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FUZZY IDEALS EXTENSTIONS OF ORDERED SEMIGROUPS

(submitted by M. M. Arslanov)

Abstract. In this paper, we introduce the concepts of the extension of fuzzy ideals , prime, semiprime and 3-prime fuzzy ideals in an ordered semigroup S, respectively. We discuss properties of fuzzy ideals extensions and the relationships between prime fuzzy ideals and 3-prime fuzzy ideals of S in terms of the extension of fuzzy ideals of S, we give an example to show that 3-prime fuzzy ideal is not necessarily prime. Moreover, for commutative ordered semigroups, we obtain some properties of the extension of fuzzy ideals in commutative ordered semigroup.

1. Introduction

The ideals of a semigroup S is a good tool for us to study the algebraic structure of S, ideals are divided into many kinds such as bi-ideals, interior ideals, prime ideals, semiprime ideals, quasi-prime ideals, etc. They have been discussed universally. Since Zadeh firstly introduced the

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concept of fuzzy set[7], Kuroki introduced fuzzy semigroups as a generalization of classical semigroups[8,9], then many people began to research fuzzy ideals of semigroups[3,5,6,8]. Xie[2,4] introduced the extensions of ideals and the n-primes ideals in ordered semigroups and studied the relationship between them. For commutative ordered semigroups, the author prove, among others, that if S has an identity element,then the n-prime ideals and $(n-1)$-prime ideals are coincide $(n \geq 3)$. If I is a semiprime ideal of S, then I is the intersection of the extensions $< x, I >$, $x \in S$. Moreover, if I is n-prime $(n \geq 3)$, then I is the intersection of the $(n-1)$-prime ideals of S containing it. Specially if S is a semilattice, then every n-prime ideal of S, $n \geq 3$, is the intersection of the $(n-1)$-prime ideals of S containing it. As a generalization, Xie[1] introduced the extensions of fuzzy ideals and fuzzy 3-prime ideals in semigroups, and discussed the fuzzy ideals extension of semigroups.

Recently, N.Kehayopulu and M. Tsingeles introduced in [11, 12,13] fuzzy subsets of an ordered semigroup S. They defined the fuzzy bi-ideals in ordered semigroups and gave the main theorem which characterizes the bi-ideals in terms of fuzzy bi-ideals. Then authors characterized the left and right simple, the completely regular, and the strongly regular ordered semigroups by means of fuzzy biideals. Authors also studied the decomposition of left and right simple ordered semigroups and of ordered semigroups having the property a6a2 for all a, in terms of fuzzy bi-ideals. This decomposition is uniquely defined. In this paper, we introduce the extension of fuzzy ideals (including prime fuzzy ideals, semiprime fuzzy ideals and 3-prime fuzzy ideals) in an ordered semigroup S. we discuss the relationships between prime fuzzy ideals and 3-prime fuzzy ideals of an ordered semigroup, and give an example to show that 3-prime fuzzy ideal is not necessarily prime. Furthermore, for the commutative ordered semigroups, we obtain some properties of the extension of fuzzy ideals.

2. Prime and semiprime fuzzy ideals

Throughout this paper, S is an ordered semigroup $(S, \cdot, \leq)$, which means $(S, \cdot)$ is a semigroup, and $(S, \leq)$ is a partially ordered set satisfying

$$(\forall x, a, b \in S)\ a \leq b \Rightarrow xa \leq xb, ax \leq bx.$$

A fuzzy subset f of S is a function from S to the unit interval [0, 1]. The set $F(S)$ of all fuzzy subsets of S with the order relation "$\subseteq$" on

$F(S)$ defined by :

$$f \subseteq g \Longleftrightarrow f(x) \leq g(x),\ \forall x \in S$$

is a complete lattice where, for a non-empty family $\{f_i|i \in I\}$ of fuzzy subset of S, the $\inf\{f_i|i \in I\}$ and the $\sup\{f_i|i \in I\}$ are fuzzy subsets of S defined by :

$$\inf\{f_i|i \in I\} : S \longrightarrow [0,1],\ x \longrightarrow \inf\{f_i(x)|i \in I\}.$$

$$\sup\{f_i|i \in I\} : S \longrightarrow [0,1],\ x \longrightarrow \sup\{f_i(x)|i \in I\}.$$

(cf. M.Tsingelis [6]).

