $\mathcal{LODE}$ is homomorphisms of modules.

Proposition 3.3. *Given $\mathcal{D}$-modules (E_1, δ_1) and (E_2, δ_2), then $(\, Hom_A(E_1, E_2), \, \delta \,)$ with*

$$\delta : Hom_A(E_1, E_2) \longrightarrow Hom_A(E_1, E_2)$$

defined by

$$\delta \,:\, F \mapsto \; \delta_2 \circ F - F \circ \delta_1 \,,$$

for $F \in Hom_A(E_1, E_2)$, is an object in the category $\mathcal{LODE}$.

Proof. Given $a \in A$, $F \in Hom_A(E_1, E_2)$ and $e \in E_1$ we see that

$$\begin{aligned} \delta F(ae) &= (\delta_2 \circ F - F \circ \delta_1)(a \cdot e) \\ &= \delta_2(a \cdot F(e)) \; - \; F(\delta_A(a) \cdot e + a \cdot \delta_1(e)) \\ &= \delta_A(a) \cdot F(e) + a\delta_2(F(e)) - \delta_A(a) \cdot F(e) - a \cdot F(\delta_1(e)) \\ &= a \cdot (\delta_2 \circ F - F \circ \delta_1)(e) \; = \; a \cdot \delta F(e) \;. \end{aligned} \tag{14}$$

So, δF is an A-homomorphism. Second, we have that

$$\begin{aligned} \delta(aF) &= \; \delta_2 \circ (a \cdot F) - a \cdot F \circ \delta_1 \\ &= \; \delta_A a \cdot F + a \cdot \delta_2 \circ F - a \cdot F \circ \delta_1 \\ &= \; \delta_A a \cdot F \; + \; a \cdot \delta F \,. \end{aligned} \tag{15}$$

Thus $(\, Hom_A(E_1, E_2)\,,\, \delta\,)$ is an object in $\mathcal{LODE}$. □

We introduce the notation

$$E^{\#}$$

for taking the kernel of δ in E, i. e. $E_1^{\#} = \ker \delta_1 \subset E_1$, and $E_2^{\#} = \ker \delta_2 \subset E_2$. Note that a δ-invariant homomorphism $F \in (\, Hom_A(E_1, E_2)\,)^{\#}$ is a morphism in the category, with

$$\delta_2 \circ F \; = \; F \circ \delta_1 \,. \tag{16}$$

How can we interpret morphisms? If (E_1, δ_1) and (E_2, δ_2) correspond to ODE systems (17) and (18) respectively,

$$\underline{h}' \; + \; A_1 \, \underline{h} \quad = \; 0 \tag{17}$$

$$\underline{u}' \; + \; A_2 \, \underline{u} \quad = \; 0 \,, \tag{18}$$

then the vector spaces $E_1^{\#} \subset E_1$ and $E_2^{\#} \subset E_2$ are isomorphic to the solution spaces of the systems (17) and (18) respectively. By requirement

$$F(\, \ker \delta_1 \,) \; \subset \; \ker \delta_2 \;, \tag{19}$$

so a morphism $F \in (\, Hom_A(E_1, E_2)\,)^{\#}$ *maps solutions of system* (17) *into solutions of system* (18). Thus solutions of the induced homomorphism equation

$$(\, Hom_A(\, E_1 \,,\, E_2 \,)\,,\, \delta \,) \tag{20}$$

give precisely linear maps that transfer solutions of equation E_1 to solutions of E_2. If system (17) is an $n \times n$ system and (18) is an $m \times m$ system, then the homomorphism equation is an $(nm) \times (nm)$ system.

Definition 3.4. *Given a $\mathcal{D}$-module (E,δ) in $\mathcal{LODE}$, the ODE corresponding to the induced $\mathcal{D}$-module of endomorphisms*

$$(\, End_{\mathcal{A}}(E)\,,\, \delta\,) \tag{21}$$

will be denoted the Lie equation, or symmetry equation of (E,δ).

Proposition 3.4. $(End_{\mathcal{A}}(E),\delta)$ *is*

(i) *an associative $\mathcal{D}$-algebra with respect to composition of endomorphisms, and,*
(ii) *a $\mathcal{D}$-Lie algebra with respect to commutators of endomorphisms.*

Proof. In (i) multiplication of endomorphisms ϕ,ψ is defined by

$$m(\phi\otimes\psi) \;=\; \phi\circ\psi\,. \tag{22}$$

Thus

$$\begin{aligned} m\left(\delta(\phi\otimes\psi)\right) &= m\left(\delta\phi\otimes\psi+\phi\otimes\delta\psi\right)\\ &= \delta\phi\circ\psi+\phi\circ\delta\psi\\ &= (\delta\circ\phi-\phi\circ\delta)\circ\psi+\phi\circ(\delta\circ\psi-\psi\circ\delta)\\ &= \delta\circ\phi\circ\psi-\phi\circ\psi\circ\delta \;=\; \delta(m(\phi\otimes\psi))\,, \end{aligned} \tag{23}$$

and whence $\delta\circ m = m\circ\delta$. In (ii) the bracket operation is defined by

$$[\,\phi\,,\,\psi\,] \;=\; m(\phi\otimes\psi-\psi\otimes\phi)\,, \tag{24}$$

so it follows from the previous computation that the bracket operation and δ satisfy

$$\delta[\,\phi\,,\,\psi\,] = [\,\delta\phi\,,\,\psi\,] + [\,\phi\,,\,\delta\psi\,]\,.$$

□

Proposition 3.5. *For any $\mathcal{D}$-module (E,δ) in the category $\mathcal{LODE}$, the dual module*

$$(\,E^* \;=\; Hom_A(E,A)\,,\,\delta\,)$$

is also an object in $\mathcal{LODE}$. The equation (E^*,δ) is the adjoint of equation (E,δ).

Proof. Assume (E,δ) is of rank n with a basis $\{e_1,\ldots,e_n\}$ over $\mathcal{A}$, and that $\delta\underline{e} = A\underline{e}$ on matrix form. Taking the dual basis $\{e_1^*,\ldots,e_n^*\}$ as basis in E^* and applying the definition of the induced δ on basis elements $e_i^*\in E^*$ yields

$$\begin{aligned} (\,\delta e_i^*\,)(e_j) &= \;(\delta\circ e_i^*)(e_j)\;-\;(e_i^*\circ\delta)(e_j)\\ &= \;\delta(\delta_i^j)\;-\;e_i^*(\sum_{s=1}^n a_{js}e_s)\;=\;0-\sum_{s=1}^n a_{js}\delta_i^s\;=\;-a_{ji} \end{aligned} \tag{25}$$

where δ_i^j denotes the Kronecker delta. Whence

$$\delta\,\underline{e}^* \;=\; -A^T\,\underline{e}^* \tag{26}$$

on matrix form and the corresponding system of ODEs is

$$\underline{h}' - A\underline{h} = 0\,, \tag{27}$$

which is the adjoint system. □

Proposition 3.6. *Given a δ-invariant representation of a group G into a $\mathcal{D}$-module (E,δ), i.e. a group representation*

$$r : G \to End_{\mathcal{A}}(E)$$

such that $\delta\circ r(g) = r(g)\circ\delta$ for any $g\in G$. Then the set of G-invariant elements in E,

$$\Sigma_G(E) \;=\; \{e\in E\,|\,r(g)(e)=e,\,\forall g\in G\}\,,$$

is a sub-$\mathcal{D}$-module of E with respect to restriction of δ, i.e. $(\Sigma_G(E),\delta)\in Ob(\mathcal{LODE})$.

Proof. $\Sigma_G(E)$ is obviously a sub-module of E, and we need only prove that $\delta(\Sigma_G(E))\subset\Sigma_G(E)$. Let $e\in\Sigma_G(E)$. Then

$$\delta(e) = \delta(r(g)(e)) = r(g)(\delta(e))$$

for any $g\in G$, thus $\delta e\in\Sigma_G(E)$. □

There are some well-known constructions that are of this type. Both symmetric and anti-symmetric tensors, $S^n(E)$ and $\bigwedge^n(E)$ arise as invariant sub-modules of $E^{\otimes n}$ with respect to representations of the symmetric group $G = S_n$. Regarding representations of S_n we have the following general result. Let E be a finite rank module over $\mathcal{A}$. A homomorphism

$$\tau : E^{\otimes 2}\to E^{\otimes 2} \tag{28}$$

with the condition $\tau^2 = 1$ determines a homomorphism

$$\tau_i \;=\; I_{i-1}\otimes\tau\otimes I_{n-i-1} : E^{\otimes n}\to E^{\otimes n} \tag{29}$$

for $n\le 2$, with τ acting only on the ith and $(i+1)$th copy of E in $E^{\otimes n}$.

Theorem 3.2. *$\tau_1,\dots,\tau_{n-1}$ generate a representation of S_n into $E^{\otimes n}$ iff τ satisfies the so called Yang - Baxter equation :*

$$(\tau\otimes 1)(1\otimes\tau)(\tau\otimes 1) \;=\; (1\otimes\tau)(\tau\otimes 1)(1\otimes\tau) \tag{30}$$

on $E^{\otimes 3}$.

The symmetric power $S^n(E) = \Sigma^n_\tau(E^{\otimes n})$ consists of elements in $E^{\otimes n}$ invariant with respect to the action of S_n given by the *twist-solution* of the Yang-Baxter equation,

$$\tau(f \otimes g) \;=\; g \otimes f\,.$$

Note that for the twist τ

$$\begin{aligned}\delta(\,\tau(f \otimes g)\,) \;&=\; \delta(\,g \otimes f\,) \;=\; \delta g \otimes f + g \otimes \delta f \\ &=\; \tau(\,\delta f \otimes g + f \otimes \delta g\,) \;=\; \tau(\delta(f \otimes g))\,,\end{aligned}$$

so this τ generates a δ-invariant representation of S_2 into $E^{\otimes n}$.

Similarly, $\bigwedge^n(E) = \Sigma^\tau_n(E^{\otimes n}) \subset E^{\otimes n}$ where τ is minus twist,

$$\tau(f \otimes g) = -g \otimes f.$$

Proposition 3.7. *The symmetrization of the kth tensor product of E, $S^k(E) \subset E^{\otimes k}$, is again an object in $\mathcal{LODE}$ together with the restriction of δ on $E^{\otimes k}$ to*

$$\delta : S^k(E) \longrightarrow S^k(E)\,.$$

The restriction of δ acts on decomposable elements of $S^k(E)$ by

$$\delta(\theta_1 \cdot \ldots \cdot \theta_k) = \delta(\theta_1) \cdot \ldots \cdot \theta_k + \ldots + \theta_1 \cdot \ldots \cdot \delta(\theta_k)\,, \tag{31}$$

where $\theta_i \in E$, and $\cdot$ is the symmetric product.

Proposition 3.8. *Any exterior power $\wedge^k(E) \subset E^{\otimes k}$ of a $\mathcal{D}$-module (E, δ) is an object in $\mathcal{LODE}$ with δ being the restriction of δ on $E^{\otimes k}$ to*

$$\delta : \wedge^k(E) \longrightarrow \wedge^k(E)\,.$$

The restriction of δ acts on decomposable k-forms by

$$\delta(\omega_1 \wedge \ldots \wedge \omega_k) = \delta(\omega_1) \wedge \ldots \wedge \omega_k + \ldots + \omega_1 \wedge \ldots \wedge \delta(\omega_k)\,, \tag{32}$$

where $\omega_i \in E$.

In Section 7 we will see examples of non-trivial representations of S_2 for second order equations.

Theorem 3.3 below is a key tool allowing us to move between studying the $\mathcal{D}$-module and the solution space of the corresponding differential equation.

Theorem 3.3. *For any $(E, \delta) \in Ob\,(\mathcal{LODE})$*

$$E \;\cong\; E^{\#} \otimes_{\mathbf{R}} \mathcal{A} \tag{33}$$

by an $\mathcal{D}$-module isomorphism

$$\phi : E^{\#} \otimes_{\mathbf{R}} \mathcal{A} \;\;\to\;\; E \tag{34}$$

defined by $\phi : v_i \otimes 1 \mapsto v_i$ *for any basis* $\{ v_1, \dots, v_n \}$ *of* $E^{\#}$.

Proof. First note that we may rephrase the statement, it is equivalent to the following:
For $(E, \delta) \in Ob\,(\mathcal{LODE})$ any basis of $E^{\#}$ over $\mathbb{R}$ is a basis of E over $\mathcal{A}$. Let E be as above, rank n, with δ-matrix A. Every element of $E^{\#}$ is on the form $h = \sum_{i=1}^{n} h_i(x) e_i$ where $\underline{h}$ solves (5). From the theory of ODEs we know that there exist a fundamental set of solutions of the system (5). Let $\underline{h}_1 = [\, h_{11}(x), \dots, h_{1n}(x)\,]^T, \dots, \underline{h}_n = [\, h_{n1}(x), \dots, h_{nn}(x)\,]^T$ be such a set. Then

$$\{ \gamma_1 = \sum_i h_{1i}(x) e_i\,,\ \dots\ , \gamma_n = \sum_i h_{ni}(x) e_i \}$$

is a basis of $E^{\#}$ over $\mathbb{R}$. The matrix $H = (h_{ij}(x))$ is the Wronskian of the system (5), hence its determinant is non-zero everywhere, and

$$\underline{\gamma} = H \underline{e}$$

constitutes a basis of E over $\mathcal{A}$. Any basis of $E^{\#}$ over $\mathbb{R}$ is on the form as the set $\{\gamma_i\}$ above, hence a basis of E over $\mathcal{A}$.

□

Corollary 3.2. *Given* (E_1, δ_1), (E_2, δ_2) *in* $Ob\,(\mathcal{LODE})$, *then,*

(i) $(E_1 \otimes_{\mathcal{A}} E_2)^{\#} = E_1^{\#} \otimes_{\mathbb{R}} E_2^{\#}$, *and,*
(ii) $(Hom_{\mathcal{A}}(E_1, E_2))^{\#} = Hom_{\mathbb{R}}(E_1^{\#}, E_2^{\#})$.

Proof. This follows directly by combining the theorem above with the definitions of the induced δ-s on $E_1 \otimes E_2$ and $Hom_{\mathcal{A}}(E_1, E_2)$ respectively. □

4. Primitive element bases

For a given equation that we wish to study, we need to be able to identify and work with the corresponding $\mathcal{D}$-module. Considering a system resolved into single equation, a convenient way to describe the corresponding module is to introduce the notion of a primitive element in E.

Definition 4.1. *Let* $(E, \delta) \in Ob(\mathcal{LODE})$, *with* $rank_{\mathcal{A}}(E) = n$. *An element* $e \in E$ *with the property that*

$$\mathcal{B} \;=\; \{e_1 = e,\, e_2 = \delta e,\, e_3 = \delta^2 e,\, \dots\, ,\, e_n = \delta^{n-1} e\} \tag{35}$$

is a basis of E *over* $\mathcal{A}$ *is called a primitive element of* E, *and* $\mathcal{B}$ *a primitive element basis of* E.

In a primitive element basis as $\mathcal{B}$ above the action of δ is completely described by n functions $a_i(x) \in \mathcal{A}$ where

$$\delta^n e \;=\; \sum_{i=1}^{n} a_i(x)\, \delta^{i-1} e \;. \tag{36}$$

In this basis the matrix form of the action of δ becomes

$$\delta \underline{e} = A \underline{e}, \tag{37}$$

where

$$\delta \begin{bmatrix} e \\ \delta e \\ . \\ . \\ . \\ \delta^{n-1} e \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & 1 & 0 \\ 0 & . & . & . & 0 & 1 \\ a_1 & a_2 & . & . & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} e \\ \delta e \\ . \\ . \\ . \\ \delta^{n-1} e \end{bmatrix}$$

The advantage of this approach is that the corresponding equation system

$$\underline{h}' \;+\; A^t \underline{h} \;=\; 0 \tag{38}$$

with $\underline{h} = [\, h_1(x), \ldots, h_n(x)\,]^T$, may then be resolved into a single equation

$$y^{(n)} + (a_n y)^{(n-1)} - (a_{n-1} y)^{(n-2)} + \ldots + (-1)^{n+1} a_1 y = 0 \;. \tag{39}$$

That is, $h_n = y(x)$ and $h_k = H_k(y)$ for $k = 1, ..., n-1$ where

$$H_k(y) = (-1)^{n-k} (\; y^{(n-k)} + \sum_{l=1}^{n-k} (-1)^{l+1} (a_{n+1-l} y)^{(n-k-l)} \;) \;. \tag{40}$$

Written as an operator, with $\partial = \frac{d}{dx}$, H_k is

$$H_k = (-1)^{n-k} \,(\; \partial^{(n-k)} + \sum_{l=1}^{n-k} (-1)^{l+1} \partial^{(n-k-l)} a_{n+1-l} \;) \;. \tag{41}$$

We may sum up as follows.

Proposition 4.1. *Given an equation on the form* (39)*, then the corresponding* $\mathcal{D}$*-module* $(E, \delta) \in Ob(\mathcal{LODE})$ *has a primitive element basis*

$$\{\; e,\, \delta e,\, \delta^2 e,\, \ldots\,,\, \delta^{n-1} e \;\}$$

with $\delta : E \to E$ *determined by the coefficients* a_i *in the equation through the relation*

$$\delta^n e \;=\; \sum_{i=1}^{n} a_i(x)\, \delta^{i-1} e \,.$$

The kernel of δ *is*

$$E^{\#} \;=\; \{\, h_y \;=\; \sum_{i=1}^{n} \; H_k(y)\, \delta^{i-1} e \mid y \text{ solves } (39) \,\} \,, \tag{42}$$

with the operators H_k *on the form* (41).

Remark: A natural concern is whether, starting with an equation on the form

$$y^{(n)} + f_n(x)\, y^{(n-1)} + f_{n-1}(x)\, y^{(n-2)} + \ldots + f_1(x)\, y \;=\; 0\,, \tag{43}$$

it is a problem to write it on the form (39), in order to be able to write down the structure of the corresponding $\mathcal{D}$-module (E, δ).

