

*V. Alferiev and E. Kuznetsov*

**THE BEST ARGUMENT FOR THE PARAMETRIC  
CONTINUATION OF SOLUTIONS OF  
DIFFERENTIAL-ALGEBRAIC EQUATIONS**

(submitted by A. M. Elizarov)

ABSTRACT. New algorithms for numerical continuation of Cauchy problem solution for different forms of DAEs, and results of their implementations are presented.

1. INTRODUCTION

Differential-algebraic equations (DAEs) differ from other problems in solutions given by smooth and continuous one-parametric sets. They combine specifics of the nonlinear algebraic or transcendental equations with ordinary differential ones in the normal form.

The first use of the parametric continuation idea for solution to nonlinear finite dimensional equations appears to be due to M. Lahaye [5, 6] (1934). Another formulation of the continuation method was given by D. Davidenko [2, 3] (1953). He was apparently the first who realized the process of solution continuation as a process of moving, and applied adequate mathematical apparatus of differential equations to it. The following development of this method was presented in [8]. There was raised a problem of choosing the best continuation parameter in it. It

---

*2000 Mathematical Subject Classification.* 34A09, 65H10, 65L80.

*Key words and phrases.* Differential-algebraic equations (DAEs), the best argument, Cauchy problem.

was found out that such a parameter was the arc length of the solution curve of the Cauchy Problem for a system of DAEs.

We consider the Cauchy problem for a system of DAEs

$$\begin{cases} F(y, \dot{y}, x, t) = 0, & y(t_0) = y_0, \\ G(y, x, t) = 0, & x(t_0) = x_0, \end{cases} \quad (1)$$

where  $y(t) = (y_1(t), \dots, y_n(t))^T$ ,  $x(t) = (x_1(t), \dots, x_m(t))^T$ ,  $F = (F_1, \dots, F_n)^T$ ,  $G = (G_1, \dots, G_m)^T$ ,  $t \in \mathbb{R}^1$ ,  $y_0 = (y_{10}, \dots, y_{n0})^T$ ,  $x_0 = (x_{10}, \dots, x_{m0})^T$ ,  $\dot{y} = \frac{dy}{dt} = \left( \frac{dy_1}{dt}, \dots, \frac{dy_n}{dt} \right)^T$ .

The vectors  $y_0$ ,  $x_0$  and value  $t_0$  must be consistent, i.e., satisfy the system of equations  $G(y_0, x_0, t_0) = 0$ , and a solution of (1) exists and unique on some interval containing  $t_0$ .

The method of solution continuation with respect to a parameter [8] for the problem (1) can be described as follows. The integral of the problem (1)

$$f(y, x, t) = 0, \quad f(y_0, x_0, t_0) = 0, \quad f = (f_1, \dots, f_{n+m})^T \quad (2)$$

specify a unique smooth integral curve  $K$  in the  $(n+m+1)$ -dimensional Euclidean space  $\mathbb{R}^{n+m+1}$ . The process of its construction may be viewed as the process of continuation of the solution  $y = y(t)$ ,  $x = x(t)$  with respect to the parameter  $t$ . Such approach brings us the problem of choosing the best parameter of solution continuation of the system (2) and, hence, the best argument of problem (1).

We will introduce the best argument locally, i.e., in a small neighborhood of each point of the integral curve  $K$ . To find the best argument we introduce in the neighborhood of the point under consideration a parameter  $\mu$  such that

$$d\mu = \alpha_i dy_i + \beta_j dx_j + \gamma dt, \quad i = \overline{1, n}, \quad j = \overline{1, m}. \quad (3)$$

Here  $\alpha_i$ ,  $\beta_j$ ,  $\gamma$  are components of the unit vector considered above  $\alpha = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m, \gamma)^T \in \mathbb{R}^{n+m+1}$  which specifies the direction with respect to which the argument  $\mu$  is measured. Hereafter the summation in products with respect to repeating indexes within given ranges is assumed.