Let S be an ordered semigroup, $\emptyset \neq T \subseteq S$, T is called *prime* if $(\forall x, y \in S) xy \in T \Rightarrow x \in T$ or $y \in T$. T is called *semiprime* if $(\forall a \in S) a^2 \in T \Rightarrow a \in T$. Let T be an ideal of S, If T is a prime subset, then T is called a *prime ideal*, and if T is a semiprime subset of S, then T is called a *semiprime ideal*.

Definition 2.1[11] A fuzzy subset f of S is called a *fuzzy ideal* if

1) $x \leq y \Rightarrow A(x) \geq A(y)$.

2) $A(xy) \geq A(x) \vee A(y), \forall x, y \in S$.

Definition 2.2 Let f be a fuzzy subset of S, f is called *prime* if

$$f(xy) = f(x) \vee f(y),\ \forall x, y \in S.$$

A fuzzy ideal f of S is called a *prime fuzzy ideal* of S if f is a prime fuzzy subset of S.

Definition 2.3 Let f be a fuzzy subset of S, f is called *semiprime*, if $f(a) \geq f(a^2), \forall a \in S$. A fuzzy ideal f of S is called a *semiprime fuzzy ideal* if f is a semiprime fuzzy subset of S.

It is easy to see that if f is a prime or semiprime fuzzy ideal of S, then we have $f(x) = f(x^2), \forall x \in S$.

Lemma 2.4(cf. [12,Proposition 2,3]) *Let S be an ordered semigroup and f be a fuzzy subset of S. Then f is a fuzzy ideal of S if and only if $(\forall t \in [0,1])$, $f_t(\neq \emptyset)$ is an ideal of S.*

By Lemma 2.4, we have

Corollary 2.5 *Let S be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is an ideal of S if and only if the characteristic function f_I is a fuzzy ideal of S.*

Theorem 2.6 *Let S be an ordered semigroup and f be a fuzzy subset of S. Then f is prime fuzzy ideal if and only if $(\forall t \in [0,1])$ $f_t(\neq \emptyset)$ is a prime ideal of S.*

Proof Let f be prime fuzzy ideal of S . Then f is a fuzzy ideal. By Lemma 2.4, $(\forall t \in [0,1])$ $f_t(\neq \emptyset)$ is an ideal of S. Let $x, y \in S$ such that

$xy \in f_t$. Then $f(xy) \geq t$. Since f is prime, we have $f(xy) = f(x) \vee f(y)$, then $f(x) \geq t$ or $f(y) \geq t$, i.e., $x \in f_t$ or $y \in f_t$, thus f_t is a prime ideal of S.

Conversely, let $f_t(\neq \emptyset)$ is a prime ideal of S for any $t \in [0,1]$. Then, by Lemma 2.4 and hypothesis, f is a fuzzy ideal. Let $x, y \in S$ and $f(xy) = t$. Since $f_t(\neq \emptyset)$ is a prime ideal of S, and $xy \in f_t$, we have $x \in f_t$ or $y \in f_t$, which implies that $f(x) \geq t$ or $f(y) \geq t$. Then $f(x) \vee f(y) \geq t = f(xy) \geq f(x) \vee f(y)$. Thus $f(xy) = f(x) \vee f(y)$. It follows that f is prime. □

Theorem 2.7 *Let S be an ordered semigroup, $\emptyset \neq I \subseteq S$. Then I is a prime ideal of S if and only if f_I is a prime fuzzy ideal of S.*

Proof Let I be a prime ideal of S, by Corollary 2.5, f_I is a fuzzy ideal of S. Moreover, $f_I(xy) = f_I(x) \vee f_I(y), \forall x, y \in S$. In fact:

i) If $xy \notin I$, since I is a prime ideal, we have $x \notin I$, $y \notin I$. Then $f_I(xy) = 0 = 0 \vee 0 = f_I(x) \vee f_I(y)$.

ii) If $xy \in I$, since I is a prime ideal, we have $x \in I$, or $y \in I$. Then $f_I(xy) = 1 = f_I(x) \vee f_I(y)$.