This is not a problem. To express the coefficients $a_i(x)$ in terms of the f_i-s and their derivatives we need only start with the highest coefficient $a_n = f_n$, and nest our way down to a_1. At each stage a_i is given in terms of derivatives of the functions $f_i, f_{i+1}, \ldots, f_n$, and the formulas are recovered by performing the derivations in the expression (39) and collect terms of the same degree of derivatives of y and compare with the form (43). The equations are on the form

$$\begin{aligned} f_n \;&=\; a_n \;, \\ f_{n-1} \;&= a_n' - a_{n-1} \;, \\ f_{n-2} \;&= \binom{n-1}{1} a_n'' - a_{n-1}' + a_{n-2} \;, \\ &\vdots \\ f_1 = \;&= a_n^{(n-2)} - a_{n-1}^{(n-3)} + \ldots + (-1)^{n-1} a_1 \,. \end{aligned}$$

Knowing $E^{\#}$, i.e. knowing solutions of the corresponding equation, means that we can produce solutions of equations corresponding to such $\mathcal{D}$-modules as

$$E^{\otimes n}\,, \;\; S^k(E)\,, \;\; \wedge^l(E)\;,$$

since $(E^{\otimes n})^{\#} \;=\; (E^{\#})^{\otimes n}$, $S^k(E)^{\#} \;=\; S^k(E^{\#})$ and $\bigwedge^l(E)^{\#} \;=\; \bigwedge^l(E^{\#})$ are completely described when $E^{\#}$ is described. We may use primitive element bases to precisely describe solutions of symmetric powers of second order equations. Let (E, δ) correspond to an equation

$$y'' + (a_2 y)' - a_1 y = 0 \tag{44}$$

with primitive element basis $\{e_1, e_2\}$. Let $\{e_1^*, e_2^*\}$ denote the dual basis of (E^*, δ). This basis generates a basis of the module $S^k(E^*)$

$$\{\; \alpha_l \;=\; (e_1^*)^{k-l+1} \cdot (e_2^*)^{l-1} \;\;,\; l = 1, \ldots, k+1 \,\}\,. \tag{45}$$

To find the equation $S^k(E^*)$ for some k simply apply δ to a general element

$$\theta \;=\; \sum_{l=1}^{k+1} g_l(x)\,\alpha_l \tag{46}$$

in $S^k(E^*)$. Recall that $\delta e_1^* = -a_1 e_1^*$, and $\delta e_2^* = -e_1^* - a_2 e_1^*$. Thus

$$\delta\,:\,\alpha_l \;\mapsto\; -a_1(k-l+1)\,\alpha_{l+1} \;-a_2(l-1)\,\alpha_l - \;(l-1)\,\alpha_{l-1}\,. \tag{47}$$

Setting

$$\delta\theta = 0$$

and collecting basis terms α_s yields a system of $k+1$ equations,

$$g_s' \,-\, a_1(k-s+2)g_{s-1} \,-a_2(s-1)g_s \,-\, s\,g_{s+1} \;=\;, \tag{48}$$

for $s = 1,\ldots,k+1$. This system resolves into a single equation in $g_1 = y(x)$. We may conclude the following about $S^k(E^*)^\#$.

Proposition 4.2. *Let (E,δ) be the $\mathcal{D}$-module corresponding to an arbitrary second order equation (44). For each $k \geq 1$ the kernel $S^k(E^*)^\#$ consists of elements*

$$\theta_y \;=\; y\alpha_1 \,+\, y'\alpha_2 \,+\, \sum_{l=3}^{k+1} g_l(y)\,\alpha_l \;\;, \tag{49}$$

where

$$g_l = \frac{1}{l-1}\left[\,g_{l-1}' \,-\, a_2(l-2)g_{l-1} \,-\, a_1(k-l+3)g_{l-2}\,\right] \tag{50}$$

for $l = 2,..,k+1$ and $g_1 = y$ solves the $S^k(E^*)$ equation, i.e. the equation in y we obtain from setting

$$\delta\theta_y \;=\; 0$$

for θ_y on the form (49), with g_l-s expressed in derivatives of y.

A list of symmetric powers of second order equations is easily produced, and particular hierarchies of this sort will be investigated in Section 10.2. We may immediately deduce the following result concerning solutions of such an hierarchy of equations.

Theorem 4.1. *Given a set of fundamental solutions $\{u,v\}$ of a second order equation corresponding to a $\mathcal{D}$-module (M,δ). Then*

$$\{\,u^k, u^{k-1}v,\ldots,uv^{k-1},v^k\,\} \;=\; \{\,u^i\,v^{k-i}\,\}_{i=0}^{k}$$

is a fundamental set of solutions of the equation corresponding to the $\mathcal{D}$-module $(S^k(M),\delta)$ for any $k \geq 2$.

Proof. We may freely choose (E,δ) such that $(M,\delta) = (E^*,\delta)$ as in Proposition 4.2. Given the solutions $\{u,v\}$ we know that $\{\theta_u,\theta_v\}$ span $(E^*)^\#$, thus

$$\{\ \theta_u^k\,,\ \theta_u^{k-1}\cdot\theta_v\,,\ \ldots\,,\ \theta_u\cdot\theta_v^{k-1}\,,\ \theta_v^k\ \}$$

span $S^k(E^*)^\#$ over $\mathbb{R}$. Also, $S^k(E^*)^\#$, is closed with respect to the symmetric product, and we need only collect the $\alpha_1 = (e_1^*)^k$ term in products $\theta_u^{k-i}\cdot\theta_v^i$ to state that

$$\theta_u^{k-i}\cdot\theta_v^i \ = \ \theta_{u^{k-i}v^i}\,, \tag{51}$$

hence, $\{\,u^i\,v^{k-i}\,\}_{i=0}^k$ span the solution space of $S^k(E^*)$. □

Note: The kth symmetric power of an equation may be defined as the equation whose fundamental solutions are spanned by precisely degree k monomials in fundamental solutions of the base equation. Theorem 4.1 connects this to the $\mathcal{D}$-module picture.

5. Differential operator view on $\mathcal{D}$-modules.

There is a third way to approach $\mathcal{D}$-modules corresponding to linear ODEs, introducing differential operators, practical for calculations with symmetries. Let $\mathcal{K}$ be the ring of linear differential operators over $\mathbb{R}$. An operator

$$P = \partial^k + c_k(x)\partial^{k-1} + \ldots + c_1(x) \tag{52}$$

where $\partial = \frac{d}{dx}$ defines a $\mathcal{D}$-module $(E_P,\delta) \in Ob(\mathcal{LODE})$ with

$$E_P \ \stackrel{\text{def}}{=}\ \mathcal{K}/(\mathcal{K}\circ P)\,, \tag{53}$$

and

$$\delta : E_P \longrightarrow E_P \quad \text{defined by} \ \ \delta\ :\ [X] \mapsto [\partial\circ X]\,. \tag{54}$$

Obviously this operation is well defined with respect to choice of representative $X\in\mathcal{K}$ modulo $(\mathcal{K}\circ P)$, and it is a derivation over ∂. For P as above

$$\{\ e_1 = e = [1],\ e_2 = \delta e = [\partial],\ \ldots\ , e_k = \delta^{k-1}e = [\partial^{k-1}]\ \} \tag{55}$$

is a primitive element basis of E_P over $\mathcal{A}$. In E_P

$$[\partial^k] \equiv -c_1 e - \ \ldots\ - c_k\delta^{k-1}e\,, \tag{56}$$

thus

$$\ker\delta \ \cong\ \ker P^t$$

To return to the situation in Section 4, considering an equation

$$L(y) \ = \ (\ \partial^k + c_k(x)\partial^{k-1} + \ldots + c_1(x)\ )(y) \ = \ 0 \tag{57}$$

we get the corresponding $\mathcal{D}$-module (E, δ) from taking

$$E = E_{L^t} \tag{58}$$

so that

$$\ker \delta \cong \ker L . \tag{59}$$

6. Geometric image of ODEs in Jet space

Consider a vector bundle $B \xrightarrow{\beta} \mathbb{R}$ of rank m with its $\mathcal{A} = C^\infty(\mathbb{R})$-module of sections $C^\infty(\beta) = \{s \in C^\infty(\mathbb{R}, \mathcal{E}) \mid \beta \circ s = I_\mathbb{R}\}$. The corresponding bundle $J^k(\beta) \xrightarrow{\pi_k} \mathbb{R}$ of k-jets of sections of β is of rank $m(k+1)$ over $\mathbb{R}$, and is equipped with the *Cartan distribution*. A system of linear k-th order ordinary differential equations is a linear subbundle

$$\mathcal{E} \xrightarrow{\alpha} \mathbb{R} \;\subset\; J^k(\beta) \xrightarrow{\pi_k} \mathbb{R} \tag{60}$$

of codimension m such that the Cartan distribution on $J^k(\beta)$ when restricted to $\mathcal{E}$, and denoted $\mathcal{C}_\mathcal{E}$

(i) is 1-dimensional, and
(ii) projects isomorphically to $\mathbb{R}$

We denote the $\mathcal{A}$-module of sections in the bundle α by $C^\infty(\alpha)$. We have a linear connection in the bundle α,

$$\nabla : \mathcal{D}(\mathbb{R}) \longrightarrow Der(C^\infty(\alpha)) \tag{61}$$

where $Der(C^\infty(\alpha))$ denotes derivations of $C^\infty(\alpha)$ over $\frac{d}{dx}$, i. e. $\mathbb{R}$-linear maps

$$D : C^\infty(\alpha) \to C^\infty(\alpha)$$

such that

$$D(fs) = f' \cdot s + f \cdot D(s) \;,$$

for any $f \in \mathcal{A}, s \in C^\infty(\alpha)$. ∇ is defined by the requirement that it lifts $\frac{d}{dx}$ on the base $\mathbb{R}$ to a generator $X \in \mathcal{D}(\mathcal{E})$ of $\mathcal{C}_\mathcal{E}$ on $\mathcal{E}$. Consider $s = \underline{s}(x) \in C^\infty(\alpha)$) as a curve in $\mathcal{E}$. Then, geometrically, $\nabla_{\frac{d}{dx}}$ on acts on s by

$$\overline{A}_{-t} \circ s \circ A_t = s + \nabla_{\frac{d}{dx}}(s) \cdot t + o(t) \tag{62}$$

where A_t is the flow generated by $\frac{d}{dx}$ on $\mathbb{R}$, and $\overline{A}_t$ is the *flow generated by X on $\mathcal{E}$*. Thus, constant sections of ∇, i.e. sections s such that

$$\nabla_Y(s) = 0 \;\;,\quad \forall\, Y \in \mathcal{D}(\mathbb{R})$$

are precisely the integral curves of $\mathcal{C}_\mathcal{E}$ on $\mathcal{E}$.

The pair $(C^\infty(\alpha), \delta = \nabla(\frac{d}{dx}))$ is a $\mathcal{D}$-module over $(\mathcal{A}, \delta_{\mathcal{A}})$, and we have the correspondence.

$$(C^\infty(\alpha) \, , \, \nabla_{\frac{d}{dx}}) \quad \Leftrightarrow \quad (E, \delta) \in Ob(\mathcal{LODE})$$

A kth order linear equation

$$y^{(k)} + f_k y^{(k-1)} + \; \ldots \; + f_1 y \; = \; 0 \tag{63}$$

has corresponding linear bundle

$$\mathcal{E} \; \stackrel{\alpha}{\to} \; \mathbb{R} \quad \subset \quad J^k(\mathbb{R}) \stackrel{\pi_k}{\to} \; \mathbb{R}$$

where

$$\mathcal{E} = \{ \, p_k = -f_k p_{k-1} - \; \ldots \; - f_1 p_0 \, \} \; \subset \; J^k(\mathbb{R})$$

with coordinates $(x, p_0, \ldots, p_{k-1})$, taking standard coordinates $(x, p_0, \ldots p_k)$ on $J^k(\mathbb{R})$.

Denote $F(x, p_0, \, ... \, , p_{k-1}) \; = \; -f_k p_{k-1} - \; \ldots \; - f_1 p_0$. The vector field

$$\mathcal{D} \; = \; \partial_x + p_1 \partial_{p_0} + \ldots p_{k-1} \partial_{p_{k-2}} + F \, \partial_{p_{k-1}}$$

is a generator of the Cartan distribution on $\mathcal{E}$, and its integral curves are on the form

$$\phi(x) \; = \; (\, x, \, y(x), \, y'(x) \, , \, \ldots \, , \, y^{(k-1)}(x) \,)$$

where $y = y(x)$ is a solution of equation(63). Here $\delta = \nabla_{\frac{d}{dx}}$, where ∇ lifts $\frac{d}{dx}$ on the base to $\mathcal{D}$ in the bundle $\mathcal{E} \; \stackrel{\alpha}{\to} \; \mathbb{R}$.

7. Classic Geometries and ODEs

7.1. **Euclidean structures.**

Definition 7.1. *By a harmonic oscillator we mean a $\mathcal{D}$-module $(E, \delta) \in Ob \, (\mathcal{LODE})$ equipped with an δ-invariant positive symmetric 2-form $g \in (\, S^2(E^*) \,)^{\#}$.*

Theorem 7.1.
(1) *For any linear ODE there exists a quadratic 1st integral.*

(2) *Any two Harmonic Oscillators (E, δ, g) and (E', δ', g') of the same dimension are equivalent in the sense that there exists an isomorphism*

$$A : E \longrightarrow E'$$

such that
(i) $A \circ \delta = \delta' \circ A$ *and*
(ii) $g'(Ax, Ay) = g(x, y)$

Proof. (1) An ODE of degree n specifies a module E of dimension n as in Theorem 3.3 , hence there exists a δ-invariant basis $\{\gamma_i\}$ of E as described in the proof of the theorem. Then $g = (\gamma_1^*)^2 + \ldots + (\gamma_n^*)^2 \in S^2(E^\#)$ and it is obviously positive definite. $g(h,h) = c$ is our quadratic first integral, for $h \in E^\#$.
(2) Let $\{\gamma_i\}$ and $\{\gamma_i'\}$ be bases of $E^\#$ and $(E')^\#$ respectively as in Theorem 3.3. In these bases g and g' are given by orthogonally diagonalisable $\mathbb{R}$-matrices G and G'. Let $\{\eta_i^*\}$ and $\{\eta_i'^*\}$ be bases of $(E^*)^\#$ and $(E'^*)^\#$ such that G and G' are diagonal. Then the map A is given by

$$A \,:\, \eta_i \;\mapsto\; \left(\frac{g(\eta_i,\eta_i)}{g'(\eta_i',\eta_i')}\,.\right)^{1/2} \eta_i'\,.$$

Since $\{\eta_i\}$ and $\{\eta_i'\}$ are bases of E and E' over $\mathcal{A}$, expand A as an $\mathcal{A}$-homomorphism $E \to E'$. □

Before moving to more specific results on Euclidean structures we include the following property of δ-invariant symmetric bilinear forms.

Proposition 7.1. *Given a $\mathcal{D}$-module (E,δ) in the category $\mathcal{LODE}$. For any $g \in S^2(E^*)$, and arbitrary $X, Y \in E$ the following are equivalent*

$$\delta g \;=\; 0 \quad \Leftrightarrow \quad g(\delta X, Y) \,+\, g(X, \delta Y) \;=\; g(X,Y)'\,. \tag{64}$$

Proof. Given $g = \sum_{i,j=1}^n g_{ij} e_i^* \cdot e_j^*$ in $S^2(E^*)$ and arbitrary $X, Y \in E$. Then

$$\begin{aligned}
\delta g(X,Y) \;&=\; \left[\sum g_{ij}' e_i^* \cdot e_j^* \,+\, \sum g_{ij}(\delta e_i^* \cdot e_j^* + e_i^* \cdot \delta e_j^*)\right] (X,Y) \\
&=\; \sum \left(g_{ij}' e_i^*(X) \cdot e_j^*(Y) \,+\, g_{ij}\left[\delta(e_i^*(X)) e_j^*(Y) + e_i^*(X) \cdot \delta(e_j^*(Y))\right]\right) \\
&-\; \sum g_{ij}[e_i^*(\delta X)\, e_j^*(Y) \,+\, e_i^*(X)\, e_j^*(\delta Y)] \\
&=\; g(X,Y)' \,-\, g(\delta X, Y) \,-\, g(X, \delta Y)\,.
\end{aligned} \tag{65}$$

□

7.1.1. *Euclidean equations of second order.* We will take a closer look at 2nd order equations and Euclidean structures. Consider a general equation of second order

$$y'' + (a_2(x)\, y)' - a_1(x) y \;=\; 0 \tag{66}$$

corresponding to a $\mathcal{D}$-module (E,δ) with primitive element basis $\{e_1 = e, e_2 = \delta e\}$ where δ is described by

$$\delta^2 e \;=\; a_1 e \,+\, a_2 \delta e\,.$$

We want to study the induced module $(S^2(E^*), \delta)$ and look for positive δ-invariant symmetric forms. Taking the dual basis $\{e_1^*, e_2^*\}$ of E^* we recall that the induced δ in the dual module is given by

$$\delta \underline{e}^* \;=\; \begin{bmatrix} 0 & -a_1 \\ -1 & -a_2 \end{bmatrix} \underline{e}^* \,. \tag{67}$$

Constructing a basis $\{(e_1^*)^2,\, e_1^* \cdot e_2^*,\, (e_2^*)^2\}$ of $S^2(E^*)$ by taking symmetric products in the basis elements of E^* and calculating the induced δ gives us a full description of $(S^2(E^*), \delta)$.

$$\delta \;:\; \begin{bmatrix} (e_1^*)^2 \\ e_1^* \cdot e_2^* \\ (e_2^*)^2 \end{bmatrix} \;\mapsto\; \begin{bmatrix} 0 & -2a_1 & 0 \\ -1 & -a_2 & -a_1 \\ 0 & -2 & -2a_2 \end{bmatrix} \begin{bmatrix} (e_1^*)^2 \\ e_1^* \cdot e_2^* \\ (e_2^*)^2 \end{bmatrix} \,. \tag{68}$$

Thus the system of equations $(S^2(E^*), \delta)$ is

$$s_1' \;-s_2 = 0 \tag{69}$$

$$s_2' - 2a_1 s_1 - a_2 s_2 - 2s_3 = 0 \tag{70}$$

$$s_3' - a_1 s_2 - 2a_2 s_3 = 0 \tag{71}$$

for

$$g \;=\; s_1(x)\,(e_1^*)^2 \;+\; s_2(x)\,e_1^* \cdot e_2^* \;+\; s_3(x)\,(e_2^*)^2 \quad \in S^2(E^*)\,.$$

So, obviously, we may attack the problem directly, and see that the system (69) - (71) can be resolved into a single governing equation

$$s''' \;+\; (-3a_2)s'' \;+\; (2a_2^2 - a_2' - 4a_1)s' \;+\; (4a_1a_2 - 2a_1')s \;=\; 0 \tag{72}$$

by setting $s_1 = s(x)$. Equation (69) implies that

$$s_2 = s_1' = s' \,,$$

and

$$s_3 = \frac{1}{2}(s'' - a_2 s' - 2a_1 s)$$

by (70). Then (71) becomes (72), which we will denote the 2nd symmetric power of the equation (E^*, δ). We may conclude that any element g in the kernel $S^2(E^*)^\# \subset S^2(E^*)$ is on the form

$$g \;=\; s\,(e_1^*)^2 \;+\; s'\,(e_1^* \cdot e_2^*) \;+\; \frac{1}{2}(s'' - a_2 s' - 2a_1 s)\,(e_2^*)^2 \,, \tag{73}$$

where $s = s(x)$ is a solution of (72).

