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ON ORDERED LEFT GROUPS

(submitted by Arslanov)

Abstract. Our purpose is to give some similarities and some differences concerning the left groups between semigroups and ordered semigroups. Unlike in semigroups (without order) if an ordered semigroup is left simple and right cancellative, then it is not isomorphic to a direct product of a zero ordered semigroup and an ordered group, in general. Unlike in semigroups (without order) if an ordered semigroup S is regular and has the property $aS \subseteq (Sa]$ for all $a \in S$, then the $\mathcal{N}$-classes of S are not left simple and right cancellative, in general. The converse of the above two statements hold both in semigroups and in ordered semigroups. Exactly as in semigroups (without order), an ordered semigroup is a left group if and only if it is regular and right cancellative.

INTRODUCTION-PREREQUISITES

A semigroup isomorphic to the direct product of a left zero semigroup and a group is called a left group. A semigroup is a left group if and only if it is left simple and right cancellative. An $\mathcal{N}$-class of a semigroup S is a left group if and only if S is regular and for every $a \in S$, $aS \subseteq Sa$. Moreover, a semigroup S is a left group if and only if S is regular and

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right cancellative, equivalently, if S is left simple and contains an idempotent [9]. The aim of this paper is to examine these results for ordered semigroups and emphasize the similarities and the differences between semigroups and ordered semigroups. In semigroups the following definitions are the two essential (equivalent) definitions of the left groups. Definition 1. A semigroup is a left group if it is isomorphic to the direct product of a left zero semigroup and a group. Definition 2. A semigroup is a left group if it is left simple and right cancellative. Unlike in semigroups for which we have these two equivalent definitions of the left group, in ordered semigroups the situation is as follows: If an ordered semigroup is isomorphic to a direct product of a left zero ordered semigroup and an ordered group, then it is left simple and right cancellative. The converse statement does not hold, in general. So in case of ordered semigroups, we have to distinguish the case of the ordered semigroup which is isomorphic to a direct product of a left zero ordered semigroup and an ordered group from the case of the ordered semigroup which is left simple and right cancellative. We introduce the concepts of the left group and the complete left group. An ordered semigroup which is left simple and right cancellative is called a left group. An ordered semigroup which is isomorphic to a direct product of a left zero ordered semigroup and an ordered group is called a complete left group. We prove the following: Each complete left group is a left group, the converse statement does not hold, in general. If every $\mathcal{N}$-class of an ordered semigroup S is a left group, then S is regular and $aS \subseteq (Sa]$ for every $a \in S$, the converse statement does not hold, in general. More precisely, if an ordered semigroup S is regular and $aS \subseteq (Sa]$ for every $a \in S$, then each $\mathcal{N}$-class of S is left simple (cf. the Remark below), it is not right cancellative, in general, so an $\mathcal{N}$-class is not necessarily a left group. An ordered semigroup is a left group if and only if it is regular and right cancellative. Finally, if an ordered semigroup S is a left group, then S is left simple and contains an element e such that $e \leq e^2$, the converse does not hold, in general. The results of the paper remain true if we replace the word "right" by "left" and conversely.

We now give the necessary definitions for ordered semigroups. An ordered semigroup $(S, ., \leq)$ is called left simple if S is the only left ideal of S. Equivalently, if $(Sa] = S$ for all $a \in S$ [4]. For $H \subseteq S$, we denote $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. Thus an ordered semigroup S is left simple if and only if for every $a, y \in S$ there exists an element $z \in S$ such that $y \leq za$. An ordered semigroup S is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. Equivalently, if $A \subseteq (ASA]$ for every $A \subseteq S$ (or $a \in (aSa]$ for every $a \in S$) [5]. This is an extended form of J. von Neumann's regularity in case of ordered semigroups. A left (resp. right) ideal of an ordered semigroup $(S, ., \leq)$ is a non-empty subset A of S which is a left (resp. right) ideal of $(S, .)$ [that is, $SA \subseteq A$ (resp. $AS \subseteq A$)] such that if $a \in A$ and $S \in b \leq a$, then $b \in S$ [2]. A filter of an ordered semigroup S, is a subsemigroup of S such that if $a \in S$ and $S \ni b \geq a$, then $b \in S$. We denote by $N(a)$ the filter of S generated by a ($a \in S$) and by $\mathcal{N}$ the semilattice congruence on S defined by $\mathcal{N} := \{(x, y) \mid N(x) = N(y)\}$.

An equivalence relation "σ" on an ordered semigroup S is called a congruence on S if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called semilattice congruence if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$. The relation $\mathcal{N}$ is a semilattice congruence and each $\mathcal{N}$-class $(a)_{\mathcal{N}}$ of S is a subsemigroup of S ($a \in S$). For further information concerning the filters we refer to [1,3,6].

THE MAIN RESULTS

For ordered semigroups, we first introduce the following concepts:

Definition 1. An ordered semigroup is called a *complete left group* if it isomorphic to a direct product of a left zero ordered semigroup and an ordered group.