The functions  $y_i(\mu)$ ,  $x_j(\mu)$ ,  $t(\mu)$  are assumed to be differentiable. Dividing equation (3) by  $d\mu$  and differentiating the first of relations (2) with respect to  $\mu$ , we obtain the following continuation equations for the

problem (1)

$$\begin{aligned}\alpha_i y_{i,\mu} + \beta_j x_{j,\mu} + \gamma t_{,\mu} &= 1, \\ f_{,y_i} y_{i,\mu} + f_{,x_j} x_{j,\mu} + f_{,t} t_{,\mu} &= 0.\end{aligned}\tag{4}$$

Hereafter  $y_{i,\mu} = dy_i/d\mu$ ,  $f_{,y_i} = \partial f/\partial y_i$ , ...

However, such approach is not constructive since integral (2) is unknown until problem (1) is solved. In this work we describe a constructive algorithm for numerical continuation of Cauchy problem solution for different forms of DAEs. The paper is organized as follows. In Section 2 a way to obtain the continuation equations is studied and the choice of the best argument is investigated. The sections 3,4 present an algorithm for numerical solution of explicit differential-algebraic equations and implicit ordinary differential equations with respect to the best argument. For the implicit DAEs two methods of numerical solution are presented in Section 5. The numerical results are shown in Section 6. The conclusions are given in Section 7.

## 2. THE CHOICE OF THE BEST ARGUMENT

The continuation equations can be obtained in another way. Let us linearize the vector function  $F$  with respect to  $\dot{y}_i$  in a neighborhood of a certain value  $\dot{y}_i = \dot{y}_i^*$  which is obtained, for example, at the previous step of the iterative process of integration procedure. Then

$$F^* + F_{,\dot{y}_i}^* (\dot{y}_i - \dot{y}_i^*) = 0, \quad i = \overline{1, n}.$$

Here the vector functions  $F^*$  and  $F_{,\dot{y}_i}^*$  are calculated at  $\dot{y}_i = \dot{y}_i^*$ .

Taking into account the first equation of the system (4), the relations  $\dot{y}_i = y_{i,\mu}/t_{,\mu}$ ;  $\dot{y}_i^* = y_{i,\mu}^*/t_{,\mu}^*$ , and differentiating the vector function  $G$  with respect to  $\mu$ , we arrive at the continuation equations:

$$\begin{aligned}\alpha_i y_{i,\mu} + \beta_j x_{j,\mu} + \gamma t_{,\mu} &= 1, \\ t_{,\mu}^* F_{,\dot{y}_i}^* y_{i,\mu} + (F_{,t_{,\mu}}^* t_{,\mu} - F_{,\dot{y}_i}^* y_{i,\mu}^*) t_{,\mu} &= 0, \\ G_{,y_i} y_{i,\mu} + G_{,x_j} x_{j,\mu} + G_{,t} t_{,\mu} &= 0.\end{aligned}\tag{5}$$

The integral curve for problem (1) can be constructed by integrating the system of ordinary differential equations obtained by solving the continuation equations (5) with respect to derivatives with the initial conditions

$$y_i(0) = y_{i0}, \quad x_j(0) = x_{j0}, \quad t(0) = t_0.\tag{6}$$

We assume here that the argument  $\mu$  is measured from the initial point of the problem (1), and the system (5) is nonsingular with respect to unknowns  $y_{i,\mu}$ ,  $x_{j,\mu}$ ,  $t_{,\mu}$ .

The conditionality of the system (5) depends on the choice of the argument  $\mu$  which, in turn, is determined by the vector  $\alpha$ . It is known [8] that the parameter that ensures the best conditionality for the system of linear continuation equations is the arc length  $\lambda$  measured along the curve of solutions of the system (2) which, in this case, is an integral curve  $K$  for the problem (1) and the continuation parameter for the system (2) is the argument of problem (1). The argument  $\mu = \lambda$  which ensures the best conditionality for the system of continuation equations (5), will be called the best argument. Note that the value of the determinant of the system divided by the product of the Euclidean norms of the matrix rows is taken as the measure of conditionality. It was shown in [8] that errors of numerical solution are minimal if the best argument is chosen.