There is a second approach to the quest of finding δ-invariant symmetric bilinear forms of an equation; we may use Proposition 7.1 to deduce properties of positive, symmetric bilinear forms on a general second order equation (66). Let $\{e_1^*, e_2^*\}$ be the dual basis of the primitive element basis $\{e, \delta e\}$ of (E, δ) as before. Consider $g = \sum_{i,j=1}^{2} g_{ij} e_i^* \cdot e_j^*$ in $S^2(E^*)$. We will require throughout that g is positive.

Step 1 We may start with the assumption that g is normalized on the primitive element, i. e.

$$g(e, e) \ = \ g_{11}(x) \ = \ 1 \, .$$

We have the requirement that g is positive, so if $g(e, e) = \alpha^2(x) > 0$, $\alpha^2 \neq 1$, we may perform a change of primitive element basis

$$\widetilde{e} = \frac{1}{\alpha} e \, . \tag{74}$$

Then $\delta\widetilde{e} = -\frac{\alpha'}{\alpha^2} e + \frac{1}{\alpha}\delta e$. Writing the transformation in matrix form yields

$$\begin{bmatrix} \widetilde{e} \\ \delta\widetilde{e} \end{bmatrix} \ = \ \begin{bmatrix} \frac{1}{\alpha} & 0 \\ -\frac{\alpha'}{\alpha^2} & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} e \\ \delta e \end{bmatrix} \, . \tag{75}$$

The determinant of the transformation matrix is $\frac{1}{\alpha^2(x)} \neq 0$, thus this is a change of basis. On the level of equations a transformation that changes the primitive element by a non-zero factor as above corresponds to a change of variable transformation of (66):

$$y \ = \ \frac{1}{\alpha} \, u \, .$$

Thus we may assume that g is normalized in e, up to a change of variable in the original ODE.

Step 2 For $g(e, e) = 1$ applying Proposition 7.1 immediately determines $g(e, \delta e)$ by

$$g(e, \delta e) + g(\delta e, e) \ = \ 2g(e, \delta e) \ = \ g(e, e)' \ = \ 0 \, ,$$

hence, $g(e, \delta e) = 0$.

Step 3 By positivity of g we have that $g(\delta e, \delta e) = \omega^2(x)$ for some non-zero $\omega(x)$. Using Proposition 7.1 again we get a requirement on $g(\delta e, \delta e)$ by

$$g(\delta e, \delta e) + g(e, \delta^2 e) \ = \ g(e, \delta e)' \ = \ 0 \, .$$

But

$$g(e, \delta^2 e) \ = \ g(e, a_1 e + a_2 \delta e) \ = \ a_1 g(e, e) + a_2 g(e, \delta e) \ = \ a_1$$

thus we get the requirement $\omega^2 = -a_1$.

Step 4 The last relation we are able to get from applying the proposition determines a relation between ω and a_2. First we have

$$g(\delta e, \delta^2 e) + g(\delta^2 e, \delta e) = 2g(\delta e, \delta^2 e) = 2g(\delta e, a_1 e + a_2 \delta e) = 2a_2 g(\delta e, \delta e)$$

and due to Proposition 7.1,

$$2\,a_2\,g(\delta e, \delta e) \;=\; 2\,a_2\,\omega^2 \;=\; g(\delta e, \delta e)' \;=\; 2\omega'\omega\,,$$

that is, $a_2 = \frac{\omega'}{\omega}$. We may sum this up as follows. In matrix form, i. e. $g = \underline{e}^{*T} G \underline{e}^{*}$, g is given by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & \omega^2 \end{bmatrix}.$$

Proposition 7.2. *An equation*

$$y'' + \left(\frac{\omega'}{\omega} y\right)' + \omega^2 y \;=\; 0 \tag{76}$$

with $\omega \neq 0$ has a quadratic first integral

$$q \;=\; [\omega^2 + \left(\frac{\omega'}{\omega}\right)^2] y^2 \;+\; 2\left[\frac{\omega'}{\omega}\right] y y' \;+\; [y']^2\,. \tag{77}$$

Proof. The calculations above determine that equation (76) has an associated positive, symmetric bilinear form $g \;=\; (e_1^*)^2 \;+\; \omega^2 (e_2^*)^2 \;\in S^2(E^*)^\#$ on its solution space $E^\#$. That is, for any $h \in E^\#$, $g(h,h) \;=\; c$, constant. But any element $h \in E^\#$ is on the form

$$h \;=\; (-y' - a_2 y) e \;+\; y \delta e, \tag{78}$$

where y is a solution of (76). Thus, setting $q \;=\; g(h,h)$ gives the desired quadratic first integral. □

There is another question to be considered here, namely, how to transform one equation with "potential" ω into another with "potential" $\widetilde{\omega}$? Theorem 7.1 in the beginning of this section shows the existence of a transformation between any two harmonic oscillators preserving the δ-invariant Euclidean structure. But the construction in the proof depends on knowing solutions of our two equations, and deals only with existence. The following result for second order equations determines a transformation independent of knowing any solutions of the equations.

Theorem 7.2. *Let (E_ω, δ) be the $\mathcal{D}$-module corresponding to the equation*

$$y'' + \left(\frac{\omega'}{\omega} y\right)' + \omega^2 y \;=\; 0\,. \tag{79}$$

The transformation of primitive element bases

$$T_\theta \;:\; \begin{bmatrix} e \\ \delta e \end{bmatrix} \;\mapsto\; \begin{bmatrix} \widetilde{e} \\ \delta \widetilde{e} \end{bmatrix} \;=\; \begin{bmatrix} \cos\theta & \frac{1}{\omega}\sin\theta \\ -(\omega + \theta')\sin\theta & \frac{(\omega+\theta')}{\omega}\cos\theta \end{bmatrix} \begin{bmatrix} e \\ \delta e \end{bmatrix} \tag{80}$$

yields an equation $(E_{\widetilde{\omega}}, \delta)$

$$y'' + \left(\frac{\widetilde{\omega}'}{\widetilde{\omega}} y\right)' + \widetilde{\omega}^2 y \; = \; 0 \; ,$$

where $\widetilde{\omega} = \omega + \theta'$. *The associated Euclidean structure is, in matrix form,*

$$\widetilde{G} \; = \; \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{\omega}^2 \end{bmatrix}$$

and is non-degenerated for $\widetilde{\omega} = \omega + \theta' \neq 0$.

Proof. The transformation can be divided into three steps. First one transforms $\{e, \delta e\}$ into a basis orthonormal with respect to g.

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \; = \; \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\omega} \end{bmatrix} \begin{bmatrix} e \\ \delta e \end{bmatrix}$$

The orthonormal basis is then "rotated" by an "angle" $\theta = \theta(x)$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \; = \; \begin{bmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Now, taking w_1 as new primitive element we find,by applying δ that

$$\delta w_1 \; = \; (\theta' + \omega) w_2$$

The total transformation is

$$T_\theta = \begin{bmatrix} 1 & 0 \\ 0 & \omega + \theta' \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\omega} \end{bmatrix} = \begin{bmatrix} \cos\theta & \frac{1}{\omega}\sin\theta \\ -(\omega + \theta')\sin\theta & \frac{(\omega+\theta')}{\omega}\cos\theta \end{bmatrix}$$

and has determinant $\frac{(\omega+\theta')}{\omega}$, which is non-zero for $(\omega + \theta') \neq 0$. Thus, if you wish to transform an equation E_ω to another $E_{\widetilde{\omega}}$, you need a $\theta(x)$ such that $\theta' = \widetilde{\omega} - \omega$.

□

Example 7.1. *The equation*

$$u'' = 0 \tag{81}$$

has the first quadratic integral

$$u^2 - 2xuu' + [x^2 + 1](u')^2 \tag{82}$$

It is obtained by a change of variable $y = u\omega^{-1/2}$ *from the equation*

$$y'' + (\frac{\omega'}{\omega} y)' + \omega^2 y = 0 \tag{83}$$

with

$$\omega = \frac{1}{x^2 + 1} \, . \tag{84}$$

7.1.2. *3rd order Euclidean equations.* Consider a third order equation

$$y''' + (a_3 y)'' - (a_2 y)' + a_1 y \ = \ 0 \tag{85}$$

with corresponding $\mathcal{D}$-module (E, δ) and primitive element basis

$$\{\, e, \, \delta e, \, \delta^2 e \,\},$$

with $\delta^3 e = a_1 e + a_2 \delta e + a_3 \delta_2 e$. Using Proposition 7.1 repeatedly we can derive requirements for a symmetric bilinear form $g \in S^2(E^*)$ to be δ-invariant. Additional requirements determine the coefficients a_i in terms of α, β, as in the step by step calculations leading to Proposition 7.2. Assuming that g is normalized in e, written in matrix form, g has to be on the form

$$G = \begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & \alpha^2 & \alpha'\alpha \\ -\alpha^2 & \alpha'\alpha & \beta^2 \end{bmatrix} \tag{86}$$

where $\alpha = \alpha(x) \neq 0$ and $\beta = \beta(x) \neq 0$.

Theorem 7.3.

$$y''' + (a_3 y)'' - (a_2 y)' + a_1 y \ = \ 0 \tag{87}$$

has a δ-invariant symmetric bilinear form given by G as in (86) *for*

$$\begin{aligned} a_1 \ &= \ -(\alpha^2)' - \alpha^2 \frac{\lambda'}{\lambda} \, , \\ a_2 \ &= \ \frac{\alpha''}{\alpha} + \frac{\alpha'}{\alpha}\frac{\lambda'}{\lambda} - \frac{\beta^2}{\alpha^2} \, , \\ a_3 \ &= \ \frac{\alpha'}{\alpha} - \frac{\lambda'}{\lambda} \, , \end{aligned} \tag{88}$$

where

$$\lambda \ = \ g(v, v)^{-1/2} \ = \ (\beta^2 - \alpha^4 - (\alpha')^2)^{-1/2}$$

for

$$v \ = \ \alpha^2 e + \frac{\alpha'}{\alpha} \delta e + \delta^2 e \, ,$$

which is orthogonal to both e and δe.

7.2. **Symplectic structures.** We may equally study equations with symplectic structure on the solution space.

Definition 7.2. *A symplectic equation is a $\mathcal{D}$-module $(E, \delta) \in Ob(\mathcal{LODE})$ of even rank $2n$ equipped with a non-degenerated δ-invariant $\omega \in \bigwedge^2 (E^*)^{\#}$.*

Thus, if we seek a symplectic structure on the solution space of an equation (E, δ) we should investigate the second exterior power of E^*.

7.2.1. *Equations of second order.* Consider a second order equation

$$y'' + (a_2 y)' - a_1 y = 0 \tag{89}$$

corresponding to the $\mathcal{D}$-module (E, δ), with the usual primitive element basis $\{e_1 = e, e_2 = \delta e\}$, $\delta^2 e = a_1 e + a_2 \delta e$. A general element in $\wedge^2(E^*)$ is on the form

$$\omega = \alpha(x)\, e_1^* \wedge e_2^* \tag{90}$$

for some $\alpha \in \mathcal{A}$. Applying δ to ω yields

$$\begin{aligned} \delta\omega &= \alpha' e_1^* \wedge e_2^* + \alpha\,[\,(-a_1 e_2^*) \wedge e_2^* + e_1^* \wedge (-e_1^* - a_2 e_2^*)\,] \\ &= (\alpha' - a_2\alpha) e_1^* \wedge e_2^*\,, \end{aligned} \tag{91}$$

so $(\bigwedge^2(E^*)\,,\,\delta)$ corresponds to the equation

$$\alpha' - a_2 \alpha = 0\,. \tag{92}$$

The 2-form is non-degenerated if and only if $\alpha \neq 0$. We may interpret this as a requirement on the coefficient a_2, and sum up as follows.

Theorem 7.4. *For $\alpha(x) \neq 0$, any $a_1(x)$ in $\mathcal{A}$, the equation*

$$y'' + (\frac{\alpha'}{\alpha} y)' - a_1 y = 0 \tag{93}$$

is a symplectic equation with the δ-invariant 2-form

$$\omega = \alpha(x)\, e_1^* \wedge e_2^* \tag{94}$$

determining the symplectic structure on $E^{\#}$.

Recall that an element $h_y \in E^{\#}$ corresponding to a solution y of (93) is on the form $h_y = (-y' - \frac{\alpha'}{\alpha} y) e_1 + y e_2$. Given two solutions y_1, y_2,

$$\begin{aligned} \omega_\alpha(h_{y_1}, h_{y_2}) &= \alpha \left[(-y_1' - \frac{\alpha'}{\alpha} y_1) y_2 - (-y_2' - \frac{\alpha'}{\alpha} y_2) y_1 \right] \\ &= \alpha\,(\,y_1' y_2 - y_1 y_2'\,)\,. \end{aligned} \tag{95}$$

7.3. **Complex and Hermitian structure.** There is a natural way to introduce a complex structure on a $\mathcal{D}$-module (E, δ).

Definition 7.3. *Let (E, δ) be a $\mathcal{D}$-module in $\mathcal{LODE}$. A complex structure on the corresponding equation is a δ-invariant $\mathcal{A}$-endomorphism $J \in End_{\mathcal{A}}(E)^{\#}$ such that*

$$J^2 = -Id_E$$

We may immediately deduce the following.

Proposition 7.3. *Denote $\widetilde{\mathcal{A}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$. We may identify the it with smooth complex valued functions in one real variable, $C^\infty(\mathbb{R}, \mathbb{C})$. Given (E, δ) with a complex structure $J \in End_{\mathcal{A}}(E)^\#$.*
(1) E is an $\widetilde{\mathcal{A}}$-module, which we may denote $\widetilde{E}$, by the following definition:

$$[u(x) + i \cdot v(x)]\, e \overset{\text{def}}{=} u(x) \cdot e + v(x) \cdot J(e) \tag{96}$$

for $u(x), v(x) \in \mathcal{A}$ and $e \in E$. (2) If J is δ-invariant, i. e. $\delta\, J = 0$, or, equivalently,

$$\delta \circ J = J \circ \delta\,.$$

Then

$$J^\# = J|_{E^\#} \;:\; E^\# \to E^\# \tag{97}$$

is a complex structure on the vector space $E^\#$.

As a digression we may stop to note that J is actually a symmetry of our base equation E that satisfies the extra condition $J^2 = -1$. Symmetries and the corresponding symmetry equations will be discussed extensively in section 8.

Proposition 7.4. *Given a second order equation with complex structure J, (E, δ, J), its solution space $E^\#$ is isomorphic to $\mathbb{C}$ as vector space, and as a field.*

Proof. We know that any basis of $E^\#$ is generated by two linearly independent solutions u_1, u_2 of the equation corresponding to (E, δ). Written in the primitive element basis of E the basis elements are on the form $h_u = (-u' - a_2 u)e + u\delta e$. Choose $u = u_1$. We know that $J(E^\#) \subset E^\#$, thus the linear independent set $\{h_u, J(h_u)\}$ is a basis of $E^\#$. Now,

$$\phi \;:\; E^\# \to \mathbb{C}$$

defined by

$$h_u \mapsto 1 \quad \text{and} \quad J(h_u) \mapsto i$$

and requiring $\mathbb{R}$-linearity, is an isomorphism of vector spaces. Defining multiplication in $E^\#$ accordingly by

$$\begin{aligned} h_u^2 &= h_u\,, \\ h_u \cdot J(h_u) &= J(h_u)\,, \quad \text{and} \\ J(h_u)^2 &= -h_u\,, \end{aligned} \tag{98}$$

yields that ϕ is also a multiplicative homomorphism, i.e.

$$\phi(h_1 \cdot h_2) = \phi(h_1) \cdot \phi(h_2)\,,$$

and thus

$$E^{\#} \cong \mathbb{C}$$

as fields by ϕ. $\square$

Definition 7.4. *Given a $\mathcal{D}$-module (E, δ) with a complex structure given by $J \in End_{\mathcal{A}}(E)^{\#}$, a Hermitian structure on E is a non-degenerate δ-invariant 2-form $H \in (E^*)^{\otimes 2}$ which satisfies the conditions*

$$H(x, y) = \overline{H(y, x)}$$

and

$$H(Jx, y) = -H(x, Jy) = iH(x, y)$$

for all $x, y \in E$.

Any Hermitian form corresponds to a pair g, ω of Euclidean and symplectic forms satisfying the relation

$$H = g + i\omega \ .$$

In the next subsection we will see examples of equations with complex structure and compatible Euclidean and symplectic structures, hence Hermitian structure.