Definition 2. An ordered semigroup is called a *left group* if it is left simple and right cancellative.

Definition 3. An element z of an ordered semigroup S is called *left zero* if $zx = z$ for all $x \in S$. It is called right zero if $xz = z$ for all $x \in S$. An ordered semigroup is called left zero if each of its elements is left zero, that is, if $zx = z$ for all $z, x \in S$.

Definition 4. (cf. [4]). An ordered semigroup $(S, ., \leq)$ is called *left simple* if S is the only left ideal of S. Equivalent Definitions: i) $(Sa] = S$. ii) For each $a, y \in S$ there exists an element $z \in S$ such that $y \leq za$.

Definition 5. (cf. [8]). An ordered semigroup S is called *right cancellative* if for each $a, b, c \in S$ such that $ac \leq bc$, we have $a \leq b$.

For two ordered semigroups S and T, the direct product $S \times T$ is the ordered semigroup defined by the multiplication and the order below:

$$(s_1, t_1)(s_2, t_2) := (s_1 s_2, t_1 t_2) \tag{1}$$

$$(s_1, t_1) \leq (s_2, t_2) \;\Leftrightarrow\; s_1 \leq s_2,\; t_1 \leq t_2 \tag{2}$$

Theorem 1. *Let $(S, ., \leq)$ be an ordered semigroup. If S is a complete left group, then it is a left group. The converse statement does not hold, in general.*

Proof. Let T be a left zero ordered semigroup, G an ordered group and

$$f : S \to T \times G$$

be an isomorphism. Then
A) S is left simple. In fact: Let $a, b \in S$. Then $f(a) := (s, g)$ and $f(b) := (t, h)$ for some $s, t \in T$ and $g, h \in G$. Since $g, h \in G$ and G is a group, there exists $y \in G$ such that $h = yg$. Since $(t, y) \in T \times G$, we have $f^{-1}((t, y)) \in S$. We put $x := f^{-1}((t, y))$. Then we have $f(x) = (t, y)$, where $x \in S$. Since $s, t \in T$ and T is left zero, we have $ts = t$, so $t \leq ts$. Since $h = yg$, we have $h \leq yg$. Thus we have

$$f(b) = (t, h) \leq (ts, yg) := (t, y)(s, g) = f(x)f(a) = f(xa),$$

then $b \leq xa$, where $x \in S$.
B) Let $a, x, b \in S$ such that $ax \leq bx$. Then $a \leq b$. Indeed: Let $f(a) := (s, g)$, $f(b) := (t, h)$, $f(x) := (u, k)$, where $s, t, u \in T$ and $g, h, k \in G$. Since $ax \leq bx$, we have $f(ax) \leq f(bx)$, then $f(a)f(x) \leq f(b)f(x)$. Hence we have $(s, g)(u, k) \leq (t, h)(u, k)$, that is

$$su \leq tu \qquad \text{and} \qquad gk \leq hk \tag{3}$$

Since $s, t, u \in T$ and T is left zero, we have $su = s$ and $tu = t$. Then $s \leq t$ (by (3)) and, since G is an ordered group, again by (3), we have $g \leq h$. Then $(s, g) \leq (t, h)$, $f(a) \leq f(b)$, and $a \leq b$. Hence S is right cancellative.

For the converse statement, we consider the ordered semigroup $(N, +, \leq)$ of natural numbers $(N := \{1, 2, \dots\})$ with the usual addition, order. This is left simple since for each $n, m \in N$ we have $m \leq m + n$ and, clearly, it is right cancellative, so it is a left group. $(N, +, \leq)$ is not a complete left group. Indeed: Suppose it is. Then there exists a left zero ordered semigroup T, an ordered group G and an isomorphism $f : N \to T \times G$. Let e be the identity of G and let $t \in T$ $(T \neq \emptyset)$. Since $(t, e) \in T \times G$, there exists a unique $n \in N$ such that $f(n) = (t, e)$. Since f is a homomorphism, we have

$$\begin{aligned} f(n+n) &= f(n)f(n) = (t, e)(t, e) = (tt, ee) \text{ (by (1))} \\ &= (t, e) \text{ (since } T \text{ is left zero)} \\ &= f(n). \end{aligned}$$

Then $n + n = n$, so $n = 0$. Impossible. $\square$

Theorem 2. *If each $\mathcal{N}$-class of an ordered semigroup S is a left group, then S is regular and $aS \subseteq (Sa]$ for every $a \in S$. The converse statement does not hold, in general.*