Conversely, if f_I is a prime fuzzy ideal of S, by Corollary 2.5, I is an ideal of S. On the other hand, if $\forall x, y \in S, xy \in I$, then $f_I(x) \vee f_I(y) = f_I(xy) = 1$. Thus $f_I(x) = 1$ or $f_I(x) = 1$, i.e., $x \in I$ or $y \in I$. Hence I is a prime ideal of S. □

Similar to that of fuzzy prime ideals of an ordered semigroup S, we have

Theorem 2.8 *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is semiprime if and only if ($\forall t \in [0,1]$) $f_t(\neq \emptyset)$ is semiprime.*

Proof Let f be semiprime fuzzy subset of S . Then ($\forall t \in [0,1]$) $f_t(\neq \emptyset)$ is semiprime. In fact: Let $x \in S$ such that $x^2 \in f_t$. Then $f(x^2) \geq t$. Since f is semiprime, we have $f(x) \geq f(x^2) \geq t$. Then $x \in f_t$. Thus f_t is a semiprime fuzzy subset of S.

Conversely, let $f_t(\neq \emptyset)$ is a semiprime subset of S for any $t \in [0,1]$. Then f is fuzzy semiprime. In fact: Let $x \in S$ and $f(x^2) = t$. Since $f_t(\neq \emptyset)$ is semiprime and $x^2 \in f_t$, we have $x \in f_t$, which implies that $f(x) \geq t = f(x^2)$. Then f is semiprime. □

Theorem 2.9 *Let S be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is semiprime if and only if f_I is a semiprime fuzzy subset of S.*

Proof Let I be semiprime. Then $(\forall a \in S) f_I(a) \geq f_I(a^2)$. In fact: If $a^2 \notin I$. Then $f_I(a) \geq 0 = f_I(a^2)$. If $a^2 \in I$. Then $a \in I$. Thus $f_I(a) = 1 = f_I(a^2)$.

Conversely, let f_I be a semiprime fuzzy subset of S, and $(\forall a \in S) a^2 \in I$. Since $f_I(a) \geq f_I(a^2) = 1$, we have $a \in I$. Then I is semiprime. □

By Theorems 2.8, 2.9 , Lemma 2.4, and Corollary 2.5, we have follows:
Theorem 2.10 *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is a semiprime fuzzy ideal of S if and only if $(\forall t \in [0,1])$ $f_t(\neq \emptyset)$ is a semiprime ideal of S.*
Theorem 2.11 *Let S be an ordered semigroup and $\emptyset \neq I \subseteq S$. Then I is a semiprime ideal of S if and only if f_I is a semiprime fuzzy ideal of S.*

3. Fuzzy ideal extensions of ordered semigroups

Definition 3.1 Let S be an ordered semigroup, f a fuzzy subset of S, $x \in S$. The fuzzy subset $< x, f >$ defined by :

$$< x, f >: S \longrightarrow [0,1] \mid y \longrightarrow < x, f > (y) = f(xy), \forall y \in S$$

is called the *extension* of f by x.
Proposition 3.2 *Let S be a commutative ordered semigroup . If f is a fuzzy ideal of S and $x \in S$, then $< x, f >$ is a fuzzy ideal of S.*
Proof Let f be a fuzzy ideal of S. Then the fuzzy subset $< x, f >$ is a fuzzy ideal of S. In fact: Since $\forall y, z \in S$,

A) $y \leq z \Rightarrow xy \leq xz \Rightarrow f(xy) \geq f(xz) \Rightarrow < x, f > (y) \geq < x, f > (z)$,
B) $< x, f > (yz) = f(xyz) \geq f(xy) = < x, f > (y)$,
C) $< x, f > (yz) = f(xyz) = f(yxz) \geq f(xz) = < x, f > (z)$,

we have $< x, f > (yz) \geq < x, f > (y) \vee < x, f > (z)$. Thus $< x, f >$ is a fuzzy ideal of S. □
Proposition 3.3 *Let S be an ordered semigroup, f a fuzzy ideal of S, $\forall x \in S$. Then we have the following:*

(1) $f \subseteq < x, f >$.
(2) $< x^n, f > \subseteq < x^{n+1}, f >$ for any $n \in N$.
(3) If $f(x) > 0$, then supp $< x, f > = S$.
(4) If $x \leq y$, then $< y, f > \subseteq < x, f >$.

Proof (1) For $\forall y \in S$, since f is a fuzzy ideal of S, we have $< x, f > (y) = f(xy) \geq f(y)$. Thus $f \subseteq < x, f >$.