7.4. **Second order equations with complex and Hermitian structures.** Investigating when a second order equation

$$y'' \ + \ (a_2 y)' \ - \ a_1 y \ = \ 0 \tag{99}$$

has complex structure yields the following. Denote the corresponding $\mathcal{D}$-module (E, δ), with primitive element basis $\{e_1 = e, e_2 = \delta e\}$. We may identify $End_{\mathcal{A}}(E)$ with $E^* \otimes E$ and write an endomorphism

$$J = F_1(x) e_1^* \otimes e_1 + F_2(x) e_1^* \otimes e_2 + F_3(x) e_2^* \otimes e_1 + F_4(x) e_2^* \otimes e_2 \,, \tag{100}$$

or, in matrix form, $J = \underline{e}^{*T} M_J^T \underline{e}$,

$$M_J \ = \ \begin{bmatrix} F_1 & F_3 \\ F_2 & F_4 \end{bmatrix} . \tag{101}$$

Then

$$M_J^2 \ = \ \begin{bmatrix} F_1^2 + F_2 \, F_3 & F_3 \, (F_1 \ + \ F_4) \\ F_2 \, (F_1 \ + \ F_4) & F_4^2 + F_2 \, F_3 \end{bmatrix} . \tag{102}$$

Requiring $M_J^2 = -I$ gives us four equations on the coefficients F_i, and one immediate requirement is that $F_1^2 = F_4^2$. Splitting the problem into

two cases we get the following classes of endomorphisms.
Class (A), characterized by $F_1 = F_4$:

$$J = \begin{bmatrix} 0 & \alpha \\ -\frac{1}{\alpha} & 0 \end{bmatrix} \tag{103}$$

where $\alpha = \alpha(x) \neq 0$.

Adding the requirement that $\delta J = 0$ gives four new equations, and for F_i-s as above they are reduced to

$$a_1(x) = -\alpha^2 \tag{104}$$

and

$$a_2(x) = \frac{\alpha'}{\alpha}\,. \tag{105}$$

Class (B), characterized by $F_1 = -F_4$:

$$J = \begin{bmatrix} \alpha & \beta \\ -\frac{1+\alpha^2}{\beta} & -\alpha \end{bmatrix}, \tag{106}$$

where $\alpha = \alpha(x)$ is any function, and $\beta = \beta(x) \neq 0$. The requirement $\delta J = 0$ with F_i-s as above is reduced to

$$a_1(x) = -\frac{\beta\,(\beta - \alpha')}{1+\alpha^2}\,, \tag{107}$$

and

$$a_2(x) = \frac{\beta'}{\beta} + \frac{2\alpha\,(\beta - \alpha')}{1+\alpha^2}\,. \tag{108}$$

Theorem 7.5. *There are two classes of second order equations that possess complex structure.*

(A) *For $\alpha(x) \neq 0$,*

$$y'' + (\frac{\alpha'}{\alpha}\, y)' + \alpha^2 y = 0\,,$$

with complex structure J determined by (103)*.*

(B) *For $\beta(x) \neq 0$ and any $\alpha(x)$,*

$$y'' + \left[(\frac{\beta'}{\beta} + \frac{2\alpha\,(\beta-\alpha')}{1+\alpha^2}\,)y \right]' + [\,\frac{\beta\,(\beta-\alpha')}{1+\alpha^2}\,]y = 0\,,$$

with complex structure J determined by (106)*.*

Corollary 7.1. *The complex structures of class (A) and (B) as above are symmetries of the respective equations.*

(i) *For equation*

$$y'' + (\frac{\alpha'}{\alpha}\,y)' + \alpha^2 y \;=\; 0 \tag{109}$$

of class (A), $\alpha \neq 0$, the complex structure J acts on solutions $y(x)$ as follows

$$O_J \,:\, y \;\mapsto\; \frac{(\alpha\, y)'}{\alpha^2} \;=\; (\frac{1}{\alpha})\, y' \,+\, (\frac{\alpha'}{\alpha^2})\, y\;. \tag{110}$$

(ii) *For equation*

$$y'' + \left[(\frac{\beta'}{\beta} \,+\, \frac{2\alpha\,(\beta - \alpha')}{1+\alpha^2}\,)y \right]' \,+\, [\,\frac{\beta\,(\beta - \alpha')}{1+\alpha^2}\,]y \;=\; 0 \tag{111}$$

of class (B), $\beta \neq 0$, any α, the complex structure J acts on solutions $y(x)$ as follows

$$O_J \,:\, y \;\mapsto\; \frac{\alpha\beta(\beta - \alpha') + \beta'(1+\alpha^2)}{\beta}\, y \;+\; \frac{(1+\alpha^2)}{\beta}\, y'\;. \tag{112}$$

The class (A) equations are precisely on the form as equations with Euclidean structure in Proposition 7.2. A compatible symplectic structure is given by the defining 2-form

$$\omega(x,y) \;=\; g(Jx,y)\,, \tag{113}$$

where g is Euclidean and J is the complex structure. The triple corresponds to a Hermitian structure

$$H = g + i\omega\,.$$

Theorem 7.6. *The equation*

$$y'' + (\frac{\alpha'}{\alpha}\,y)' + \alpha^2 y \;=\; 0 \tag{114}$$

where $\alpha = \alpha(x) \neq 0$ has Euclidean, complex, symplectic and Hermitian structures.

$$g = (e_1^*)^2 \;+\; \alpha^2\,(e_2^*)^2\;, \tag{115}$$

$$J = \;-\,\frac{1}{\alpha}\, e_1^* \otimes e_2 \;+\; \alpha\, e_2^* \otimes e_1\;, \tag{116}$$

$$\omega = \;-\,\alpha\, e_1^* \wedge e_2^*\,, \tag{117}$$

$$H = g \;+\; i\omega\;, \tag{118}$$

in matrix form,

$$g \;=\; \begin{bmatrix} 1 & 0 \\ 0 & \alpha^2 \end{bmatrix}\;,$$

$$J = \begin{bmatrix} 0 & \alpha \\ -\frac{1}{\alpha} & 0 \end{bmatrix},$$

$$\omega = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}.$$

Example 7.2. *The equation*

$$y'' + y = 0 \tag{119}$$

is on the form as in Theorem 7.6 for $\alpha^2 = 1$. We may take $\alpha = -1$ and get the standard complex structure with matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Choosing $\alpha = 1$ only alters the sign of J. The solution space is spanned by $u = \cos(x)$, $v = \sin(x)$, which in turn determines the basis

$$\begin{aligned} h_{\cos(x)} &= \quad \sin(x)e + \cos(x)\delta e \\ J(h_{\cos(x)}) &= -\cos(x)e + \sin(x)\delta e \;=\; h_{\sin(x)} \end{aligned} \tag{120}$$

of $E^{\#}$ over $\mathbb{R}$, and of E over $\mathcal{A}$.

7.5. S_n **representations from Yang-Baxter solutions, examples.** Proposition 3.6 states that if we have a δ-invariant group action into a $\mathcal{D}$-module, then the invariant elements of this action constitute a $\mathcal{D}$-sub-module, which in turn corresponds to a new ODE. In Theorem 3.2 we saw that, given a $\mathcal{D}$-module (E, δ), any solution $\tau : E \otimes E \to E \otimes E$ of the Yang-Baxter equation with the property $\tau^2 = 1$ gives a representation of S_n into $E^{\otimes n}$ for all $n \geq 2$. Recall that τ being plus and minus twist gave us sub-modules $S^n(E)$ and $\bigwedge^n(E)$ of $E^{\otimes n}$. The twist operation may be used to construct other solutions of the Yang-Baxter equation.

Proposition 7.5. *Given a splitting of a $\mathcal{D}$-module $E = E_0 \oplus E_1$, we may introduce the following map on $E^{\otimes 2}$ given by a combination of $+$ and $-$ twisting:*

$$\tau : \begin{cases} f_0 \otimes g_i \mapsto \quad g_i \otimes f_0, & i = 0, 1 \\ f_i \otimes g_0 \mapsto \quad g_0 \otimes f_i, & i = 0, 1 \\ f_1 \otimes g_1 \mapsto -g_1 \otimes f_1 \end{cases}$$

This τ is a solution if the Yang - Baxter equation (30)*, and thus induces a representation of the symmetric group S_n into $E^{\otimes n}$.*

Given a $\mathcal{D}$-module (E, δ) corresponding to the second order equation

$$y'' + (a_2 y)' - a_1 y \;=\; 0 \tag{121}$$

with primitive element basis $\{e_1, e_2\}$, we may try to find a splitting of E by means of an operator $A : E \to E$ with the property

$$A^2 = 1,$$

such that E splits into two one-dimensional modules

$$E = E_0 \oplus E_1 = \ker(A-1) \oplus \ker(A+1)\,.$$

Ensuring that the splitting preserves the $\mathcal{D}$-module structure we require that

$$\delta A = 0\,. \tag{122}$$

We get two classes of non-trivial splittings.
Class (1) consists of equations

$$y'' + (\frac{\alpha'}{\alpha} y)' - \alpha^2 y \;=\; 0 \tag{123}$$

with $\alpha \neq 0$. This equation splits non-trivially into

$$E \;=\; E_0 \oplus E_1 \;=\; \ker(A-1) \oplus \ker(A+1)$$

for

$$A \;=\; \alpha e_2^* \otimes e_1 + \frac{1}{\alpha} e_1^* \otimes e_2,$$

with $E_0 =< \eta_0 = \alpha e_1 + e_2 >_{\mathcal{A}}$ and $E_1 =< \eta_1 = -\alpha e_1 + e_2 >_{\mathcal{A}}$. Restricting δ to E_0 and E_1 yields

$$\delta\eta_0 \;= (\tfrac{\alpha'}{\alpha} + \alpha)\eta_0\,, \quad \text{and} \tag{124}$$

$$\delta\eta_1 \;= (\tfrac{\alpha'}{\alpha} - \alpha)\eta_1\,. \tag{125}$$

This means that $E^{\#} = \{\, u\eta_0 + v\eta_1 \;=\; \alpha(u-v)e_1 + (u+v)e_2 \,\}$ where u and v solve equations

$$(E_0, \delta): \qquad u' \;\; +(\frac{\alpha'}{\alpha} + \alpha)u = 0 \tag{126}$$

$$(E_1, \delta): \qquad v' \;\; +(\frac{\alpha'}{\alpha} - \alpha)v = 0 \tag{127}$$

respectively, i.e. sums of solutions $y = u + v$ give all solutions of equation E.

But we also have an action into $E \otimes E$ of the symmetric group S_2 given by τ as in Proposition 7.5. Invariants of this action $\Sigma^2_\tau(E) = \ker(\tau - 1) \subset E \otimes E$ is generated over $\mathcal{A}$ by

$$\gamma_1 = \eta_0 \otimes \eta_0 \;\;, \quad \gamma_2 = \eta_0 \otimes \eta_1 + \eta_1 \otimes \eta_0\,.$$

Investigating derivatives we get that

$$\delta\gamma_1 \;=\; \left(2\tfrac{\alpha'}{\alpha} + 2\alpha\right)\,\gamma_1 \tag{128}$$

$$\delta\gamma_2 \;=\; \left(2\tfrac{\alpha'}{\alpha}\right)\,\gamma_2 \tag{129}$$

thus,

$$\Sigma^2_\tau(E) \;=\; M_1 \oplus M_2 \;\subset\; E \otimes E,$$

where (M_1, δ) corresponds to the equation

$$u' + (2\alpha + 2\frac{\alpha'}{\alpha})u \;=\; 0,$$

and (M_2, δ) corresponds to the equation

$$u' + (2\frac{\alpha'}{\alpha})u \;=\; 0.$$

On the other hand, we may take $-\tau$ to generate an action of S_2, and $\Sigma^2_{-\tau}(E) = \ker(\tau + 1) \subset E \otimes E$ is generated over $\mathcal{A}$ by

$$\gamma_3 = \eta_1 \otimes \eta_1\;, \quad \gamma_4 = \eta_0 \otimes \eta_1 - \eta_1 \otimes \eta_0.$$

Taking derivatives we get that

$$\delta\gamma_3 \;=\; (2\tfrac{\alpha'}{\alpha} - 2\alpha)\,\gamma_3 \tag{130}$$

$$\delta\gamma_4 \;=\; (2\tfrac{\alpha'}{\alpha})\,\gamma_4 \tag{131}$$

thus,

$$\Sigma^2_{-\tau}(E) \;=\; M_3 \oplus M_4$$

where (M_3, δ) corresponds to the equation

$$u' + (2\alpha - 2\frac{\alpha'}{\alpha})u \;=\; 0$$

and (M_4, δ) corresponds to the equation

$$u' + (4\alpha)u \;=\; 0.$$

We may sum up as follows:

$$E \otimes E \;=\; \Sigma^2_\tau(E) \oplus \Sigma^2_{-\tau}(E) \;=\; M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

and

$$(E \otimes E)^\# \;=\; \{\, m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 + m_4\gamma_4 \,\},$$

where m_i solves equation M_i.

Class (2) consists of equations

$$y'' + \left[(\frac{\beta'}{\beta} + 2\frac{\alpha(\beta - \alpha')}{(\alpha^2 - 1)})\,y\right]' - \frac{\beta(\beta - \alpha')}{(\alpha^2 - 1)}y \;=\; 0 \tag{132}$$

with $\beta \neq 0$, $\alpha \neq -1, 1$. It splits with respect to the operator

$$A \;=\; \alpha e_1^* \otimes e_1 + \beta e_2^* \otimes e_1 + \frac{(1-\alpha^2)}{\beta} e_1^* \otimes e_2 - \alpha e_2^* \otimes e_2$$

into a direct sum $E = E_0 \oplus E_1 = \ker(A-1) \oplus E_1$ as for class (1)-equations, with $E_0 =<\eta_0 = \beta e_1 + (1-\alpha) e_2>_{\mathcal{A}}$, and $E_1 =<\eta_0 = \beta e_1 - (1+\alpha) e_2>_{\mathcal{A}}$. The corresponding equations are

$$E_0: \qquad u' + \left(\tfrac{\beta'}{\beta} + \tfrac{(\beta-\alpha')}{(\alpha+1)}\right) u \;=\; 0, \tag{133}$$

$$E_1: \qquad v' + \left(\tfrac{\beta'}{\beta} + \tfrac{(\beta-\alpha')}{(\alpha-1)}\right) v \;=\; 0. \tag{134}$$

Thus

$$E^{\#} = \{\, u\eta_0 + v\eta_1 \;=\; \beta(u+v)e_1 + [(u-v) - \alpha(u+v)]e_2 \,\}$$

so solutions of equation E are $y = (u-v) - \alpha(u+v)$ for solutions u, v of E_0, E_1. Repeating the study of the representation of S_2 into $E \otimes E$ as for Class (1) equations yields that

$$\Sigma^2_\tau(E) \;=\; M_1 \oplus M_2 \;\subset\; E \otimes E$$

and

$$\Sigma^2_{-\tau}(E) \;=\; M_3 \oplus M_4 \;\subset\; E \otimes E,$$

with

$$E \otimes E \;=\; \Sigma^2_\tau(E) \oplus \Sigma^2_{-\tau}(E) \;=\; M_1 \oplus M_2 \oplus M_3 \oplus M_4.$$

The modules M_i correspond to the following equations

$$M_1: \qquad m_1' \quad +2\left(\tfrac{(\beta-\alpha')}{(\alpha+1)} + \tfrac{\beta'}{\beta}\right) m_1 \quad = 0 \tag{135}$$

$$M_2: \qquad m_2' \quad +2\left(\tfrac{\alpha(\beta-\alpha')}{(\alpha^2-1)} + \tfrac{\beta'}{\beta}\right) m_2 \quad = 0 \tag{136}$$

$$M_3: \qquad m_3' \quad +2\left(\tfrac{(\beta-\alpha')}{(\alpha-1)} + \tfrac{\beta'}{\beta}\right) m_3 \quad = 0 \tag{137}$$

$$M_4: \qquad m_4' \quad -8\beta\left(\tfrac{\alpha(\beta-\alpha')}{(\alpha^2-1)} + \tfrac{\beta'}{\beta}\right) m_4 \quad = 0 \tag{138}$$

8. Symmetries and representations

In this section we study symmetries of equations, in particular through symmetry operators. Section 8.2 contains results for linear operator symmetries, most of which is discussed in detail in [11]. The most important addition to these results is the description on how this embeds into the category $\mathcal{LODE}$, through Proposition 8.3 and Theorem 8.2.

Symmetry operators are convenient tools for calculations with symmetries in $\mathcal{D}$-modules, and are important to ensure full applicability of solving strategies developed in Sections 9 and 10.

8.1. **Symmetry algebras and representations.** In the category $\mathcal{LODE}$ a *symmetry* of an equation (E, δ) is an endomorphism of E which is δ-invariant, that is an element

$$X \in \mathrm{End}_{\mathcal{A}}(E)^{\#} \tag{139}$$

Such an X is simply a map of the module E into itself such that it maps solutions to solutions,

$$X : E \to E \quad \text{such that} \quad X(E^{\#}) \subset E^{\#} \tag{140}$$

We thus arrive at a natural way to introduce symmetry algebras of equations in our picture, in terms of representation theory.

Definition 8.1. *A Lie algebra $\mathfrak{g}$ is a Lie algebra of linear symmetries of an equation $(E, \delta) \in Ob(\mathcal{LODE})$ if there is a representation*

$$\rho \ : \ \mathfrak{g} \ \longrightarrow \ \mathrm{End}_{\mathcal{A}}(E)$$

such that

$$\rho(g) \circ \delta \ = \ \delta \circ \rho(g) \ \ , \ \ \forall g \in \mathfrak{g} \ ,$$

i.e. ρ maps $\mathfrak{g}$ into δ-invariant endomorphisms of E,

$$\rho(\mathfrak{g}) \ \subset \ \mathrm{End}_{\mathcal{A}}(E)^{\#} \ .$$

Proposition 8.1. *If $\mathfrak{g}$ is a symmetry algebra of (E, δ) with associated representation ρ, then*

$$\rho \ : \ \mathfrak{g} \ \rightarrow \ \mathrm{End}_{\mathbb{R}}(E^{\#})$$

is a representation into the $\mathbb{R}$-vector space $E^{\#}$.

We just need to recall that taking the kernel of δ commutes with the algebraic constructions in our category, $\mathrm{End}_{\mathcal{A}}(E)^{\#} = \mathrm{End}_{\mathbb{R}}(E^{\#})$, ref. Proposition 3.2.

The consequences of combining Proposition 3.2 and Theorem 3.3 are immediate, this enables us to make use of results from the rich theory of representations of Lie algebras into vector spaces. In particular decomposition theorems from the theory concerning semisimple Lie algebras.