By Kramer's rule, the solution of the system (5) in this case can be represented in the form

$$\frac{dy_i}{d\lambda} = \frac{\Delta_i}{\Delta}, \quad \frac{dx_j}{d\lambda} = \frac{\Delta_{n+j}}{\Delta}, \quad \frac{dt}{d\lambda} = \frac{\Delta_{n+m+1}}{\Delta}, \quad (7)$$

$$i = \overline{1, n}, \quad j = \overline{1, m},$$

where  $\Delta$  is the determinant of the system;  $\Delta_k = (-1)^{k+1}\delta_k$ , ( $k = \overline{1, n+m+1}$ );  $\delta_k$  is the determinant of the matrix that is obtained from the matrix of the last  $n + m$  equations of the system by deleting its  $k$ th column. These determinants satisfy the equation

$$\Delta^2 = \Delta_k \Delta_k, \quad (k = \overline{1, n+m+1}). \quad (8)$$

This equation shows that the Euclidean norm of the right hand side of the system of ordinary differential equations (7) is always equal to one. If the argument  $\lambda$  is measured from the initial point of the problem (1), the initial conditions take the form (6).

Thus we have proved the following

**Theorem.** *In order to formulate the Cauchy problem (1) for the system of differential-algebraic equations with respect to the best argument, it is necessary and sufficient to choose the arc length  $\lambda$  measured along the integral curve of the problem as this argument. In this case the problem (1) is transformed into the problem (7), (6) and the right hand sides of the problem (7) satisfy the relation (8).*

## 3. EXPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS

The Cauchy problem for the system of explicit DAEs is given by

$$\begin{cases} \frac{dy}{dt} = f(y, x, t), & y(t_0) = y_0, \\ G(y, x, t) = 0, & x(t_0) = x_0, \end{cases} \quad (9)$$

$y : \mathbb{R}^1 \longrightarrow \mathbb{R}^n$ ,  $x : \mathbb{R}^1 \longrightarrow \mathbb{R}^m$ ,  $f : \mathbb{R}^{n+m+1} \longrightarrow \mathbb{R}^n$ ,  $G : \mathbb{R}^{n+m+1} \longrightarrow \mathbb{R}^m$ ,  $G(y_0, x_0, t_0) = 0$ .

This problem is a special case of the problem (1). Let us formulate it with respect to the best argument  $\lambda$  assuming that functions  $y = y(\lambda)$ ,  $x = x(\lambda)$ ,  $t = t(\lambda)$  are differentiable. Introduce the notation

$$\frac{dy}{d\lambda} = Y, \quad \frac{dx}{d\lambda} = X, \quad \frac{dt}{d\lambda} = T, \quad (10)$$

$$Y = (Y_1, \dots, Y_n)^T, \quad X = (X_1, \dots, X_m)^T.$$

Differentiating the vector function  $G$  with respect to  $\lambda$  and taking into account relations (10) and the definition of the best argument, let us write the system (9) in the form

$$\begin{cases} Y_i & - & f_i T & = & 0, \\ G_{,y_i} Y_i & + & G_{,x_j} X_j & + & G_{,t} T & = & 0, \\ Y_i Y_i & + & X_j X_j & + & T T & = & 1, \\ & & i = \overline{1, n}, & & j = \overline{1, m}. \end{cases} \quad (11)$$

Because of the last equation, this system is nonlinear with respect to functions  $Y$ ,  $X$ ,  $T$ . However, we can represent this system in linear form in the iterative process using the solution obtained at the previous  $(k-1)$ -th step. In order to get this representation, we rewrite the system (11) in the form

$$\begin{cases} Y_i^{(k)} & - & f_i T^{(k)} & = & 0, \\ G_{,y_i} Y_i^{(k)} & + & G_{,x_j} X_j^{(k)} & + & G_{,t} T^{(k)} & = & 0, \\ Y_i^{(k-1)} Y_i^{(k)} & + & X_j^{(k-1)} X_j^{(k)} & + & T^{(k-1)} T^{(k)} & = & 1, \end{cases} \quad (12)$$

where  $k = 1, 2, \dots$

Let us denote by  $Z^{(k)} = (Y^{(k)}, X^{(k)}, T^{(k)})^T$  a  $n+m+1$ -dimensional vector. Because of the structure of the system (12), this vector is tangential to the integral curve  $K$  of problem (9) at the point corresponding to the  $k$ th step. Thus the last equation of the system (12) is a scalar

product of vectors  $Z^{(k)}$  and  $Z^{(k-1)}$  tangential to the integral curve at  $k$ th and  $(k-1)$ -th steps. This equation states that projection of the vector  $Z^{(k)}$  onto the direction of the unit vector  $Z^{(k-1)}$  is equal to one. Replacing the unknown vector  $Z^{(k)}$  with the known vector  $Z^{(k-1)}$  in (12), we ensure a local choice of an argument which is close to the best one.