Proof. Let $a \in S$. Since $a \in (a)_{\mathcal{N}}$ and $(a)_{\mathcal{N}}$ is left simple, there exists $z \in (a)_{\mathcal{N}}$ such that $a \le za$. Then $a^2 \le a(za) = (az)a$. Since $a, az \in (a)_{\mathcal{N}}$ and $(a)_{\mathcal{N}}$ is right cancellative, we have $a \le az$. Since $a, z \in (a)_{\mathcal{N}}$ and $(a)_{\mathcal{N}}$ is left simple, there exists $x \in (a)_{\mathcal{N}}$ such that $z \le xa$. Then we have $a \le az \le axa$, where $x \in (a)_{\mathcal{N}} \subseteq S$. Thus S is regular. Let $a, s \in S$. Then $as \in (Sa]$. In fact: Since S is regular and $sa \in S$, there exists $z \in S$ such that

$$N(sa) \ni sa \le sazsa \tag{4}$$

Then $sazsa \in N(sa)$, $zsa \in N(sa)$ and $N(zsa) \subseteq N(sa)$. On the other hand, since $zsa \in N(zsa)$, we have $sa \in N(zsa)$ and $N(sa) \subseteq N(zsa)$. Since $N(zsa) = N(sa)$, we have $(zsa, sa) \in \mathcal{N}$, so $zsa \in (sa)_{\mathcal{N}}$. By (4), we have

$$as(zsa) = (asz)sa \le (asz)sazsa = (aszsa)zsa.$$

Since $(as, sa) \in \mathcal{N}$, we have $as \in (sa)_{\mathcal{N}}$. Since $as, zsa \in (sa)_{\mathcal{N}}$ and $(sa)_{\mathcal{N}}$ is a subsemigroup of S, we have $aszsa \in (sa)_{\mathcal{N}}$. Since $(sa)_{\mathcal{N}}$ is right cancellative, we have $as \le aszsa \in Sa$. Thus $as \in (Sa]$.

For the converse statement, we consider the ordered semigroup $S := \{a, b, e\}$ defined by the multiplication $xy = e$ for all $x, y \in S$ and the order

$$\le = \{(a,a), (a,e), (b,b), (b,e), (e,e)\}.$$

This is regular (since $x \le xex$ for each $x \in S$), and $xS = \{e\} \subseteq S = (Sx]$ for every $x \in S$. Since $N(a) = N(b) = N(e) = S$, S is the only $\mathcal{N}$-class of S. On the other hand, S is not right cancellative. Indeed: Suppose it is. Then, since $ae = e \le e = be$, we have $a \le b$, which is impossible. So S is not a left group. □

Remark. In semigroups we have the following: Every $\mathcal{N}$-class of a semigroup S is left simple if and only if $x \in Sx^2$ and $xS \subseteq Sx$ for every $x \in S$ [9; II.4.9 Theorem]. For ordered semigroups, we have the following: Every $\mathcal{N}$-class of an ordered semigroup S is left simple if and only if $x \in (Sx^2]$ and $xS \subseteq (Sx]$ for every $x \in S$ [7; Proposition 3]. As a consequence, if an ordered semigroup S is regular and $xS \subseteq (Sx]$ for every $x \in S$, then each $\mathcal{N}$-class of S is left simple. Indeed: Let $x \in S$ and $y \in S$ such that $x \le xyx$. Since $xy \in (Sx]$, we have $xy \le sx$ for some $s \in S$. Then $x \le sx^2 \in Sx^2$, so $x \in (Sx^2]$. As we have already seen in Theorem 2, under the same hypotheses, the $\mathcal{N}$-classes are not cancellative, in general, so S is not a left group, in general.

Theorem 3. *An ordered semigroup S is a left group if and only if it is regular and right cancellative.*

Proof. $\Rightarrow$. Since S is a left group, it is right cancellative. Let now $a \in S$. Since S is left simple, there exists $y \in S$ such that $a \le ya$. Then $a^2 \le aya$ and, since S is right cancellative, $a \le ay$. Since $a, y \in S$ and S is left simple, there exists $x \in S$ such that $y \le xa$. Then $a \le ay \le axa$, and S is regular.
$\Leftarrow$. Let $a, b \in S$. Then there exists $x \in S$ such that $b \le xa$. Indeed: Since $ba \in S$ and S is regular, there exists $y \in S$ such that $ba \le bayba$. Since S is right cancellative, we have $b \le bayb$, then $byb \le bybayb$. Since S is right cancellative, we have $b \le (byb)a$, where $byb \in S$. Thus S is left simple. $\square$

Theorem 4. *If an ordered semigroup S is a left group, then S is left simple and there exists an element $e \in S$ such that $e \le e^2$. The converse statement does not hold, in general.*

Proof. By the definition, S is left simple. By Theorem 3, S is regular. Let $a \in S$ ($S \neq \emptyset$). Since S is regular, there exists $x \in S$ such that $a \le axa$. Then $ax \le axax$. For the element $e := ax \in S$, we have $e \le e^2$. For the converse statement, let S be the ordered semigroup considered in Theorem 2. This is not a left group. Since $ee := e$, we have $e \le e^2$. On the other hand, S is left simple. Indeed: Let L be a left ideal of S. Let $x \in L$ ($L \neq \emptyset$). We have $e := ex \in SL \subseteq L$, $S \ni a \le e \in L$, and $a \in L$. Similarly $b \in L$. Hence $L = S$. $\square$

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