8.2. **Symmetry operators.** Most of the results in this section concerning theory of linear differential operators are found in [11].
Proposition 8.3 and Theorem 8.2 enables the incorporation of these results in the $\mathcal{D}$-module perspective. In particular, Proposition 8.3 and Theorem 8.2 explain precisely how a symmetry operator induces a δ-invariant endomorphism of the relevant $\mathcal{D}$-module, using the formulations of Section 5, and determining its action on actual solutions of the

equation. Let $\mathcal{K}$ still denote the ring of linear differential operators over $\mathbb{R}$.

Definition 8.2. *A symmetry operator of an equation*

$$L(y) \;=\; y^{(k)} + c_k(x)y^{(k-1)} + \ldots + c_1(x)y \;=\; 0 \tag{141}$$

is a linear operator

$$\Delta \;=\; b_0(x) + b_1(x)\partial + \ldots + b_{k-1}(x)\partial^{k-1} \;\in\; \mathcal{K}$$

with the property that there exists $\nabla \in \mathcal{K}$ *such that*

$$L \circ \Delta = \nabla \circ L\,. \tag{142}$$

Note that $\Delta(\ker L) \subset \ker L$, i. e. it maps solutions to solutions. **Remark:** Recall the discussion in Section 6 on the jet space approach to ODEs. Studying an equation (141) we make the following connection. Associated to an operator $\Delta = b_0(x) + b_1(x)\partial + \ldots + b_{k-1}(x)\partial^{k-1}$ is the function

$$f_\Delta \;=\; b_0p_0 + b_1p_1 + \ldots + b_{k-1}p_{k-1} \;\in\; C^\infty(\mathcal{E})$$

where $\mathcal{E} \xrightarrow{\alpha} \mathbb{R}$ is the linear subbundle in $J^k(\mathbb{R})$ corresponding to (141). The function f solves the *Lie equation* for (141) and generates a shuffling symmetry of the Cartan distribution $\mathcal{C}_\mathcal{E}$ on $\mathcal{E} \subset J^k\mathbb{R}$ if and only if Δ is a symmetry in the above sense. See [3] for a more detailed discussion on symmetries of Cartan distributions and their generating functions. We denote the set of symmetries Δ by $Sym(L)$.
Consider more generally

$$\Sigma(L) = \{\, P \in \mathcal{K} \mid \exists\, Q \in \mathcal{K} \text{ such that } L \circ P = Q \circ L \,\}\,. \tag{143}$$

Proposition 8.2. $\Sigma(L)$ *is*

(1) *an associative* $\mathbb{R}$*-algebra with respect to composition of operators, and*

$$L \circ (P_1 \circ P_2) = (Q_1 \circ Q_2) \circ L.$$

(2) *a Lie algebra with respect to commutators of operators, and*

$$L \circ [P_1, P_2] = [Q_1, Q_2] \circ L.$$

Lemma 8.1. *For any* $P \in \mathcal{K}$ *of order* l*,* L *as in (141) there are uniquely determined operators* C_P *and* R_P *in* $\mathcal{K}$*, of order* $\leq l-k$ *and* $\leq k$ *respectively such that*

$$P = C_P \circ L + R_P\,. \tag{144}$$

This is proved by induction and order arguments, as done in [11]. An analogous argument shows that there are unique operators c_P and r_P such that

$$P = L \circ c_P + r_P\,. \tag{145}$$

Theorem 8.1. *For $P \in \Sigma(L)$ the remainder R_P from right division by L as in Lemma 8.1 is an element of $Sym(L)$. The map*

$$R : \Sigma(L) \to Sym(L) \;\text{ with }\; R(P) = R_P$$

induces

(1) *associative $\mathbb{R}$-algebra structure on $Sym(L)$ by*

$$m(\Delta_1, \Delta_2) = R_{\Delta_1 \circ \Delta_2}$$

(2) *Lie algebra structure*

$$\begin{aligned} [\,\cdot\,,\,\cdot\,]_L \;&:\; Sym(L) \times Sym(L) \to Sym(L) \\ \text{by} \quad [\Delta_1, \Delta_2]_L \;&=\; R_{[\Delta_1,\Delta_2]} \end{aligned}$$

Proof. Let $\Delta \in Sym(L)$ with associated operator ∇ s.t. $L \circ \Delta = \nabla \circ L$. Decompose $\Delta = C_\Delta \circ L + R_\Delta$ according to Lemma 8.1, and likewise $\nabla = L \circ c_\nabla + r_\nabla$ by left division by L. Then the symmetry property implies

$$L \circ (c_\nabla - C_\Delta) \circ L \;=\; L \circ R_\Delta - r_\nabla \circ L\,.$$

If $c_\nabla - C_\Delta \neq 0$ the left hand side is an operator of order $> 2k$, which is impossible since the operator on the right hand side is of maximum order $2k$, whence $c_\nabla = C_\Delta$ and $L \circ R_\Delta = r_\nabla \circ L$, which proves that $R_\Delta \in Sym(L)$. Statements (1) and (2) follow directly. □

In the category $\mathcal{LODE}$ we viewed symmetries of an equation E as δ-invariant endomorphisms of E. We will make the link between symmetries viewed as endomorphisms and symmetry operators in the following way.

Proposition 8.3. *Let $E = E_{L^t}$ be the factor $\mathcal{D}$-module corresponding to an equation*

$$L(y) \;=\; 0.$$

A symmetry operator $\Delta \in \mathcal{K}$ of the equation induces a δ-invariant endomorphism $\overline{\Delta}$ of E

$$\overline{\Delta} \;:\; E \;\to\; E \tag{146}$$

defined by

$$[X] \;\mapsto\; [X \circ \nabla^t]\,, \tag{147}$$

where $\nabla \in \mathcal{K}$ such that $L \circ \Delta = \nabla \circ L$.

Proof. Note primarily that right composition by ∇^t is well defined with respect to choice of representative modulo L^t :

$$(X + A \circ L^t) \circ \nabla^t \;=\; X \circ \nabla^t \,+\, A \circ (\nabla \circ L)^t \;= \\ =\; X \circ \nabla^t \,+\, A \circ (L \circ \Delta)^t \;=\; X \circ \nabla^t \,+\, (A \circ \Delta^t) \circ L^t\,.$$

Moreover, $\overline{\Delta}$ is an $\mathcal{A}$ homomorphism, and obviously commutes with δ:

$$(\,\overline{\Delta} \circ \delta\,)[X] \;=\; [\,\partial \circ X \circ \nabla^t\,] \;=\; (\,\delta \circ \overline{\Delta}\,)[X]$$

Thus, $\overline{\Delta} \in End_{\mathcal{A}}(E)^{\#}$. □

Theorem 8.2. *Given a symmetry operator* Δ *of the equation* $L(y) = 0$ *the corresponding* δ*-invariant endomorphism* $\overline{\Delta} : E \to E$ *acts as follows when restricted to* $\ker\delta = E^{\#} \subset E$*:*

$$\overline{\Delta} \;:\; [X_y] \;\mapsto [X_{\Delta(y)}]\,,$$

where $[X_y] \in E^{\#}$ *is generated by a solution* y *of* $L(y) = 0$*.*

Proof. We start by noting that for a representative X of a class $[X] \in \ker\delta$ there is an associated operator A such that

$$\partial \,\circ\, X \;=\; A \circ L^t\,. \tag{148}$$

Recall from Section 4 on primitive element bases that a solution y of $L(y) = 0$, generates an element $[X_y] \in E^{\#}$ on the form

$$[X_y] \;=\; H_k(y)e + H_{k-1}(y)\delta e + \ldots + H_2(y)\delta^{k-2}e + y\,\delta^{k-1}e\,,$$

that is, with a representative

$$X_y = H_k(y) + H_{k-1}(y)\partial + \ldots + H_2(y)\partial^{k-2} + y\,\partial^{k-1} \;\in \mathcal{K}\,,$$

with primitive element $e = [1]$, $\delta e = [\partial]$ etc. For choice of representative X_y the left hand side in (148) is of degree k with highest degree coefficient equal to y, whence the operator A is of degree zero, and equal to y, that is $\partial \circ X_y = yL^t$. Further we know that

$$X_y \circ \nabla^t \;=\; X_u + B \,\circ\, L^t \tag{149}$$

for some solution u of $L(y) = 0$, and some operator B. Again, applying δ to this representative yields

$$\partial \circ (X_u + B \,\circ\, L^t) \;=\; (u + \partial \circ B) \,\circ\, L^t\,.$$

But $\partial \circ (X_y \circ \nabla^t) = (\partial \circ X_y) \circ \nabla^t = (yL^t) \circ \nabla^t = y\,\Delta^t \circ L^t$, whence

$$u + \partial \circ B = y\,\Delta^t\,,$$

i.e. $u + B \circ \partial = \Delta \circ y$. Collecting terms of degree zero yields precisely that

$$u = \Delta(y)\,.$$

$\square$

Remark: Certainly we wish to be able to calculate the action of $\overline{\Delta}$ directly for a known symmetry Δ, whence ∇^t should be directly retrieved from Δ. If we look to the condition (142) of Δ being a symmetry of the equation $L(y) = 0$, we see the following: Recall that $L = \partial^k + c_k\partial^{k-1} + \ldots + c_1$. Let

$$\begin{aligned}
\Delta &= A_1 + A_2\partial + \ldots + A_k\partial^{k-1} \\
\nabla &= B_1 + B_2\partial + \ldots + B_k\partial^{k-1}
\end{aligned}$$

On the one hand we get

$$L \circ \Delta = [A_k]\partial^{2k-1} + \sum_{l=1}^{k-1} [A_{k-l} + \alpha_{k-l}(A,c)]\partial^{2k-l-1} + \sum_{l=1}^{k} \phi_{k-l}(A,c)\partial^{k-l}$$

where α_{k-l} depends on the coefficient functions $c_i(x)$ of L, and A_j-s and their derivatives for $j > k-l$. The functions ϕ_{k-l} depend on A_i-s and c_i-s. Likewise,

$$\nabla \circ L = [B_k]\partial^{2k-1} + \sum_{l=1}^{k-1} [B_{k-l} + \beta_{k-l}(B,c)]\partial^{2k-l-1} + \sum_{l=1}^{k} \psi_{k-l}(B,c)\partial^{k-l}$$

where, similarly β_{k-l} depends on the coefficient functions $c_i(x)$ of L, and B_j-s and their derivatives for $j > k-l$. The functions ψ_{k-l} depend on B_i-s and c_i-s. Thus, by setting $L \circ \Delta = \nabla \circ L$ and collecting terms of the same order in ∂ we arrive at $2k$ equations. The first k equations determine the B_i-s in terms of A_j-s and c_i-s:

$$\begin{array}{lll}
(i) & B_k & = A_k \\
(ii) & B_{k-1} & = A_{k-1} + \alpha_{k-1}(A_k, c) - \beta_k(B_k, c) \\
 & \vdots & \\
(l+1) & B_{k-l} & = A_{k-l} + \alpha_{k-l}(A_{k-l+1}, \ldots, A_k, c) - \beta_{k-l}(B_{k-l+1}, \ldots, B_k, c) \\
 & \vdots & \\
(k) & B_1 & = A_1 + \alpha_1(A_2, \ldots, A_k, c) - \beta_1(B_2, \ldots, B_k, c)
\end{array}$$

Starting with equation (i) and successively substituting into the following equations we find the B_i-s in terms of the coefficients A_i of Δ, and the coefficients c_i of L. Thus ∇, and subsequently $\overline{\Delta}$ is derived directly from Δ. The last k equations are the differential equations

$$\begin{aligned} (k+1) \quad \psi_{k-1}(B,c) &= \phi_{k-1}(A,c) \\ &\vdots \\ (2k) \quad \psi_0(B,c) &= \phi_0(A,c) \end{aligned}$$

that determine conditions on A_j-s for Δ to be a symmetry, *Lie equations* for L.

8.3. **Skew- and self- adjoint equations.** Note that the map

$$\phi \;:\; Sym(L) \;\longrightarrow\; Sym(L^t)$$

$$\Delta \;\mapsto\; \nabla^t \tag{150}$$

is an isomorphism. Whenever L is skew- or self-adjoint, i.e $L^t = \pm L$, we note that

$$\phi \;:\; \Sigma(L) \;\longrightarrow\; \Sigma(L) \tag{151}$$

and likewise

$$\phi \;:\; Sym(L) \;\longrightarrow\; Sym(L) \tag{152}$$

gives us an involution on symmetries, $\phi^2 = Id$. Whence the $\Sigma(L)$ and the symmetry space decompose into

$$\begin{aligned} \Sigma(L) &= \Sigma_0(L) \oplus \Sigma_1(L) \;, \\ Sym(L) &= Sym_0(L) \oplus Sym_1(L) \;, \end{aligned}$$

where

$$\Sigma_0(L) \;=\; \{\, \Delta \,|\, L \circ \Delta \;=\; \Delta^t \circ L \} \,, \tag{153}$$

$$\Sigma_1(L) \;=\; \{\, \Delta \,|\, L \circ \Delta \;=\; -\Delta^t \circ L \} \;, \tag{154}$$

and $Sym_a(L) = \Sigma_a(L) \cap Sym(L)$.

Theorem 8.3. *Let L be skew- or self-adjoint. Then*

$$Sym(L) = Sym_0(L) \oplus Sym_1(L)$$

is a $\mathbb{Z}_2$-graded Lie algebra, i.e.

$$[\, Sym_a(L) \;,\; Sym_b(L) \,] \;\subset\; Sym_{a+b}(L)$$

$a, b \in \mathbb{Z}_2$.

8.4. **Symmetries of second order equations.** We will investigate in detail the symmetry equation of a second order equations, using both the operator approach and direct calculation in the module of endomorphisms of the equation, and thus illustrate both methods. Consider the equation

$$L(y) \;=\; y'' + (a_2 y)' - a_1 y \;=\; 0 \tag{155}$$

Note that the corresponding module (E, δ) has primitive element basis $\{e_1 = e, e_2 = \delta e\}$ with $\delta^2 e = a_1 e + a_2 \delta e$. The module of endomorphisms $(End_{\mathcal{A}}(E), \delta)$ may be identified with $(E^* \otimes E, \delta)$, and we may write a general endomorphism

$$F \;=\; F_1(x) e_1^* \otimes e_1 + F_2(x) e_1^* \otimes e_2 + F_3(x) e_2^* \otimes e_1 + F_4(x) e_2^* \otimes e_2 \,. \tag{156}$$

Thus

$$\delta F \;=\; 0$$

if and only if the coefficient functions F_i satisfy the system

$$\begin{array}{cccccccc} (i) & F_1' & & +a_1 F_2 & -F_3 & & = & 0 \\ (ii) & F_2' & +F_1 & +a_2 F_2 & & -F_4 & = & 0 \\ (iii) & F_3' & -a_1 F_1 & & -a_2 F_3 & +a_1 F_4 & = & 0 \\ (iv) & F_4' & & -a_1 F_2 & +F_3 & & = & 0 \end{array} \tag{157}$$

Adding (i) and (iv) yields

$$F_4' = -F_1'$$

Integrating, we get that

$$F_4 = -F_1 + c$$

for some constant $c \in \mathbb{R}$. Denoting

$$F_2 = p(x) \tag{158}$$

equation (ii) implies that

$$F_1 = \frac{1}{2}(c - p' - a_2 p) \,, \tag{159}$$

and thus,

$$F_4 = \frac{1}{2}(c + p' + a_2 p) \,. \tag{160}$$

Finally, from (i) we get that

$$F_3 = \frac{1}{2}(-p'' - (a_2\, p)' + 2a_1 p) \tag{161}$$

and (iii) becomes

$$p''' + (2a_2' - a_2^2 - 4a_1)p' + (a_2'' - a_2 a_2' - 2a_1')p \;=\; 0 \tag{162}$$

This may be summed up as follows. In matrix form, F is given by

$$M_F = \begin{bmatrix} \frac{1}{2}(c - p' - a_2 p) & a_1 p - \frac{1}{2}((a_2\, p)' + p'') \\ p & \frac{1}{2}(c + p' + a_2 p) \end{bmatrix} \tag{163}$$

where $p = p(x)$ solves equation (162) and $c \in \mathbb{R}$. Note that this tells us that, as a $\mathcal{D}$-module,

$$End_A(E) \;\cong\; E_0 \oplus E_1$$

where (E_0, δ_0) and (E_1, δ_1) are the $\mathcal{D}$-modules corresponding to equations

$$s' \;=\; 0$$

and

$$p''' + (2a_2' - a_2^2 - 4a_1)p' + (a_2'' - a_2 a_2' - 2a_1')p \;=\; 0$$

respectively. (E_0, δ_0) contributes with the trivial part of our invariant F, the constant c. This part $\frac{c}{2}(e_1^* \otimes e_1 + e_2^* \otimes e_2)$ only acts by multiplying an element in E by $\frac{c}{2}$.

Calculating the symmetry equation using the operator approach yields the following. Our equation is given by

$$L = \partial^2 + \partial a_2 - a_1 \,.$$

Recall that a first order linear operator

$$P = P_1 + P_2 \partial$$

is a symmetry of the equation $Ly = 0$ if there is an operator

$$Q = Q_1 + Q_2 \partial$$

such that

$$L \circ P = Q \circ L \,. \tag{164}$$

Setting $L \circ P = Q \circ L$ and collecting terms of the same order gives four equations, of which the two first determine the functions Q_1 and Q_2 in terms of P_1 and P_2, as promised in Section 8.2 :

$$Q_2 \;=\; P_2 \,, \tag{165}$$

$$Q_1 \;=\; P_1 - a_2 Q_2 + 2P_2' + a_2 P_2 \,, \tag{166}$$

thus

$$Q_1 \;=\; P_1 + 2P_2' \tag{167}$$

The two last equations are

$$P_2'' + (a_2 P_2)' + a_1 P_2 + 2P_1' + a_2 P_1 - a_2 Q_1 - a_1 Q_2 - 2a_2' Q_2 \;= 0 \tag{168}$$

$$P_1'' + (a_2 P_1)' + a_1 P_1 - a_2'' Q_2 - a_1' Q_2 - a_2' Q_1 - a_1 Q_1 \;= 0 \,. \tag{169}$$

They become

$$P_2'' - (a_2P_2)' + 2P_1' \ = 0 \ , \tag{170}$$

$$P_1'' + a_2P_1' - 2a_1P_2' - 2a_2'P_2' - (a_1' + a_2'') \ = 0 \ , \tag{171}$$

of which the first may be integrated to give us

$$P_1 = \frac{1}{2}(a_2P_2 - P_2') + c \tag{172}$$

for $c \in \mathbb{R}$. Setting $P_2 = p(x)$ we arrive at the following. Any symmetry operator of equation (155) is on the form

$$P \ = \ c + \ \frac{1}{2}(a_2p - p') \ + \ p\,\partial \tag{173}$$

where $p = p(x)$ solves

$$p''' + (2a_2' - a_2^2 - 4a_1)p' + (a_2'' - a_2a_2' - 2a_1')p \ = \ 0 \ . \tag{174}$$

This equation is precisely equation (162), which we arrived at when considering endomorphisms of our equation.