Clearly, the vector  $Z^{(k)}$  which satisfies the system of linear equations (12), in general, is not a unit vector, as it is required by the system (11). Therefore, after finding a solution of the system (12) the obtained vector  $Z^{(k)}$  should be normalized:

$$Z_i^{*(k)} = \frac{Z_i^{(k)}}{\sqrt{Z_j^{(k)} Z_j^{(k)}}}, \quad i, j = \overline{1, n+m+1}. \quad (13)$$

This yields a solution to the system (11). Below we will omit the asterisk in (13).

Since the initial point usually is not a limiting point with respect to  $t$ , we can take initial approximation of the vector  $Z$  in the form

$$Z^{(0)} = (0, \dots, 0, 1)^T. \quad (14)$$

Assuming that the argument  $\lambda$  is counted from the initial point of the problem (9), the following algorithm for its solution can be proposed.

The solution of differential equations (10) satisfying the initial conditions

$$y(0) = y_0, \quad x(0) = x_0, \quad t(0) = t_0. \quad (15)$$

is obtained. Right hand sides of the system (10) are determined from the solution of the system of linear equations (12) by Gauss elimination method. This solution is normalized by formulas (13).

Such an approach allows not only to overcome the difficulties associated with vanishing of the Jacobian  $G_{,x_j}$  but also to solve systems (9) in which the right hand sides  $f$  of differential equations become infinite at some points. To overcome difficulties associated with the latter case, it is sufficient to rewrite, where it is possible, first  $n$  equations of the system (11) in the form  $Q_\alpha Y_\alpha - P_\alpha T = 0$ . Here  $\alpha = \overline{1, n}$ , summation with respect to this index is not performed, and functions  $Q_\alpha$ ,  $P_\alpha$  are finite.

If the functions at the right hand side of the system of differential equations (9) are finite, the dimension of the system (12) can be reduced

by  $n$ , then we write this system in the form

$$\begin{cases} G_{,x_j} X_j^{(k)} + (G_{,t} + G_{,y_i} f_i) T^{(k)} = 0, \\ X_j^{(k-1)} X_j^{(k)} + (1 + f_i f_i) T^{(k-1)} T^{(k)} = 1. \end{cases} \quad (16)$$

Given a solution to the system (16), the values  $Y_i^{(k)}$  are defined by the formulas  $Y_i^{(k)} = f_i T^{(k)}$ . Then, the vector  $Z^{(k)}$  should be normalized according to (13), and the values obtained should be used as the right hand sides for the system (10).

#### 4. IMPLICIT ORDINARY DIFFERENTIAL EQUATIONS

Let us consider the problem

$$f(y, \dot{y}, t) = 0, \quad y(t_0) = y_0, \quad (17)$$

where  $f = (f_1, \dots, f_n)^T$ ,  $\dot{y} = dy/dt$ . If the Jacoby matrix  $\partial f / \partial \dot{y}$  is not singular then this system of implicit equations is a DAE of index zero.

We consider algorithm for numerical solution of the problem (17) without its transformation to the form (9). Clearly, the problem (17) is a particular case of problem (1). Let us formulate problem (17) in terms of the best argument.