(2) For $\forall n \in N, y \in S$, since f is a fuzzy ideal of S, we have

$$< x^{n+1}, f > (y) = f(x^{n+1}y) = f(xx^n y) \geq f(x^n y) = < x^n, f > (y).$$

Thus $< x^n, f > \subseteq < x^{n+1}, f >$.

(3) Let $f(x) > 0$. Since $< x, f >$ is a fuzzy subset of S, we have *supp* $< x, f > \subseteq S$. Since f is a fuzzy ideal of S, we have $< x, f > (y) = f(xy) \geq f(x) > 0, \forall y \in S$. Then $y \in$ *supp* $< x, f >$. Thus *supp* $< x, f > = S$.

(4) Let $x \leq y, \forall x, y \in S$. Since S is an ordered semigroup, we have $xz \leq yz, \forall z \in S$. Then $< x, f > (z) = f(xz) \geq f(yz) =< y, f > (z)$. Thus $< y, f > \subseteq < x, f >$. □

Lemma 3.4 *Let S be an ordered semigroup, f be a prime fuzzy subset of S, then $< x, f >$ is a prime fuzzy subset of S and $< x, f >=< x^2, f >$, $\forall x \in S$.*

Proof (1) For $\forall y, z \in S$,

$$\begin{aligned} < x, f > (yz) &= f(xyz) = f(xy) \vee f(z) = f(x) \vee f(y) \vee f(z) \\ &= (f(x) \vee f(y)) \vee (f(x) \vee f(z)) = f(xy) \vee f(xz) \\ &= < x, f > (y) \vee < x, f > (z). \end{aligned}$$

Thus $< x, f >$ is a prime fuzzy subset of S.

(2) For $\forall x \in S$, since f is a prime fuzzy subset of S, we have $f(x^2) = f(x)$. Then

$$< x, f > (y) = f(xy) = f(x) \vee f(y) = f(x^2) \vee f(y) = f(x^2 y) =< x^2, f > (y).$$

Thus $< x, f >=< x^2, f >$. □

By Proposition 3.2, Lemma 3.4, the following theorem is easy.

Theorem 3.5 *Let S be an ordered semigroup, f a prime fuzzy ideal of S, then $< x, f >$ is a prime fuzzy ideal of S.*

Theorem 3.6 *Let S be an ordered semigroup. If f is a prime fuzzy subset of S and $x \in S$ such that $f(x) = \inf_{y \in S} f(y)$, then $< x, f >= f$. Conversely, let f be a fuzzy ideal of S. Suppose that for any $y \in S$ such that $f(y)$ is not maximal in $f(S)$, we have $< y, f >= f$. Then f is prime.*

Proof Let f be a prime fuzzy subset of S and $x \in S$ such that $f(x) = \inf_{y \in S} f(y)$. $\forall y \in S$, Since f is a prime, we have $< x, f > (y) = f(xy) = f(x) \vee f(y) = f(y)$, thus $< x, f >= f$.

Conversely, since f is a fuzzy ideal, we have $f(ab) \geq f(a) \vee f(b), \forall a, b \in S$. Suppose that $f(ab) > f(a) \vee f(b)$ for some a, b in S. In this case, $f(a)$ is not maximal, by hypothesis, $< a, f > (b) = f(b)$, then $f(b) = f(ab) > f(a) \vee f(b)$, impossible. It follows that $f(ab) = f(a) \vee f(b), \forall a, b \in S$, that is, f is prime. □

Corollary 3.7 *Let S be an ordered semigroup, I an ideal of S. Then I is prime if and only if for any $x \in S$ such that $x \notin I$, $< x, f_I >= f_I$.*

Proof Let I be a prime ideal of S. Then by Lemma 2.10, f_I is a prime fuzzy ideal of S. For any $x \in S$ such that $x \notin I$, we have $f_I(x) = 0 = inf_{y \in S} f_I(y)$. By Theorem 3.6, $< x, f_I >= f_I$.