Again,

$$Sym(L) \ \cong \ Sym_{tr} \oplus Sym_{eq}$$

where Sym_{tr} is the trivial part, i.e $P = c \in \mathbb{R}$, a solution of the equation $s' = 0$.

There is another property to note from the non-trivial symmetry equation (162). It is the symmetric 2-power, $S^2(E_0)$, of the equation

$$L_0 \ = \ \partial^2 + W(x)\,, \tag{175}$$

where

$$W(x) \ = \ \frac{1}{2}a_2' - \frac{1}{4}a_2^2 - a_1\,. \tag{176}$$

From Theorem 4.1 we know that if $\{u, v\}$ is a set of fundamental solution $\{u, v\}$ of the equation

$$L_0(u) = 0\,, \tag{177}$$

then $\{u^2, uv, v^2\}$ are fundamental solutions of

$$S^2(L_0)(p) = 0\,. \tag{178}$$

Thus, we may produce symmetries from solutions of

$$u'' + (\,\frac{1}{2}a_2' - \frac{1}{4}a_2^2 - a_1\,)u \ = \ 0 \tag{179}$$

by operators P_{u^2}, P_{uv} and P_{v^2}.

9. Solvable symmetry algebras and quadratures

It is a general opinion that to solve an ODE by quadratures with the use of symmetries one needs a solvable algebra of symmetries of dimension equal to the order of the equation. In [3] it is shown that knowing a solvable k-dimensional transversal Lie algebra of symmetries of a kth order ODE (in general non-linear) one can find the general solution by quadratures. In this geometric approach the method consists of finding a complete set of first integrals of the Cartan distribution of the equation by integrating closed 1-forms and solving functional equations, where the solvability of the algebra is crucial to recover the appropriate 1-forms.

Our approach here is somewhat different, and we shall see that whether we are able to solve an ODE directly by quadratures is not dependent on the order of the equation or the dimension of its symmetry algebra, but rather on eigenvalues of symmetries viewed as endomorphisms of the corresponding $\mathcal{D}$-module. Thus it may happen that a single symmetry is sufficient to solve an equation, conditions for this are stated in in Theorem 9.1.

In Theorem 9.7 a sufficient condition for an equation with a solvable symmetry algebra to be solved directly by quadratures is given, with no requirement on the dimension of the algebra.

9.1. **Decomposition of equations by eigenspaces of symmetries.** We begin this section with a result that should be kept in mind whenever working with symmetries of equations. It is not limited to any particular type of symmetry algebra, and even states that a single symmetry may be enough to solve an equation by quadratures, regardless of order of the equation, the only factor being eigenspaces of the action of the symmetry. Given an equation (E, δ) with a symmetry $X \in End_{\mathcal{A}}(E)^{\#}$, for $\lambda \in \mathbb{R}$ denote

$$E_\lambda \;=\; \{\, h \in E \mid X(h) = \lambda h \,\}, \tag{180}$$

For a non-empty E_λ we call λ a eigenvalue of X.

Proposition 9.1. *E_λ is a sub-$\mathcal{D}$-module of E.*

Proof. E_λ is obviously a sub-module of E. For $h \in E$,

$$X(\delta h) \;=\; \delta(X(h)) \;=\; \delta(\lambda h) = \lambda \cdot \delta h$$

since X commutes with δ, thus $\delta(E_\lambda) \subset E_\lambda$, and E_λ is a $\mathcal{D}$-module. . □

The rank of the module E_λ over $\mathcal{A}$ we will call the multiplicity of λ.

Theorem 9.1. *Let (E, δ) be a $\mathcal{D}$-module of rank n, and X a symmetry of E. If X has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, all of multiplicity 1, then*

$$E = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_n} , \tag{181}$$

and E is solvable by quadratures.

Proof. The E_{λ_i} are all non-empty by definition, and are sub-$\mathcal{D}$-modules of E. Since the eigenvalues are distinct, the E_i-s don't intersect, and they span the whole of E, since there are n. Each E_{λ_i} corresponds to a first order equation which is identified by applying δ to an arbitrary element $w_i \in E_{\lambda_i}$. If $\delta w_i = \alpha_i w_i$, then the corresponding equation is

$$u_i' + \alpha_i u_i = 0$$

and elements $e_i = u_i w_i$, where the u_i solve the E_{λ_i}-equations span the solution space $E^{\#} \subset E$. □

The Theorem describes a situation where we get a maximal decomposition of the module E by pure algebraic calculations, due to the fact that multiplicities equal 1 for all eigenvalues. We may encounter situations where we have eigen-module decomposition of E with multiplicities of eigenvalues larger than 1.

Theorem 9.2. *Let (E, δ) be a $\mathcal{D}$-module of rank n, and X a symmetry of E. If X has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, of multiplicities m_i respectively, with $\sum_{i=1}^{k} m_i = n$, then*

$$E = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k} . \tag{182}$$

Proof. The E_{λ_i} are non-empty, non-intersecting sub-$\mathcal{D}$-modules of E. The sum of their ranks over $\mathcal{A}$ equals the rank of E, thus they span the whole of E and their direct sum equals E. □

Note: Knowing a decomposition of E as in Theorem 9.2 corresponds to knowing a set of equations of lower order whose solution spaces span the solution space of the original equation. Assume that we know a decomposition of E into modules E_{λ_j}, i.e. we know a set of "eigenvectors" $\{w_1^j, \ldots, w_{m_j}^j\}$ in E that span E_{λ_j}. To identify the equation corresponding to (E_{λ_i}, δ), simply apply δ to the w_i^j. If

$$\delta \, \underline{w}^j = A_j(x) \, \underline{w}^j$$

for some matrix $A_j(x)$, then the corresponding ODE is the $m_j \times m_j$ system

$$\underline{f}' + A_j^T \underline{f} = 0 \tag{183}$$

If $\underline{f}_i^j$, $i = 1, \ldots, m_j$ are solutions of the respective (E_{λ_j}, δ) systems, then $E^{\#}$ is spanned by

$$E^{\#} \;=\; \mathrm{Span}_{\mathbb{R}}\{\; \underline{f}_i^j \cdot \underline{w}^{jT} \;,\; i = 1, .., m_j \;,\; j = 1, .., k \;\} \;\subset\; E\,.$$

9.2. **Solving and decomposition procedures for solvable algebras.** We turn to study $\mathcal{D}$-modules (E, δ) with a solvable symmetry algebra. Following the nature of solvable Lie algebras, we shall describe a procedure whose aim is to identify a chain $E_i \subset E_{i-1}$ of sub $\mathcal{D}$-modules in E of codimension 1 at each step so as to solve the total equation by combined algebraic operations and quadratures. We begin by recalling the Lie Theorem for representations of solvable Lie algebras.

Theorem 9.3. *(Lie) Let $\mathfrak{g}$ be a solvable Lie algebra over a base field F, $char(F) = 0$ and F algebraically closed. Given a representation $\mathfrak{g} \to End_F(V)$, $V \neq 0$ a finite dimensional vector space over F, there exists a non-zero $v \in V$ such that it is a common eigenvector for the whole action of $\mathfrak{g}$. I. e.*

$$\rho(g)(v) = \lambda(g)v\ , \quad \forall g \in \mathfrak{g}$$

for a weight $\lambda \in \mathfrak{g}^*$.

As seen earlier, structures on the vector space level $V = E^{\#}$ can be lifted to the $\mathcal{D}$-module E, due to Theorem 3.3. To apply Lie's Theorem in full generality we need the base field, F, to be algebraically closed, so if needed we may assume that we work with $\mathbb{C}$-valued smooth real functions, i.e with $A_C = C^\infty(\mathbb{R}, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} A$, δ_{A_C} being the usual derivative in x. The analogue of the Lie Theorem for $\mathcal{D}$-modules then reads as follows.

Theorem 9.4. *($\mathcal{D}$-Lie) Let $\mathfrak{g}$ be a solvable algebra under the conditions in Theorem 9.3, and a symmetry algebra of (E, δ), a $\mathcal{D}$-module over (A_C, δ_{A_C}). Then there exists $\beta \in E$ and $\lambda \in \mathfrak{g}^* \otimes \mathbb{C}$ such that*

$$\rho(g)(\beta) \;=\; \lambda(g)\beta\ ,\;\; \forall g \in \mathfrak{g}\,.$$

Proof. Applying Lie's Theorem to the representation

$$\rho : \mathfrak{g} \to End_{\mathbb{C}}(V),$$

with $V = E^{\#}$, ensures that there exists an element $v \in V$ which is a common eigenvector of the dual representation, with a corresponding $\lambda \in \mathfrak{g}^* \otimes \mathbb{C}$. Any multiple $\beta = f(x)v \in E$, $f(x) \in A_{\mathbb{C}}$ satisfies $g(\beta) = \lambda(g)\beta$, the action of $\mathfrak{g}$ being linear in functions. ∎

Definition 9.1. *Given a symmetry algebra* $\mathfrak{g}$ *of an equation* (E, δ) *with a corresponding representation* $\rho : \mathfrak{g} \to End_{A_C}(E)$*. An element* $\lambda \in \mathfrak{g}^* \otimes \mathbb{C}$ *such that the associated sub-module*

$$E_\lambda \;=\; \{\; \beta \in E \mid \; \rho(g)(\beta) = \lambda(g)\beta \;\; \forall g \in \mathfrak{g} \;\} \;\subset\; E \tag{184}$$

is non-empty, is called a weight of the representation. The rank of E_λ *over* A_C *is called the multiplicity of* λ*, and* E_λ *the associated eigen-sub module.*

Proposition 9.2. *For a weight* λ *of a representation of a symmetry algebra* $\mathfrak{g}$ *of an equation* (E, δ)*,* E_λ *is a sub-*$\mathcal{D}$*-module of* E*.*

Proof. The proof is almost identical to the proof of Proposition 9.1. E_λ is obviously a sub-module, and $\delta(E_\lambda) \subset E_\lambda$ since

$$\rho(g)(\delta h) \;=\; \delta(\rho(g)(h)) \;=\; \delta(\lambda(g)h) = \lambda(g) \cdot \delta h$$

for any $g \in \mathfrak{g}$, $h \in E_\lambda$. Thus E_λ is a $\mathcal{D}$-module. □

The next step will be to recognize that the existence of a common eigenvector of the *dual representation* of $\mathfrak{g}$ into E^* ensures that there exists a sub-module $E_1 \subset E$ of codimension 1.

Corollary 9.1. *There exists* $\alpha \in E^*$ *and* $\lambda \in \mathfrak{g}^* \otimes \mathbb{C}$ *such that*

$$\rho^*(g)(\alpha) \;=\; \lambda(g) \cdot \alpha \quad \forall g \in \mathfrak{g}\,.$$

If the multiplicity of λ*, i. e.* $rank_{\mathcal{A}_C} E^*_\lambda$*, is* 1*, then*

$$E' \;=\; \ker\alpha \;\subset\; E \tag{185}$$

is a codimension 1 *sub-*$\mathcal{D}$*-module of* E*, which is stable under the representation of* $\mathfrak{g}$*.*

Proof. The existence of α follows directly from Theorem 9.4. $\ker\alpha = Ann(E^*_\lambda)$ is a sub-module in E, due to the A_C-linearity of $\alpha : E \to A_C$. Also,

$$< \alpha, \rho(g)h > = < \rho^*(g)\alpha, h > = \lambda(g) < \alpha, h > \,, \tag{186}$$

so $\rho(g)(\ker\alpha) \subset \ker\alpha$ for all $g \in \mathfrak{g}$. E^*_λ is of rank 1, so α spans E^*_λ, with

$$\delta(\alpha) = f(x)\alpha$$

for some $f(x) \in A_C$. By definition

$$\delta\alpha = \alpha \circ \delta - \delta \circ \alpha$$

so, combining the two yields

$$\alpha(\delta h) \;=\; (\delta\alpha)(h) + \delta(\alpha(h)) \;=\; (f(x) + \delta\,)(\alpha(h)).$$

Whence, $h \in \ker \alpha$ implies that $\delta(h) \in \ker \alpha$, thus $\ker \alpha$ is a $\mathcal{D}$-module. □

Theorem 9.5. *Given a $\mathcal{D}$-module (E, δ) with a sub-$\mathcal{D}$-module*

$$E' \subset E$$

of codimension 1 *for which we know a full set of solutions. Then we can solve the whole of E by quadratures.*

Proof. Let $h_1, \dots h_{n-1} \in E^{\#}$ be a basis of E'. Pick any element $h_n \in E \setminus E'$, $\delta h_n = \sum_{i=1}^{n} f_i(x) h_i$. For a general element $s = \sum_{i=1}^{n} s_i(x) h_i \in E$ we get

$$\delta s = \sum_{i=1}^{n} (s_i' + f_i s_n) h_i \ .$$

Thus $\delta s = 0$ if and only if

$$\begin{aligned} s_1' + f_1 s_n &= 0 \\ s_2' + f_2 s_n &= 0 \\ &\vdots \\ s_n' + f_n s_n &= 0 \end{aligned}$$

i. e. $s_i' = -f_i v$ where v solves the last equation, $v' + f_n v = 0$, and $\{s, h_1, \dots, h_{n-1}\} \subset E^{\#}$ is a basis of the whole module E over A. □

So, if we have a procedure for stepwise identifying sub-modules of codimension 1, starting with $E_1 \subset E$, we can solve the equation by quadratures.

Theorem 9.6. *Let (E, δ) be a $\mathcal{D}$-module with a solvable Lie algebra of symmetries. Then there exists a filtration of E by sub-$\mathcal{D}$-modules*

$$0 \subset E_n \subset \dots \subset E_2 \subset E_1 \subset E \ ,$$

where $E_{i+1} \subset E_i$ is of codimension 1 *at each step.*

Proof. Applying Lie's Theorem for vector spaces to the dual representation

$$\rho^* : \mathfrak{g} \to End_{\mathbb{C}}(V^*)$$

ensures that there exists an element $v^* \in V^*$, $V^* = (E^*)^{\#}$, which is a common eigenvector of the dual representation, with a corresponding $\lambda \in \mathfrak{g}^* \otimes \mathbb{C}$. It is clear that

$$V_1 = \ker v^* \subset V$$

is a sub-space of codimension 1 of V. Denote by E_1 the sub-module of E generated by $kerv^* \subset V^*$ over $\mathcal{A}_C$,

$$E_1 \;=\; \{ \sum f_i(x) v_i \,|\, f_i \in \mathcal{A}_C, v_i \in \ker v^* \} \;.$$

The module E_1 is in fact a $\mathcal{D}$-module, since $\delta(E_1) \subset E_1$, by

$$\delta(\sum f_i v_i) \;=\; \sum (f_i' v_i + f_i \delta v_i) \;=\; \sum f_i' v_i \;\in E_1 \;,$$

for a general element $\sum f_i v_i$ in E_1.
Moreover, E_1, as well as $\ker v^*$, is stable under the action of $\mathfrak{g}$:

$$< v^*, \rho(g)(w) > = < \rho^*(g) v^*, w > = \; \lambda(g) < v^*, w > ,$$

so $w \in \ker v^* \subset V$ implies that $\rho(g)(w) \in \ker v^*$, for any $g \in \mathfrak{g}$. The $\mathcal{A}_C$-linearity of v^* gives the same result for E_1, thus the representation of $\mathfrak{g}$ restricts to E_1. Repeating the procedure $n = rank(E)$ times, starting with E_1, proves that the desired filtration exists. □

Note to Theorem 9.6: Theorem 9.6 merely says something about the *existence* of such a filtration, it uses the underlying vector space V^*, which is of course, in general unavailable to us, since knowing it corresponds to having solved the equation in the first place. In practice we will always work with the representation into the module E. The main obstruction in this algorithm to reduce the problem to quadratures, is to get codimensions 1 for the desired sub-$\mathcal{D}$-modules with pure *algebraic* tools.

Theorem 9.7. *Let (E, δ) be a $\mathcal{D}$-module of rank n with a solvable symmetry algebra $\mathfrak{g}$. If there are n distinct weights $\lambda_1, \ldots, \lambda_n$ of multiplicity 1 of the dual representation of $\mathfrak{g}$ into E^*, then a filtration*

$$0 \subset E_n \subset \ldots \subset E_2 \subset E_1 \subset E \;, \quad codim(E_i \subset E_{i-1}) = 1 \;,$$

can be found directly, whence E can be solved directly by quadratures.

Proof. Given a filtration of sub-$\mathcal{D}$-modules as above, Theorem 9.5 explains how to stepwise solve E by quadratures, starting with the first order equation E_n. To find the filtration, start with an arbitrary eigenvalue λ_1 of the dual representation, find $E^*_{\lambda_1} \;=\; \{ \alpha \in E^* \,|\, \rho^*(g)(\alpha) = \lambda_1(g)(\alpha) \}$, and take E_1 to be the annihilator of $E^*_{\lambda_1}$ in E,

$$E_1 \;=\; Ann(E^*_{\lambda_1}) \;=\; \{ e \in E \,|\, \alpha(e) = 0 \,,\, \forall \alpha \in E^*_{\lambda_1} \} \,.$$

The $\mathcal{D}$-module E_1 is of codimension 1 in E. An arbitrary remaining λ_i will produce a sub-$\mathcal{D}$-module $E^*_{\lambda_i}$ with $E^*_{\lambda_i} \cap E^*_{\lambda_1} = 0$. We may choose λ_2, and take

$$E_2 \;=\; Ann(E^*_{\lambda_2}) \;\subset\; E_1 \subset E \,,$$

which is again necessarily of codimension 1 in E_1. Repeat for the remaining eigenvalues, and get the whole filtration. □

10. Equations with semisimple symmetry algebras

Semisimple algebras are popular in representation theory, as there is a general theory on how to decompose representations of semisimple Lie algebras into irreducible representations, and up to isomorphisms more or less everything is known about irreducible representations for the classical (semi)simple Lie algebras. By using Theorem 3.3 we are now ready to transfer results on representations of Lie algebras into vector spaces, to $\mathcal{D}$-modules and ODEs. We find that for a number of equations with semisimple symmetry algebras we obtain solvability by algebraic methods. An algorithm to decompose and solve equations is provided.