Let  $y_i$  and  $t$  be functions of the best argument  $\lambda$  measured from the initial point of problem (17). Let us introduce the notation

$$\frac{dy_i}{d\lambda} = Y_i, \quad \frac{dt}{d\lambda} = T, \quad i = \overline{1, n}. \quad (18)$$

It follows from the meaning of the best argument that the right hand sides of these relations satisfy the equation

$$Y_i Y_i + T^2 = 1. \quad (19)$$

We linearize the system (17) at  $k-1$ -th step of iteration with respect to  $\dot{y}_i$  and equation (19) with respect to functions  $Y_i$  and  $T$ . Then, in view of relations  $y_i = Y_i/T$ , we obtain the system of linear equations in  $Y_i^{(k)}$  and  $T^{(k)}$  at the  $k$ -th step of the iteration process

$$\begin{cases} Y_i^{(k-1)} Y_i^{(k)} + T^{(k-1)} T^{(k)} = 1, \\ T^{(k-1)} f_{, \dot{y}_i}^* Y_i^{(k)} + (f^* T^{(k-1)} - f_{, \dot{y}_i}^* Y_i^{(k-1)}) T^{(k)} = 0. \end{cases} \quad (20)$$

Here the asterisk marks the vector functions calculated for

$$\dot{y}_i = Y_i^{(k-1)} / T^{(k-1)}.$$

Whenever possible it is recommended to write the system (20) in the form that does not include relations that tend to infinity and the terms containing  $Y_i$  and  $T$  in denominators.

If the initial point is not singular, then the value of the vector  $Z = (Y_1, \dots, Y_n, T)^T$  can be taken in the form

$$Z^{(0)} = (0, \dots, 0, 1). \quad (21)$$

Thus, the problem reduces to the integration of the system of ordinary differential equations (18) that satisfy the initial conditions

$$y_i(0) = y_{i0}, \quad t(0) = t_0. \quad (22)$$

The right hand sides of the system (18) are determined by the solution of the system (20) normalized by formulas of the type (13).

Clearly, the last  $n$  equations of the system (20) determine the Newton-Raphson procedure, thus, this system of equations should be solved until it converges to a given accuracy  $\varepsilon : \|Z^{(k)} - Z^{(k-1)}\| < \varepsilon$ . Note that if the system of ordinary differential equations (18) is solved by the program PC1 [8], this condition is ensured by the predictor-corrector method. The solution calculated at a certain step of the integration process is taken as the initial approximation for the iteration procedure at the next step.

Obviously, the system (20) takes the simplest form when the system of ordinary differential equations (17) is linear with respect to the derivatives  $\dot{y}_i$ , i.e., it is given by

$$a_{ij}(y_1, \dots, y_n, t) \frac{dy_j}{dt} + g_i(y_1, \dots, y_n, t) = 0, \quad i, j = \overline{1, n}.$$

It was mentioned above that, though the last equation in the system (20) is approximate, we obtain, after normalization, the solution of the nonlinear system

$$\begin{cases} a_{ij} Y_j^{(k)} + a_{i \ n+1} T^{(k)} = 0, \\ Y_i^{(k)} Y_i^{(k)} + T^{(k)} T^{(k)} = 1, \quad a_{i \ n+1} = g_i, \end{cases}$$

which does not depend on  $Y_i^{(k-1)}, T^{(k-1)}$ .

Another algorithm can be proposed for solving the nonlinear problem (17). This algorithm does not involve the linearization of equations but requires an additional differentiation. Taking the relation  $\dot{y}_i = Y_i/T$  into account, let us rewrite equation (17) in the form

$$F(t, y_1, \dots, y_n, Y_1, \dots, Y_n, T) = 0, \quad F = (F_1, \dots, F_n)^T, \quad (23)$$

such that, whenever possible, the terms containing  $Y_i$  and  $T$  as the divisors and the relations tending to infinity are eliminated.

Differentiating equations (19) and (23) with respect to  $\lambda$ , we obtain the system of linear equations

$$\begin{cases} F_{,Y_i} Y_i' + F_{,T} T' = -F_{,y_i} Y_i - F_{,t} T, \\ Y_i Y_i' + T T' = 0, \end{cases} \quad (24)$$

in the functions

$$\frac{dY_i}{d\lambda} = Y_i', \quad \frac{dT}{d\lambda} = T', \quad i = \overline{1, n}. \quad (25)$$