Conversely, let I be an ideal of S. By Lemma 2.8, f_I is a fuzzy ideal of S. Let $x \in S$ such that $x \notin I$, then $f_I(x) = 0$, is not maximal, moreover,

$< x, f_I >= f_I$, by Theorem 3.6, f_I is prime. Therefore, by Lemma 2.9, I is prime. □

Theorem 3.8 *Let S be a commutative ordered semigroup and f a fuzzy subset of S such that $< x, f >= f$ for every $x \in S$. Then f is constant.*

Proof $\forall x, y \in S$, since $< x, f >= f$, we have

$$f(y) =< x, f > (y) = f(xy) = f(yx) =< y, f > (x) = f(x).$$

Thus f is constant. □

Proposition 3.9 *Let S be a commutative ordered semigroup and f a semiprime fuzzy ideal of S. Then $< x, f >$ is a semiprime fuzzy ideal of S.*

Proof Let f be a semiprime fuzzy ideal of S. By Proposition 3.2, $< x, f >$ is a fuzzy ideal of S, $\forall a \in S$, then

$$\begin{aligned} < x, f > (a) &= f(xa) \geq f((xa)^2) = f(xaxa) \\ &= f(xa^2x) \geq f(xa^2) =< x, f > (a^2) \end{aligned}$$

Thus $< x, f >$ is a semiprime fuzzy ideal of S. □

Corollary 3.10 *Let S be a commutative ordered semigroup, $\{f_i\}_{i \in I}$ be a non-empty family of semiprime fuzzy ideals of S, and let $f = inf\{f_i | i \in I\}$. Then for any $x \in S$, $< x, f >$is a semiprime fuzzy ideal of S.*

Proof Clearly f is a fuzzy subset of S. Furthermore, f is a fuzzy ideal of S. In fact:

A) If $x \leq y$, then $f_i(x) \geq f_i(y)$ $\forall i \in I$. Thus

$$\begin{aligned} f(x) &= inf\{f_i \mid i \in I\}(x) = inf\{f_i(x) \mid i \in I\} \\ &\geq inf\{f_i(y) \mid i \in I\} = inf\{f_i \mid i \in I\}(y) = f(y). \end{aligned}$$

B) Since

$$\begin{aligned} f(xy) &= inf\{f_i \mid i \in I\}(xy) = inf\{f_i(xy) \mid i \in I\} \\ &\geq inf\{max\{f_i(x), f_i(y)\} \mid i \in I\} \\ &\geq max\{inf\{f_i(x) \mid i \in I\}, inf\{f_i(y) \mid i \in I\}\} = max\{f(x), f(y)\}. \end{aligned}$$

and f_i is semiprime, we have

$$\begin{aligned} f(a) &= inf\{f_i \mid i \in I\}(a) = inf\{f_i(a) \mid i \in I\} \\ &\geq inf\{f_i(a^2) \mid i \in I\} = inf\{f_i \mid i \in I\}(a^2) = f(a^2). \end{aligned}$$

Thus f is a semiprime fuzzy ideal of S. By Proposition 3.9, $< x, f >$ is a semiprime fuzzy ideal of S. □

Corollary 3.11 *Let S be a commutative ordered semigroup, $\{P_i\}_{i\in I}$ a non-empty family of semiprime fuzzy ideals of S, and $A = \bigcap_{i\in I} P_i \neq \emptyset$. Then $< x, f_A >$, $\forall x \in S$ is a semiprime fuzz ideal of S.*

Proof It is clear that A is a semiprime ideal of S. By Lemma 2.12, f_A is a semiprime fuzzy ideal. Then, by Proposition 3.9, $< x, f_A >$ is a semiprime fuzzy ideal of f. □

Corollary 3.12 *Let S be a commutative ordered semigroup, f a prime fuzzy subset of S. If f is not constant, then f is not a maximal prime fuzzy subset of S.*

Proof Let f is a prime fuzzy subset of S. By Lemma 3.4, $< x, f >$ is a prime fuzzy subset of S. Furthermore, we have $f \subset < x, f >$ for some $x \in S$. Otherwise, if $f =< x, f >, \forall x \in S$, by Theorem 3.8, f is constant, a contradiction. □

If f is a fuzzy ideal of S, we denote by f_ρ the equivalent relation on S defined by:

$$f_\rho = \{(x, y) \mid < x, f >=< y, f >\}$$

Proposition 3.13 *Let S be a commutative ordered semigroup and f a fuzzy ideal of S. Then*

(1) f_ρ is a congruence on S.

(2) If f is semiprime, then f_ρ is a semilattice congruence on S.

(3) If f is prime and $x \leq y$, then $(x, xy) \in f_\rho$.