10.1. **General results for semisimple symmetry algebras.** Some results from representation theory of Lie algebras into vector spaces depend on having an algebraically closed base field, as seen in the section on solvable algebras. This problem occurs whenever we encounter eigenvalue calculations; to be able to say something about roots of characteristic polynomials in general, we need algebraic closure of the coefficient field. And this is certainly a crucial part of studying representations of semisimple algebras, where calculating roots and weights is more or less the whole trick. Thus, we may, as in Section 9, choose to work with modules over complex valued functions, $\mathcal{A}_C = C^\infty(\mathbb{R}, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{A}$, with δ_{A_C} being the usual derivative in real variable x.

Let $\mathfrak{g}$ be a semisimple algebra (over $\mathbb{C}$ whenever algebraic closure of the base field is necessary), and $R(\mathfrak{g})$ the associated Grothendieck ring of isomorphism classes of finite dimensional vector space representations of $\mathfrak{g}$.

Definition 10.1. *Denote by $\mathcal{D}(\mathfrak{g})$ the ring of isomorphism classes of $\mathcal{D}$-modules with a semisimple symmetry algebra $\mathfrak{g}$. We shall refer to $\mathcal{D}(\mathfrak{g})$ as the symmetry ring of $\mathfrak{g}$.*

Let $\omega_1, \ldots, \omega_n$ be a set of fundamental weights for $\mathfrak{g}$, and let $\Gamma_1, \ldots, \Gamma_n$ denote the corresponding isomorphism classes in $R(\mathfrak{g})$ with highest weights $\omega_1, \ldots, \omega_n$. Recall the following result from the theory of representations of semisimple Lie-algebras, see e.g. [4].

Theorem 10.1. *The representation ring $R(\mathfrak{g})$ is a polynomial ring in the variables $\Gamma_1, \ldots, \Gamma_n$.*

Combining Theorems 10.1 and 3.3 we immediately get the following.

Theorem 10.2. *For a semisimple Lie algebra $\mathfrak{g}$ the symmetry ring $\mathcal{D}(\mathfrak{g})$ is a polynomial ring in classes of $\mathcal{D}$-modules $E_1, \dots, E_n$ such that for each $i = 1, \dots n$ $E_i^{\#}$ is isomorphic to Γ_i as a $\mathfrak{g}$-space.*

We will call $\mathcal{D}$-modules $E_1, \dots, E_n$ generators of $\mathcal{D}(\mathfrak{g})$.

Corollary 10.1. *Let (E, δ) be a $\mathcal{D}$-module with a semisimple algebra of symmetries $\mathfrak{g}$. Then E is isomorphic, as a $\mathfrak{g}$-module, to a polynomial $P(E_i)$ in generators $E_1, \dots, E_n$ of $\mathcal{D}(\mathfrak{g})$.*

An equation that corresponds to an irreducible representation Γ_i, associated to a highest weight ω_i of $\mathfrak{g}$ as above we will call a *model equation* for this symmetry algebra.

10.2. $\mathfrak{sl}_2$ **equations.** Representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ has a special place in the view of symmetric powers of second order equations, given that all irreducible representations of $\mathfrak{sl}_2(\mathbb{R})$ are isomorphic to symmetric powers of the standard two dimensional representation.

10.3. **Schrödinger equations.** Equations on the following type we will denote as being of Schrödinger type, with potential $W(x)$.

$$y'' + W(x)y = 0 \tag{187}$$

It is self adjoint, hence the corresponding module E is isomorphic to E^*. Conditions for

$$\Delta = a_1 + a_2\partial$$

to be a symmetry operator of (187) are by direct calculation found to be that a_2 solves

$$z''' + 4Wz' + 2W'z = 0 \tag{188}$$

and that $a_1 = \frac{-a_2'}{2}$. Thus symmetries are given by generating functions that solve (188), i.e.

$$\Delta_a = -\frac{a'}{2} + a\partial \ , \quad a \in \mathrm{Sol}(188)$$

In Section 8.4 we studied the symmetry equation of general second order equations in detail. Recall that the non-trivial symmetries of an equation $Ly = 0$ were generated by solutions of the equation $S^2(L_0)$ where

$$L_0 = \partial^2 + (\,\frac{1}{2}a_2' - \frac{1}{4}a_2^2 - a_1\,)$$

For a Schrödinger equation (187) $a_2 = 0$, $a_1 = -W(x)$, so

$$L_0 \;=\; \partial^2 - a_1 \;=\; \partial^2 + W \;=\; L \tag{189}$$

Theorem 10.3. *For a Schrödinger equation*

$$L(y) \;=\; y'' + Wy \;=\; 0 \tag{190}$$

the symmetry equation is its second symmetric power, i. e.

$$Sym(L) \;=\; \{\, \Delta_a \;=\; -\frac{a'}{2} + a\partial \mid a \text{ solves } S^2(L)(z) = 0 \,\}$$

where

$$S^2(L) = \partial^3 + 4W\partial + 2W' \tag{191}$$

Moreover,

$$Sym(L) \;\cong\; \mathfrak{sl}_2 \tag{192}$$

as Lie algebras, where the Lie bracket operation in $Sym(L)$ is the commutator of symmetry operators.

Proof. The calculations in Section 8.4 and discussions above prove that $Sym(L) \cong S^2(E)^{\#} \cong Sol(188)$. $Sym(L)$ is a Lie algebra with respect to commutators of operators, as discussed in Section 8.2, on symmetry operators. To prove that it is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ one may calculate symbolically with a set of fundamental solutions of (188), using differential relations, and get the desired result. However, the proposition that follows enables us to prove this in yet another way. □

Proposition 10.1. *The commutator of symmetry operators in Theorem 10.3 corresponds to the following Lie bracket $\langle \cdot, \cdot \rangle$ on the solution space of the symmetry equation* (188)*, given by*

$$[\Delta_a, \Delta_b] = \Delta_{\langle a,b \rangle} \tag{193}$$

which yields simply

$$\langle a, b \rangle = ab' - a'b$$

for $a, b \in Sol(188)$. Denote the equivalent Lie bracket on $S^2(E)^{\#}$

$$[\,\cdot\,,\,\cdot\,] \;:\; S^2(E)^{\#} \times S^2(E)^{\#} \to S^2(E)^{\#} \tag{194}$$

where

$$[\,\theta_a\;,\;\theta_b\,] \;=\; \theta_{\langle a,b \rangle} \tag{195}$$

for elements $\theta_a, \theta_b \in S^2(E)^{\#}$ generated by solutions a, b of (188)*. The solution space of* (188) *is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ with respect to this bracket.*

Proof. Calculating the commutator $[\Delta_a, \Delta_b]$ directly gives precisely the formula $\langle a, b\rangle = ab' - a'b$. Theorem 4.1 asserts that solutions a, b of the $S^2(E)$-equation are linear combinations of solutions u^2, uv, v^2 where u, v are linear independent solutions of the Schrödinger equation. Whence, we may calculate all brackets of elements from a basis $\{u^2, uv, v^2\}$ of the solution space of the symmetry equation (188), and will find that as a Lie algebra it is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. □

Corollary 10.2. *The Lie bracket* $[\,\cdot\,,\,\cdot\,]$ *on* $S^2(E)^\#$ *in Proposition 10.1 extends by* $\mathcal{A}$*-linearity to a bracket*

$$[\,\cdot\,,\,\cdot\,]\ :\ S^2(E) \times S^2(E) \to S^2(E) \tag{196}$$

with respect to which $(S^2(E), \delta)$ *is a* $\mathcal{D}$*-Lie-algebra.*

Proof. Theorem 3.3 states that $S^2(E)^\#$ spans $S^2(E)$ over $\mathcal{A}$, thus the bracket extends in a well-defined way by $\mathcal{A}$-linearity to $S^2(E)$, with the Lie-bracket properties intact. We need only check that $[\,\cdot\,,\,\cdot\,]$ commutes with δ in accordance with Definition 3.3. Let $X = f_1\theta_1, Y = f_2\theta_2$, with $f_i \in \mathcal{A}$, $\theta_i \in S^2(E)^\#$. Then

$$\begin{aligned}
\delta[\,X, Y\,] \ &=\ \delta[\,f_1\theta_1, f_2\theta_2\,] \ =\ \delta(\,f_1 f_2\,[\,\theta_1, \theta_2\,]) \\
&=\ (f_1 f_2)'\,[\,\theta_1, \theta_2\,] + f_1 f_2\,\delta[\,\theta_1, \theta_2\,] \\
&=\ (f_1' f_2 + f_1 f_2')\,[\,\theta_1, \theta_2\,] \ =\ [\,f_1'\theta_1, f_2\theta_2\,] + [\,f_1\theta_1, f_2'\theta_2\,] \\
&=\ [\,\delta(f_1\theta_1), f_2\theta_2\,] + [\,f_1\theta_1, \delta(f_2\theta_2)\,] \\
&=\ [\,\delta X, Y\,] + [\,X, \delta Y\,]
\end{aligned}$$

whence $S^2(E)$ is a $\mathcal{D}$-Lie algebra. □

10.4. **Symmetric powers of a Schrödinger equation.** We shall see that the Schrödinger equations have special properties. From our basic equation we can derive a whole hierarchy of new equations $S^k(E)$. Throughout this section we will work with $S^k(E^*)$, choosing to work with the dual module E^* merely simplifies calculations, and generates exactly the same equations as the module E.

In Section 4 symmetric powers of second order equations were discussed in some detail. Let (E, δ) be the $\mathcal{D}$-module corresponding to

$$y'' + Wy = 0 \tag{197}$$

with basis $\{e_1^*, e_2^*\}$ dual to the primitive element basis of E. As before, denote the induced basis of $S^k(E^*)$

$$\{\ \alpha_l \ =\ (e_1^*)^{k-l+1} \cdot (e_2^*)^{l-1}\ \ ,\ l = 1, \ldots, k+1\ \} \tag{198}$$

For Schrödinger equations with $a_1 = -W(x)$, $a_2 = 0$ we get that

$$\delta \,:\, \alpha_l \;\mapsto\; (k-l+1)W\,\alpha_{l+1} \;-\; (l-1)\,\alpha_{l-1} \tag{199}$$

for $l = 1, \ldots, k+1$. For a general element

$$\theta \;=\; \sum_{l=1}^{k+1} g_l(x)\,\alpha_l \tag{200}$$

in $S^k(E^*)$, the requirement $\delta\theta = 0$ results in the system of $k+1$ equations

$$g_s' \;+\; (k-s+2)W g_{s-1} \;-\; s\,g_{s+1} \;=\; 0 \tag{201}$$

$s = 1, \ldots, k+1$. Thus, for Schrödinger equations Proposition 4.2 has the following form.

Proposition 10.2. *For each k the kernel $S^k(E^*)^{\#}$ consists of elements*

$$\theta_y \;=\; y\alpha_1 \;+\; y'\alpha_2 \;+\; \sum_{l=3}^{k+1} g_l(y)\,\alpha_l \;\;, \qquad y \in \mathrm{Sol}(k) \tag{202}$$

where

$$g_l \;=\; \frac{1}{l-1}[\,(k-l+3)\,W \cdot g_{l-2} \;+\; g_{l-1}'\,] \tag{203}$$

for $l = 2, .., k+1$, where $g_1 = y$ solves the $S^k(E^*)$ equation, i.e. the equation in y we get from setting

$$\delta\theta_y \;=\; 0$$

for θ_y on the form (202)*, with g_l-s expressed in derivatives of y.*

Fix (k) below to denote the equation $S^k(E^*)$. They are of the form

$y'' + Wy = 0$

$y''' + 4Wy' + 2W'y = 0$

$y^{(4)} + 10Wy'' + 10W'y' + (9W^2 + 3W'')y = 0$

$y^{(5)} + 20Wy''' + 30W'y'' + [64W^2 + 18W'']y' + [64WW' + 4W''']y = 0$

and so on. Focusing on the hierarchy of symmetric powers of Schrödinger equations we shall now see that the bracket operation on $S^2(E^*)$ extends to the whole hierarchy. The bracket in Corollary 10.2 can be obtained in a different way.

Proposition 10.3. *Let (E, δ) be the $\mathcal{D}$-module corresponding to the Schrödinger equation* (197)*. The equation corresponding to its second exterior power, $(\bigwedge^2(E^*), \delta)$, is*

$$u' \;=\; 0\,.$$

that is, $\Omega = e_1 \wedge e_2$ is δ-invariant. Here $\{e_1, e_2\}$ is the standard primitive element basis of E. Moreover, Ω determines a δ-invariant skew-symmetric, $\mathcal{A}$-linear bracket operation on E^* defined by

$$[\,\theta_1\,,\,\theta_2\,] = \langle\,\theta_1\,\wedge\,\theta_2\,,\,\Omega\,\rangle$$

for $\theta_1, \theta_2 \in E^*$.

Proof. The skew-symmetric form $\Omega = e_1 \wedge e_2 \in \bigwedge^2(E)$, is in the kernel of δ:

$$\delta\Omega \,=\, e_2 \wedge e_2 + e_1 \wedge (a_1 e_1 + a_2 e_2) = a_2 e_1 \wedge e_2 = 0$$

since $a_1 = -W, a_2 = 0$. The bracket is thus δ-invariant and obviously skew-symmetric and $\mathcal{A}$-linear, due to the properties of Ω. □

Proposition 10.4. *Given a $\mathcal{D}$-module (E, δ) corresponding to an equation of Schrödinger type as in Theorem 10.3, there is a unique skew-symmetric bracket*

$$[\,\cdot\,,\,\cdot\,] : S^m(E^*) \times S^n(E^*) \to S^{m+n-2}(E^*)$$

for all $m, n \geq 1$ which is

(i) *$\mathcal{A}$-linear.*
(ii) $[\,f \cdot g\,,\,h\,] = f \cdot [\,g\,,\,h\,] + g \cdot [\,f\,,\,h\,] \qquad \forall f, g, h \in S^{\cdot}(E^*)$
(iii) *For $n = m = 1$ the bracket coincides with the bracket in Proposition 10.3.*

Proof. Given the bracket operation in Proposition 10.3 the properties $(i)-(ii)$ determine its extension to symmetric powers $S^m(E^*) \times S^n(E^*)$. □

We immediately observe that Ω being δ-invariant implies that so is the extended bracket $[\,\cdot\,,\,\cdot\,]$. Thus, it restricts to kernels of δ-s as follows.

Proposition 10.5. *The bracket operation in Proposition 10.4 restricts to kernels $(S^k(E^*))^\# = S^k(V)$, where $V = (E^*)^\#$. The bracket*

$$[\,\cdot\,,\,\cdot\,] \;:\; S^m(V) \times S^n(V) \;\longrightarrow\; S^{m+n-2}(V)$$

has the properties

(i) *$\mathbb{R}$-linearity*
(ii) $[\,f \cdot g\,,\,h\,] = f \cdot [\,g\,,\,h\,] + g \cdot [\,f\,,\,h\,]\,, \quad \forall f, g, h \in S^{\cdot}(V)$

This is obviously equivalent to a bracket on the solution spaces of the symmetric power equations,

$$[\,\cdot\,,\,\cdot\,] : \mathrm{Sol}(m) \times \mathrm{Sol}(n) \to \mathrm{Sol}(m + n - 2)$$

with

$$[\,\theta_y\,,\,\theta_z\,] \;=\; \theta_{[y,z]}$$

for solutions y, z of $S^m(E^*)$ and $S^n(E^*)$ equations respectively. This means that solutions of the $S^2(E^*)$ equation (188) produce symmetries of *all* equations $S^k(E^*)$, and not only E^*.

Theorem 10.4. *Any solution $a \in \mathrm{Sol}(S^2(E^*))$ produces a symmetry*

$$\mathcal{O}^m_{\theta_a} \;\stackrel{\mathrm{def}}{=}\; [\,\theta_a\,,\,\cdot\,] \;:\; S^m(E^*) \;\longrightarrow\; S^m(E^*)$$

The corresponding symmetry operator is

$$\mathcal{O} \;=\; \mathcal{O}^m_a \;:\; \mathrm{Sol}(m) \longrightarrow \mathrm{Sol}(m) \tag{204}$$

with the correspondence

$$\mathcal{O}^m_{\theta_a}(\beta_y) \;=\; \beta_{\mathcal{O}^m_a(y)}$$

The precise expression is

$$\mathcal{O}^m_a \;=\; \frac{1}{2}(\,-ma' + 2a\partial\,) \tag{205}$$

for any $m \geq 1, a \in \mathrm{Sol}(S^2(E^*))$. Due to Theorem 3.3 we know that two linearly independent solutions

$$u, v \in \mathrm{Sol}(\,E^*\,) \;\Rightarrow\; \text{basis } \{\,\theta_u, \theta_v\,\} \;\subset V \text{ of } \; E^* \; \text{ over } \; \mathcal{A} \tag{206}$$

Hence, for any $k \geq 1$,

$$\{\,\theta_{u^{k-l}v^l} \;=\; \theta_u^{k-l}\cdot\theta_v^l\,\} \;\;\subset\;\; S^k(V)\,, \qquad 0 \leq l \leq k \tag{207}$$

is the basis of $S^k(E^*)$ over $\mathcal{A}$ corresponding to the fundamental set of solutions

$$u^{k-l}v^l \;\in\; \mathrm{Sol}(k)\,, \quad 0 \leq l \leq k$$

It is now easy to calculate the action of the symmetries

$$\mathcal{O}^k_1 = \mathcal{O}^k_{u^2}\,,\;\; \mathcal{O}^k_2 = \mathcal{O}^k_{uv}\,,\;\; \mathcal{O}^k_3 = \mathcal{O}^k_{v^2} \tag{208}$$

on basis elements $\theta_{u^{k-l}v^l}$ just in terms of brackets of the generating functions.