Now, the problem is to integrate the system of ordinary differential equations (18), (25) that satisfy the initial conditions (22) and the following conditions

$$Y_i(0) = Y_{i0}, \quad T(0) = T_0, \quad i = \overline{1, n}. \quad (26)$$

The right hand sides of equations (25) are determined by the solution of the linear system (24) and the initial values (26) of the functions  $Y_{i0}$  and  $T_0$  are obtained from the following system of equations

$$\begin{cases} F(t_0, y_{10}, \dots, y_{n0}, Y_{10}, \dots, Y_{n0}, T_0) = 0, \\ Y_{i0} Y_{i0} + T_0^2 = 0. \end{cases} \quad (27)$$

## 5. IMPLICIT DIFFERENTIAL-ALGEBRAIC EQUATIONS

Consider implicit differential-algebraic equations defining the Cauchy problem in the form (1), i.e.

$$\begin{cases} F(y, \dot{y}, x, t) = 0, & y(t_0) = y_0, \\ G(y, x, t) = 0, & x(t_0) = x_0, \end{cases} \quad (28)$$

where  $y(t) = (y_1(t), \dots, y_n(t))^T$ ,  $x(t) = (x_1(t), \dots, x_m(t))^T$ .

By introducing new variables  $z_i = \dot{y}_i$  one can transform this problem into the explicit problem (9) in the extended space of variables. It was shown above, however, that such an approach may result in computational difficulties.

We consider an algorithm for solving problem (28) without transformation of it to the form (9). For problem (1), or (28), we have proved Theorem 1 which determines the best argument  $\lambda$ . Let us formulate the problem with respect to this argument.

Let  $y_i = y_i(\lambda)$ ,  $x_j = x_j(\lambda)$ ,  $t = t(\lambda)$  be differentiable functions of the argument  $\lambda$  that is counted along the integral curve from the initial point of the Cauchy problem (28).

Taking into account the notation (10) and the meaning of the best argument we obtain the equations

$$Y_i Y_i + X_j X_j + T^2 = 1. \quad (29)$$

Let us linearize the system (28) and the equation (29) with respect to derivatives  $\dot{y}_i$  and quadratic terms, respectively. We also differentiate the vector function  $G$  with respect to  $\lambda$ . Thus, we obtain the system of equations that are linear with respect to the functions  $Y_i, X_j, T$  calculated at the  $k$ -th step of the iteration process

$$\begin{cases} T^{(k-1)} f_{, \dot{y}_i}^* Y_i^{(k)} + \left( f^* T^{(k-1)} - f_{, \dot{y}_i}^* Y_i^{(k-1)} \right) T^{(k)} = 0, \\ G_{, y_i} Y_i^{(k)} + G_{, x_j} X_j^{(k)} + G_{, t} T^{(k)} = 0, \\ Y_i^{(k-1)} Y_i^{(k)} + X_j^{(k-1)} X_j^{(k)} + T^{(k-1)} T^{(k)} = 1. \end{cases} \quad (30)$$

Here the asterisk marks the functions calculated at the previous step, i.e., for  $\dot{y}_i = Y_i^{(k-1)} / T^{(k-1)}$ .

If the initial point is not a singular point, initial value of the vector  $Z = (Y, X, T)^T$  can be taken in the form (14). Now the problem is to integrate a system of ordinary differential equations (10) the right hand sides of which are obtained by solving the system of linear equations (30) with the help of the Newton – Raphson method followed by the normalization (13) of the solution. Since the argument  $\lambda$  is measured from the initial point of problem (28), the initial conditions for the system (10) have the form

$$y_i(0) = y_{i0}, \quad x_j(0) = x_{j0}, \quad t(0) = t_0, \quad i = \overline{1, n}, \quad j = \overline{1, m}. \quad (31)$$

Clearly, if the system (28) is linear with respect to the derivatives  $\dot{y}_i$ , then the solution of (30), which is obtained by means of this approach, does not require iterative improvement, does not depend on the solution found at the previous step, and satisfies the relation (29).

This algorithm was implemented in the DA1ILN program [8] in which a system of differential equations is integrated by PC1 program [8] and the system of linear equations is solved by the Gauss elimination method.

Note that, when equations of problems (17), or (28), are nonlinear with respect to the derivatives  $\dot{y}_i$ , the procedure of prediction-correction method in PC1 program provides more precise iterative determination of the solution of the linearized systems (20).