Proof (1) We need to show that f_ρ is compatible. Let$(x, y) \in f_\rho, c \in S$. Then $(\forall z \in S) < xc, f > (z) = f(xcz) =< x, f > (cz) =< y, f >_{cz}= f(ycz) =< yc, f > (z)$. Thus $(xc, yc) \in f_\rho$. Similarly $(cx, cy) \in f_p$. Therefore f_ρ is a congruence on S.

(2) Let S be a commutative ordered semigroup and f be semiprime. Then

$$(\forall z \in S)(x^2, f > (z) = f(x^2 z) \leq f((xz)^2) \leq f(xz) =< x, f > (z),$$

and so $< x^2, f >\subseteq< x, f >$. By Proposition 3.3, $< x, f >\subseteq< x^2, f >$. Thus $< x, f >=< x^2, f >$, i.e., $(x, x^2) \in f_\rho$. Hence f_ρ is a semilattice congruence on S.

(3) Let f be prime and $x \leq y$. Then $f(x) \geq f(y)$. Thus $(x, xy) \in f_\rho$. In fact:

$$\begin{aligned}(\forall z \in S) \ < x, f > (z) &= f(xz) = f(x) \vee f(z) \\ &= f(x) \vee f(y) \vee f(z) = f(xy) \vee f(z) \\ &= f(xyz) =< xy, f > (z).\end{aligned}$$

Then $< x, f >=< y, f >$, that is, $(x, xy) \in f_\rho$. □

Proposition 3.14 *Let S be an ordered semigroup and f a prime fuzzy ideal of S. Then $f_\rho = f_\sigma$, where*

$$f_\sigma = \{(x, y) \mid f(x) = f(y) \text{ or } f(xz) = f(yz), \forall z \in S\}.$$

Proof Let f be a prime fuzzy ideal of an ordered semigroup S, and $x, y) \in f_\rho$. Then $< x, f >=< y, f >$. Thus

$$< x, f > (z) = f(xz) =< y, f > (z) = f(yz), \forall z \in S.$$

Consequently, $(x, y) \in f_\sigma$.

Conversely, let $(x, y) \in f_\sigma$. Then $f(x) = f(y)$, or $f(xz) = f(yz), \forall z \in S$.

A) If $f(x) = f(y)$, then

$$(\forall z \in S)\ f(xz) = f(x) \vee f(z) = f(y) \vee f(z) = f(yz).$$

Thus $< x, f >=< y, f >$, which implies that $(x, y) \in f_\rho$.

B) If $f(xz) = f(yz), \forall z \in S$, then $< x, f >=< y, f >$. Hence $(x, y) \in f_\rho$. □

4. 3-prime fuzzy ideals

Definition 4.1 A fuzzy subset f of an ordered semigroup S is called *3-prime* if for any $a, b, c \in S$,

$$\begin{aligned} f(abc) &= max\{f(ab), f(ac)\} \\ &= max\{f(bc), f(ba)\} \\ &= max\{f(ca), f(cb)\}. \end{aligned}$$

Proposition 4.2 *Let S be an ordered semigroup and f a fuzzy ideal of S. If f is prime, then f is 3-prime.*

Proof Let f be a prime fuzzy ideal of S. Then for any $a, b, c \in S$,

$$\begin{aligned} f(abc) &= max\{f(a), f(bc)\} \\ &\leq max\{f(ab), f(bc)\} \\ &\leq f(abc). \end{aligned}$$

Thus $f(abc) = max\{f(ab), f(bc)\} = max\{f(bc), f(ba)\}$. Since

$$f(bac) = max\{f(ba), f(c)\} = max\{f(ab), f(c)\} = f(abc),$$

we have

$$\begin{aligned} f(abc) &= max\{f(ab), f(c)\} \leq max\{f(ab), f(ac)\} \\ &= max\{f(ba), f(ac)\} \leq max\{f(abc), f(bac)\} \\ &= f(abc). \end{aligned}$$

Thus $f(abc) = max\{f(ab), f(ac)\}$. Similar to above, we have

$$f(abc) = max\{f(ca), f(cb)\}.$$

Consequently, f is 3-prime. □

In general the 3-prime fuzzy ideals are not necessarily prime, we illustrate it with following example:

Example 4.3 Let S be a semigroup with the multiplication table:

.	a	b	c
a	c	b	c
b	b	b	c
c	c	c	c

and $\leq := \{(a, a), (b, b), (c, c)\}$. It is easy to verify that $(S, \cdot, \leq)$ is an ordered semigroup. Define a fuzzy subset f as follows:

$$f(a) = f(b) = 0, f(c) = 0.3.$$

we can show that f is a fuzzy ideal of S. Since $f(aa) = f(c) = 0.3 \neq f(a) \vee f(a)$, we have f is not prime. But f is a 3-prime fuzzy ideal of S. In fact: For any $x_1, x_2, x_3 \in S$, since S is commutative, we have

$$x_1x_2x_3 \in \{a^3, a^2b, a^2c, b^3, b^2a, b^2c, c^3, c^2a, c^2b, abc\}$$

Let one of x_1, x_2, x_3 be c. Since $xc = cx = c$ for any $x \in S$, we have

$$\begin{aligned} f(x_1x_2x_3) = f(c) &= max\{f(x_1x_2), f(x_1x_3)\} \\ &= max\{f(x_2x_3), f(x_2x_1)\} \\ &= max\{f(x_3x_1), f(x_3x_2)\} \end{aligned}$$

Then we need to show the cases of a^3, a^2b, b^3, b^2a.

$$\begin{aligned} f(a^3) &= f(c) = max\{f(a^2), f(a^2)\}. \\ f(a^2b) &= f(c) = max\{f(a^2), f(ab)\} = max\{f(ba), f(ba)\}. \\ f(b^3) &= f(b) = max\{f(b^2), f(b^2)\}. \\ f(b^2a) &= f(b) = max\{f(b^2), f(ba)\} = max\{f(ab), f(ab)\}. \end{aligned}$$

Theorem 4.4 *Let S be an ordered semigroup and f a fuzzy subset of S. If f is 3-prime, then any extension of f is prime. In particular, if S is* *commutative, then* f *is* $3-prime$ *if and only if any extension of* f *is prime.*

Proof Let f be 3-prime. Then for any $x, y, z \in S$,

$$\begin{aligned} < x, f > (yz) &= f(xyz) = max\{f(xy), f(xz)\} \\ &= max\{< x, f > (y), < x, f > (z)\}. \end{aligned}$$

Thus f is prime.

Conversely, let S be commutative and any extension of f be prime fuzzy subset of S. Then $\forall x, y, z \in S$.

$$\begin{aligned} f(xyz) =< x, f > (yz) &= max\{f(xy), f(xz)\} \\ =< y, f > (xz) &= max\{f(yz), f(yx)\} \\ =< z, f > (xy) &= max\{f(zx), f(zy)\} \end{aligned}$$

Thus $< x, f >$ is 3-prime. □

Corollary 4.5 *Let* S *be an ordered semigroup with an identity* e *and* f *a fuzzy subset of* S*. If* f *is 3-prime, then* f *is prime.*

Proof Let e be the identity element of S. Then $< e, f >= f$. Since f is 3-prime, by Theorem 4.4, we have $< e, f >$ is prime. Thus f is prime. □

By Proposition 4.2 and Corollary 4.5, in an ordered semigroup with the identity element, the prime fuzzy subsets and the $3-prime$ fuzzy subsets coincide.

Theorem 4.6 *If* S *is an ordered semigroup and* f *a semiprime fuzzy ideal of* S*. Then* $f = inf\{< x, f > |x \in S\}$.

Proof Let f be a semiprime fuzzy ideal of S. By Proposition 3.3(1), $f \subseteq < x, f >, \forall x \in S$. Let g be a fuzzy subset of S such that $g \subseteq < x, f >, \forall x \in S$, and let $y \in S$. Since f is semiprime, we have

$$g(y) \leq < y, f > (y) = f(y^2) \leq f(y).$$

Then $g \subseteq f$. Thus $f = inf\{< x, f > |x \in S\}$. □

Corollary 4.7 *Let* S *be an ordered semigroup and* f *a fuzzy ideal of* S*. If* f *is 3-prime and semiprime, then* f *is the infimum of all prime fuzzy ideals of* S *containing* f*.*

Proof It follows by Theorem 4.4 and Theorem 4.6. □

Corollary 4.8 *Let* S *be a semilattice. Then any 3-prime fuzzy ideal* f *is expressible as the infimum of all prime fuzzy ideals of* S *containing* f*.*

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