Theorem 10.5. *For any $k \geq 1$ the symmetries of $S^k(E^*)$*

$$X_+ = -\frac{1}{2c}\mathcal{O}^k_3\,,\;\; X_- = \frac{1}{2c}\mathcal{O}^k_1 \;\text{ and }\; H = \frac{1}{c}\mathcal{O}^k_2$$

where $c = \langle u, v\rangle \in \mathbb{R}$ constitute a basis of the $\mathfrak{sl}_2(\mathbb{R})$-algebra of symmetries $\cong$ Sol(ii) with commutators

$$[X_+, X_-] = H\,,\;\; [H, X_+] = 2X_+\,,\;\; [H, X_-] = -2X_-$$

Hence, $S^k(E^*)$ decomposes into rank 1 sub-$\mathcal{D}$-modules corresponding to different eigenvalues of H

$$S^k(E^*) \;=\; \langle\theta_{u^k}\rangle_{\mathcal{A}} \,\oplus\, \langle\theta_{u^{k-1}v}\rangle_{\mathcal{A}} \,\oplus\, \ldots \,\oplus \langle\theta_{v^k}\rangle_{\mathcal{A}}$$

$-k, -k+2, \ldots, k-2, k$

Proof. The commutator relations are calculated directly in terms of the brackets

$$[v^2, u^2] = -4cuv \;, \quad [uv, v^2] = 2cv^2 \;, \quad [uv, u^2] = -2cu^2$$

and knowing the form of the operators $\mathcal{O}_i^k$ from (205). Furthermore,

$$H(\theta_{u^{k-l}v^l}) = (2l-k)\theta_{u^{k-l}v^l}$$

for $0 \le l \le k$. Certainly

$$\begin{aligned} X_+ \;&:\; Eig_\lambda(H) \;\to\; Eig_{\lambda+2}(H) \\ X_- \;&:\; Eig_\lambda(H) \;\to\; Eig_{\lambda-2}(H) \end{aligned}$$

where $Eig_\lambda(H)$ denotes the eigen-submodule of H corresponding to the eigenvalue λ. It is generated over $\mathcal{A}$ by the eigenspace in $S^k(V)$ of H. □

Theorem 10.6. *A $\mathcal{D}$-module (E, δ) with a representation of symmetries*

$$\mathfrak{sl}_2(\mathbb{R}) \to End_{\mathcal{A}}(E)$$

is decomposable into a direct sum of $\mathcal{D}$-modules

$$E = \bigoplus_{i=1}^{m} E_i$$

where each E_i is an irreducible subrepresentation of E. Moreover, each E_i is isomorphic to $S^{n_i}(M_i)$ as a $\mathcal{D}$-module and as an $\mathfrak{sl}_2(\mathbb{R})$-module, for a rank 2 $\mathcal{D}$-module M_i.

Proof. The representation of symmetries into E restricts to a representation of $\mathfrak{sl}_2(\mathbb{R})$ in the $\mathbb{R}$-vector space $V = E^{\#} \subset E$, hence it decomposes into a direct sum of representations

$$V = \oplus_{i=1}^m V_i$$

where the V_i are subspaces of V such that restricted to V_i the representation is irreducible. But any irreducible representation of $\mathfrak{sl}_2(\mathbb{R})$ into a vector space of dimension n_i $(< \infty)$ is isomorphic to the $(n_i - 1)$th symmetric power of the standard two dimensional representation, i.e. each $V_i \cong S^{n_i-1}(W_i)$, where $\dim_{\mathbb{R}}(W_i) = 2$ for all i. Recall that the symmetries commute with δ in E, hence $E_i = \mathcal{A} \cdot V_i$, the $\mathcal{A}$-module generated of V_i over $\mathcal{A}$. Due to Theorem 3.3 the vector space isomorphism

$V_i \cong S^{n_i-1}(W_i)$ lifts to an isomorphism of $\mathcal{D}$-modules $E_i \cong S^{n_i}(M_i)$, where $S^{n_i}(M_i)$ is generated of $S^{n_i}(W_i)$ over $\mathcal{A}$.

□

Corollary 10.3. *Let (E,δ) be a $\mathcal{D}$-module with an $\mathfrak{sl}_2(\mathbb{R})$ algebra of symmetries as in Theorem 10.6. If the irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules E_i in its decomposition are of distinct ranks, then the equation corresponding to E can be solved by algebraic operations and quadrature.*

Corollary 10.4. *Let (E,δ) be a $\mathcal{D}$-module with an $\mathfrak{sl}_2(\mathbb{R})$ algebra of symmetries as in Theorem 10.6. If there are irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules E_i of ranks $m_1,\dots,m_k > 1$ in its decomposition, then the obstruction to solve the equation corresponding to E by algebraic operations and quadrature is k first order systems of ODEs, with size $m_1 \times m_1,\dots,m_k \times m_k$ respectively.*

See Subsection 10.5 for a detailed account on how to decompose and solve $\mathfrak{sl}_2(\mathbb{R})$-equations.

Example 10.1. *A Schrödinger equation*

$$y'' + W(x)y = 0$$

is a model equation of $\mathfrak{sl}_2(\mathbb{R})$.

10.5. **Algorithm to solve $\mathfrak{sl}_2(\mathbb{R})$-equations.** The calculations preceding Theorem 10.6 tell us how we should approach $\mathfrak{sl}_2(\mathbb{R})$-modules in order to find its complete reduction, identify sub-$\mathcal{D}$-modules M_i as in Theorem 10.6, and eventually solve the original equation. Theorem 4.1 tells us that

$$\text{Solutions of } M_i \;\Rightarrow\; \text{Solutions of } S^{l_i}(M_i) \;\Rightarrow\; \text{Solutions of } E$$

An outline of the algorithm is as follows. **Step 1** Given an $\mathfrak{sl}_2(\mathbb{R})$-$\mathcal{D}$-module (E,δ) as in Theorem 10.6, find a basis $\{X_+, X_-, H\}$ of its $\mathfrak{sl}_2(\mathbb{R})$-algebra of symmetries that satisfies the commutator relations in Theorem 10.5. Calculate the eigen-sub-modules E_{λ_i} in E corresponding to weights $\{\lambda_1,\dots,\lambda_k\}$ of the diagonal element H. This yields a decomposition

$$E = \oplus_{i=1}^k E_{\lambda_i}$$

such that

$$\begin{aligned} X_+ &: E_{\lambda_i} \to E_{\lambda_{i+2}} \quad \text{and} \\ X_- &: E_{\lambda_i} \to E_{\lambda_{i-2}} \end{aligned}$$

The rank of E_{λ_i} over $\mathcal{A}$ is the multiplicity of the weight λ_i, which we denote m_i. Given a decomposition of E as in 10.6, then the values of the weights are precisely integers $\lambda_j = 2j - n_i$, $0 \le j \le n_i$, $i = 1, \ldots, m$. **Step 2** Identify all λ_i-s of multiplicity $m_i = 1$. For each weight λ_i of multiplicity 1 any non-zero $w \in \mathrm{E}_{\lambda_i}$ with $X_+(w) = 0$ generates an irreducible $\mathfrak{sl}_2(\mathbb{R})$-module in E

$$\langle X_-^{n_i}(w)\rangle \oplus \ldots \oplus \langle X_-(w)\rangle \oplus \langle w\rangle, \tag{209}$$

where n_i is the smallest integer such that $X_-^{n_i}(w) = 0$. Moreover, this is a sub-$\mathcal{D}$-module in E, and it is isomorphic to $S^{n_i}(M_i)$, for a rank 2 "Schrödinger" module M_i as a $\mathcal{D}$-module and a $\mathfrak{sl}_2(\mathbb{R})$-module. Recall from Theorem 10.5 the structure of symmetric powers of a Schrödinger equations:

$$S^{n_i}(M_i) \;=\; \langle\theta_{u^{n_i}}\rangle_{\mathcal{A}} \,\oplus\, \langle\theta_{u^{n_i-1}v}\rangle_{\mathcal{A}} \,\oplus\, \ldots \,\oplus\, \langle\theta_{v^{n_i}}\rangle_{\mathcal{A}}\,,$$

for a fundamental set of solutions u, v of the Schrödinger equation corresponding to M_i.
To identify M_i take the fraction of the last coefficients in w and $X_-(w)$ respectively. It is a fraction

$$\alpha = \frac{v}{u} \tag{210}$$

of fundamental solutions of the M_i-equation. Denote the potential of that equation by $W(x)$. Differentiating and using the differential relations

$$u'' + W(x)u \;=\; 0 \;\;,\quad v'' + W(x)v \;=\; 0 \tag{211}$$

yields the following expression for W

$$W \;=\; \gamma^2 + \gamma' \tag{212}$$

where

$$\gamma \;=\; \frac{\ln(\alpha)'' \,-\, (\ln\alpha')^2}{2\,\ln\alpha'} \tag{213}$$

For $w = f(x)\theta_{v^k}$, $\delta w = f'(x)w$, so from $\delta w/w$ we get $\eta = (\ln f)'$. Integrating, we get

$$f = e^{\int \eta dx}, \tag{214}$$

and the last coefficient of w/f is v^k, from which we deduce v. Then, $u = v/\alpha$. **Step 3** "Remove" the irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules in Step 2 from the module E, i.e. work in their complement in E. For each weight λ denote the complement in E_λ of these sub-modules by $\widetilde{E}_\lambda$. If there are weights of "remaining" multiplicity 1, i.e. $rank(\widetilde{E}_\lambda) = 1$, repeat Step 2 for those weights. Identify the weight with highest integer value, denote it λ, and its remaining multiplicity $m > 1$. We still have that a non-zero $w \in \widetilde{E}_\lambda$ with $X_+(w) = 0$ generates an irreducible $\mathfrak{sl}_2(\mathbb{R})$-module in E, but we are no longer guaranteed that this is also a sub-$\mathcal{D}$-module. If w in addition satisfies the condition

$$\delta w \;=\; f(x)\, w\,, \tag{215}$$

then it generates a sub-$\mathcal{D}$-module in E as in Step 2. Given a basis $\{w_1, \ldots, w_m\}$ of $\widetilde{E}_\lambda$, applying δ yields

$$\underline{w} = A_i \underline{w} \tag{216}$$

for some matrix A_i with coefficients in $\mathcal{A}$. Solving the corresponding first order system

$$\underline{h}' + A_i^T \underline{h} = 0 \tag{217}$$

is the obstruction to identify the sub-$\mathcal{D}$-modules that are irreducible $\mathfrak{sl}_2(\mathbb{R})$-modules, as obtained in Step 2 above. Repeat for highest value weights successively to get the decomposition of E.

10.6. **Schrödinger equations with shared symmetries.** Returning to a base equation with potential $W(x)$

$$y'' + W(x)y = 0 \tag{218}$$

we may discuss, as is done in [11], which potentials share a symmetry $\Delta_z = -\frac{z'}{2} + z\partial$, z being a solution of the symmetry equation (188). Let $W_0(x)$ be a potential of an equation with symmetry Δ_z, fixed z. Then W_o is a particular solution of the symmetry equation viewed as a first order equation for $W(x)$:

$$z''' + 4Wz' + 2W'z = 0 \tag{219}$$

Integrating the separable homogeneous equation yields that equations with potentials on the form

$$W(x) = W_0(x) + \frac{c}{z^2}\,, \quad c \in \mathbb{R} \tag{220}$$

share the symmetry Δ_z. Recall also that a fundamental set of solutions $\{u, v\}$ of the base equation (218) with potential $W_0(x)$ generate a fundamental set of solutions $\{u^2, uv, v^2\}$ of the symmetry equation (219), hence all equations with potentials

$$W(x) = W_0(x) + \frac{c}{(\, c_1u^2 + c_2uv + c_3v^2 \,)^2} \tag{221}$$

where $c, c_i \in \mathbb{R}$ such that the denominator in the fraction is non-zero, are integrable (by quadratures).

Example 10.2. *Let* $W_0 = 0$*,* $\{u = a_1 + a_2 x\,,\, v = b_1 + b_2 x\}$*. Then equations with potentials*

$$W = \frac{c}{(c_1 u^2 + c_2 uv + c_3 v^2)^2}$$

are integrable in quadratures, symmetries $\Delta_{u^2}, \Delta_{uv}$ *and* Δ_{v^2}*.*

Example 10.3. *Let* $W_0 = \omega^2$*, with* $\{u = \cos(\omega x)\,,\, v = \sin(\omega x)\}$*. Then equations with potentials*

$$W = \omega^2 + \frac{c}{(c_1 \cos^2(\omega x) + c_2 \cos(\omega x)\sin(\omega x) + c_3 \sin^2(\omega x))^2}$$

are integrable in quadratures, symmetries

$$\Delta_{\cos^2(\omega x)}, \Delta_{\cos(\omega x)\sin(\omega x)} \text{ and } \Delta_{\sin^2(\omega x)}$$

or, equivalently

$$\Delta_1, \Delta_{\cos(2\omega x)} \text{ and } \Delta_{\sin(2\omega x)}$$

This is a way to generate new integrable base Schrödinger equations from simpler ones, with shared symmetries, which we may in turn take symmetric products and direct sums of and arrive at new solvable higher order equations.

10.7. **Model equations for $\mathfrak{sl}_3$.** Recall that Schrödinger equations are precisely the equations of order two that have the standard δ-invariant "volume form" $\Omega = e_1 \wedge e_2 \in \wedge^2(E)^{\#}$, which in turn corresponds nicely to the fact that $\mathfrak{sl}_2(\mathbb{R})$ is connected to the preservation of a volume form on a two dimensional space. We may expect that the geometric properties of the classic Lie algebras are reflected in the associated model equations. A search for model equations for $\mathfrak{sl}_3$ should thus point us towards third order equations with a δ-invariant standard volume form. A third order equation

$$y''' + f_1(x)y'' + f_2(x)y' + f_3(x)y = 0$$

with associated $\mathcal{D}$-module (E, δ) has invariant volume forms on the form $g(x)\Omega \in \wedge^3(E)$ for any $g(x) \in \mathcal{A}$ that solves

$$g' + f_1(x)g = 0$$

Hence any third order equation on the form

$$y''' + f(x)y' + g(x)y = 0$$

has an invariant standard volume form $\Omega = e_1 \wedge e_2 \wedge e_3 \in \wedge^3(E)$. A sub-example of the above third order equations is a general skew-adjoint equation

$$y''' - f(x)y' - \frac{1}{2}f'(x)y = 0\ . \tag{222}$$

For $L = \partial^3 - f(x)\partial - \frac{1}{2}f'(x)$ $Sym(L)$ decomposes into

$$Sym(L) = Sym_0 \oplus Sym_1$$

thus we may split calculation of symmetries to consider Sym_0 and Sym_1 separately. The symmetry equations separate into

$$p''' - fp - \frac{f'}{2}p = 0 \tag{223}$$

$$s^{(5)} - 5fs''' - \frac{15}{2}f's'' + (4f^2 - \frac{9}{2}f'')s' + (4ff' - f''')s = 0 \tag{224}$$

in the sense that

$$Sym_0(L) = \{\ \Delta^0_p \ = \ -p' + p\partial \ \mid \ p \text{ solves } (223)\ \}, \qquad \textit{and} \tag{225}$$

$$Sym_1(L) = \{\ \Delta^1_s \ = \ (\frac{s''}{6} - \frac{2fs}{3}) \ - \ \frac{s'}{2}\partial \ + \ s\,\partial^2 \ \mid \ s \text{ solves } (224)\ \} \tag{226}$$

We recall the graded structure of $Sym(L)$ from Section 8.2, and note that the commutators

$$[\Delta^0_p, \Delta^0_q] \ = \ \Delta^0_{\{p,q\}_{00}} \tag{227}$$

$$[\Delta^1_s, \Delta^0_p] \ = \ \Delta^1_{\{s,p\}_{01}} \tag{228}$$

$$[\Delta^1_s, \Delta^1_w] \ = \ \Delta^0_{\{s,w\}_{11}} \tag{229}$$

induce the following structure on the solution spaces of the symmetry equations:

$$\{p, q\}_{00} = pq' - p'q \tag{230}$$

$$\{s, p\}_{01} = 2sp' - s'p \tag{231}$$

$$\{s, w\}_{11} = \frac{1}{6}(s'''w - w'''s) + \frac{1}{4}(s'w'' - s''w') + \frac{2}{3}f(sw' - s'w) \tag{232}$$

where

$$\{\ \cdot\ ,\ \cdot\ \}_{ab} \ : \ Sol_a \times Sol_b \to Sol_{a+b}\ .$$

The Sym_0-equation (223) is equal to the original equation $Ly = 0$, which is again the second symmetric power of the Schrödinger equation (233)

$$B(y) \ = \ y'' - \frac{1}{4}fy = 0, \tag{233}$$

whereas the Sym_1-equation (224) is both the second symmetric power of (223) and the fourth symmetric power of (233). This observation makes calculations of the symmetry algebra easier, as we may do them in terms of powers of solutions to the basic Schrödinger equation (233). For any set $\{u, v\}$ of independent solutions of (233)

$$\{u^2, uv, v^2\} \quad \textit{and} \quad \{u^4, u^3v, u^2v^2, uv^3, v^4\}$$

are independent solutions of (223) and (224) respectively, and generate a full basis of $Sym(L)$. Considering the commutators of functions asserts that

$$Sym(L) \cong \mathfrak{sl}_3$$

and that the subalgebra $Sym_0(L) \subset Sym(L)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Its action on Sym_1 is precisely as described in **Section 10.2** on $\mathfrak{sl}_2(\mathbb{R})$-equations, of $S^2(B)$ into $S^4(B)$. For $\mathfrak{g} = \mathfrak{sl}_3$ the representation ring $R(\mathfrak{g})$ is generated by V and $\bigwedge^2(V) \cong V^*$, where $V = F^3$, $F = \mathbb{R}$ or $\mathbb{C}$ denotes the standard (matrix) representation.

Theorem 10.7. *Equation* (222) *with corresponding* $\mathcal{D}$*-module* (E, δ) *is a model equation for the standard representation of* $\mathfrak{sl}_3$. $(E^*, \delta) \cong (\bigwedge^2(E), \delta)$ *also correspond to equation* (222).

Acknowledgements

Thanks to Professor V. V. Lychagin for sharing his ideas and engaging in fruitful discussions.

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Received April 13, 2005