Another algorithm for solving problem (28) is as follows. Taking into account the relations  $\dot{y}_i = Y_i / T$  and following the rules formulated in the

previous section, we rewrite the first vector equation of the problem in the form

$$F(t, y_1, \dots, y_n, x_1, \dots, x_m, Y_1, \dots, Y_n, T) = 0. \quad (32)$$

Let us differentiate equations (29) and (32) with respect to  $\lambda$  once and with respect to the vector function  $G$  of the system (28) twice. Then we obtain the following system of linear equations for the derivatives of  $Y_i$ ,  $X_j$  and  $T$

$$\begin{cases} F_{,Y_i} Y_i' + F_{,T} T' = -(F_{,y_i} Y_i + F_{,x_j} X_j + F_{,t} T), \\ G_{,y_i} Y_i' + G_{,x_j} X_j' + G_{,t} T' = -(G'_{,y_i} Y_i + G'_{,x_j} X_j + G'_{,t} T), \\ Y_i Y_i' + X_j X_j' + T T' = 0. \end{cases} \quad (33)$$

Here prime denotes differentiation with respect to  $\lambda$ :

$$\frac{dY_i}{d\lambda} = Y_i', \quad \frac{dX_j}{d\lambda} = X_j', \quad \frac{dT}{d\lambda} = T'. \quad (34)$$

Thus, the problem is to solve the system of ordinary differential equations (10), (34) satisfying initial conditions (31) and the following conditions:

$$\begin{aligned} Y_i(0) = Y_{i0}, \quad X_j(0) = X_{j0}, \quad T(0) = T_0, \\ i = \overline{1, n}, \quad j = \overline{1, m}. \end{aligned} \quad (35)$$

The right hand sides of equations (34) satisfy the system of linear equations (33), and initial conditions (35) are determined as solutions of nonlinear equations

$$\begin{cases} F(t_0, y_{10}, \dots, y_{n0}, x_{10}, \dots, x_{m0}, Y_{10}, \dots, Y_{n0}, T_0) = 0, \\ G_{,y_i}^0 Y_{i0} + G_{,x_j}^0 X_{j0} + G_{,t}^0 T_0 = 0, \\ Y_{i0} Y_{i0} + X_{j0} X_{j0} + T_0^2 = 1, \end{cases}$$

where the superscript zero in the vector function  $G$  means that the derivative is calculated at the initial point of problem (28).

## 6. NUMERICAL EXAMPLES

**Example 1.** We consider numerical integration of the implicit system of Euler's kinematic equations [8]

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \sin \theta \sin \varphi & 0 & \cos \varphi \\ \sin \theta \cos \varphi & 0 & -\sin \varphi \\ \cos \theta & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_{,t} \\ \varphi_{,t} \\ \theta_{,t} \end{pmatrix}.$$

The matrix of the system tends to be singular in neighborhood of the value  $\theta = 0$ , and solving this system by DE1ILN program [8] with conditions  $\omega_1 = -100$ ,  $\omega_2 = 1$ ,  $\omega_3 = 0$ ,  $\psi = \varphi = t = 0$ ,  $\theta = \pi/100$  fails.

Performing the  $\lambda$ -transformation of the system, we write it as follows:

$$\begin{pmatrix} \sin \theta \sin \varphi & 0 & \cos \varphi & -\omega_1 \\ \sin \theta \cos \varphi & 0 & -\sin \varphi & -\omega_2 \\ \cos \theta & 1 & 0 & -\omega_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \begin{pmatrix} \psi, \lambda \\ \varphi, \lambda \\ \theta, \lambda \\ t, \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

This system of equations is solved successfully for conditions  $\omega_1 = -100$ ,  $\omega_2 = 1$ ,  $\omega_3 = 0$ ,  $\psi = \varphi = t = 0$ ,  $\theta = \pi/100$ , as well as for the latter condition  $\theta = 0$ , if the initial value of vector  $Z$  is  $Z^{(0)} = (1, 0, 0, 0)$ .

Note that in the same way one can solve the following system of kinematic equations for airplane angles [7]

$$\begin{pmatrix} \sin \vartheta & 0 & 1 \\ \cos \vartheta \cos \gamma & \sin \gamma & 0 \\ -\cos \vartheta \sin \gamma & \cos \gamma & 0 \end{pmatrix} \begin{pmatrix} \psi, t \\ \vartheta, t \\ \gamma, t \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

where  $\psi$  is the angle of yaw,  $\vartheta$  is the pitch angle, and  $\gamma$  is the angle of bank.

The matrix of this system becomes singular at  $\vartheta = \pi/2$ . Performing the  $\lambda$ -transformation, we write the latter system as follows:

$$\begin{pmatrix} \sin \vartheta & 0 & 1 & -\omega_1 \\ \cos \vartheta \cos \gamma & \sin \gamma & 0 & -\omega_2 \\ -\cos \vartheta \sin \gamma & \cos \gamma & 0 & -\omega_3 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix} \begin{pmatrix} \psi, t \\ \vartheta, t \\ \gamma, t \\ t, \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

**Example 2.** The initial problem on the segment  $t \in [0, 1]$ , for linear DAEs  $A(t)\dot{x} + B(t)x = f(t)$  with

$$A(t) = \begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}, \quad B(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f(t) = \begin{pmatrix} (t+1)^2 \\ (t+1)^2 - 1 \end{pmatrix},$$

and  $x(0) = (1; 2)^T$ . The exact solution of the problem is

$$x^0(t) = ((t+1)^2, (t+1)^2 + 1)^T.$$

This system has the singular matrix  $A(t)$  and the differentiation index 1. This example was considered in [1] and [4].

We used the method of solution continuation with respect to a parameter. The system of linear equations (12) in this case takes the form

$$\begin{cases} Y^{(k)} + tX^{(k)} + (-y - (t+1)^2)T^{(k)} = 0, \\ Y^{(k)} + -X^{(k)} = 0, \\ Y^{(k-1)}Y^{(k)} + X^{(k-1)}X^{(k)} + T^{(k-1)}T^{(k)} = 1. \end{cases}$$

The linear system (11) was solved by the Gauss method and the modified Euler method was used to solve the system of differential equations (10). To obtain inaccuracy 0,005 5 iterations were applied.

**Conclusion.** The method of solution continuation with respect to a parameter can be applied to a great number of equations. The solutions of ordinary differential equations and differential-algebraic equations given in the present paper demonstrate that using this method one can overcome difficulties which appear when some of the famous numerical methods are used, and obtain enough high accuracy of calculation. In particular, by this method one can solve differential-algebraic equations of index 1 and higher.

#### REFERENCES

- [1] V. Chistyakov, Singular systems of ordinary differential equations (Nauka, Novosibirsk, 1982)(Russian).
- [2] D. Davidenko, On a new method for the numerical solution of systems of nonlinear equations, Dokladi Akademii Nauk of Russia, Vol. 88, 601-602.
- [3] D. Davidenko, On the approximate solution of systems nonlinear equations, Ukr. Mat. Zh., Vol. 5 (1953) 196-206.
- [4] V. Gorbunov, V. Petrishchev, Development of the normal spline method for linear differential equations, Zh.Vychisl. Mat. Mat. Fiz. 43 (2003) 1161-1170.
- [5] M. Lahaye, Une metode de resolution d'une categorie d'equations transcendentales, Compter Rendus hebdomataires des seances de L'Academie des sciences, Vol. 198 (1934) 1840-1842.
- [6] M. Lahaye, Solution of system of transcendental equations, Acad. Roy. Belg. Bull. Cl. Sci., Vol. 5 (1948) 805-822.
- [7] A. Lebedev, L. Chernobrovkin, The dynamics of the flying (Mashinostroenie, 1973)(Russian).
- [8] V. Shalashilin, E. Kuznetsov, Parametric Continuation and Optimal Parametrization in Applied Mathematics and Mechanics (Kluvert Academic Publishers, Dordrecht, Boston, London, 2003).

MOSCOW AVIATION INSTITUTE, VOLOKOLAMSKOE SH. 4, 107065, MOSCOW, RUSSIA

*E-mail address:* lesma@mail.ru

Received October 29, 2005