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APPLICATION OF MATRIX POLYNOMIALS TO INVESTIGATION OF SINGULAR EQUATIONS

(submitted by A. M. Elizarov)

Abstract. A class of matrix polynomials having some dominant properties is described, and some characteristic properties of such polynomials have been found. Reduction of integral and integro-differential equations having singular matrices at the main part to systems with nonsingular ones is proposed. For systems of nonlinear finite-dimensional equations with singular Jacobi matrix, the definition for multiplicity of a solution is introduced. Reduction methods of such systems to the systems with isolated solutions, which can be numerically solved by well-known methods, are suggested.

1. Introduction

Investigation of matrix polynomials is widely applied to the study of differential-algebraic equations (DAE). For instance, basing on K.Weierstrass's article [21] on reduction of the regular matrix pencil to the canonical form it was ascertained that the general solution of linear DAE with constant coefficients can depend not only on the right-hand side but also

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on the derivatives. All further developments concerning numerical methods for solving the linear DAE with constant coefficients also take into account the matrix pencils structure.

In the present paper we investigate matrix polynomial applications to the higher order DAEs, to the singular systems of integral and integro-differential equations, and to the systems of nonlinear algebraic equations having multiple solutions.

2. Properties of semi-inverse matrices and λ-matrices

In this section we give several definitions and auxiliary statements from the matrix polynomial theory.

Definition 1 [3], [20]. *An $(n \times m)$-matrix A^- is called semi-inverse with respect to an $(m \times n)$-matrix A if the matrix A^- satisfies the equation*

$$AA^-A = A.$$

For any nonsingular square matrix A the semi-inverse matrix coincides with the inverse one. If the matrix A is arbitrary then there exists a semi-inverse $(n \times m)$-matrix A^-, which is nonunique, in general [20].

The pseudo-inverse matrix, call it A^+, is one of the semi-inverse matrices of the matrix A. Unlike the semi-inverse matrix, the pseudo-inverse matrix is uniquely determined. In [20] one can find many constructive algorithms for computing the matrix A^+ and also an extended list of bibliography on the problem.

Definition 2 [17]. *The expression $A(\lambda) = \lambda^k A_0 + \lambda^{k-1} A_1 + \cdots + A_k$, where $A_0, A_1 \cdots A_k$, are constant matrices of the same dimension, λ is a scalar and $A_0 \neq 0$, is called the λ-matrix of degree k.*

Definition 3. *The matrix $A(\lambda)$ is said to be regular if there exists a scalar λ such that* $\det A(\lambda) \neq 0$.

Definition 4 [11]. *The pencil of $(n \times n)-$matrices $\lambda A + B$ has the simple structure (satisfies the "rank-degree" criterion, or has the index 1) if*

$$\deg\det(\lambda A + B) = \mathrm{rank} A.$$

Hereafter the symbol $\deg(P(\lambda))$ denotes the degree of a polynomial $P(\lambda)$.

Lemma 1 [6], [9]. *The matrix pencil $\lambda(E - AA^-) + E$ satisfies the rank-degree criterion (has index one).*

Definition 5. *The λ -matrix*

$$A(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i, \ A_0 \neq 0,$$

will be said to possess the dominant property (DP) if

$$\deg\det A(\lambda) \geq k\,\mathrm{rank} A_0.$$

For instance, the matrix $\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$ does not possess this property, while the matrix $\begin{pmatrix} \lambda^2 & \lambda \\ \lambda & d \end{pmatrix}$ for $d \neq 1$ does.

Let us construct the sequence of matrices

$$A^{(i)}(\lambda) = (E + \lambda(E - A_0^{(i-1)} A_0^{(i-1)-}))A^{(i-1)}(\lambda), \tag{1}$$

where $A_0^{(0)} = A_0$, $A^{(0)}(\lambda) = A(\lambda)$, the upper index of the matrices A denotes the number of iteration, and $A_0^{(i)}$ is the coefficient at the higher degree of λ in $A^{(i)}(\lambda)$.

For the subsequent arguments the following result is important.

Theorem 1. *Let the matrix $A(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i$ possess the DP and $\mathrm{rank} A_0 = r \leq n$. Then, for the matrix $A^{(k)}(\lambda)$ defined by the recurrent formula* (1), $\det A_0^{(k)} \neq 0$.

Proof. If $\deg\det A(\lambda) = kr + S \geq kr$, from Lemma 1 we get the following equality

$$\deg\det A^{(1)}(\lambda) = kr + S + n - r \geq k(r + s_1),$$

where $r + s_1 = \mathrm{rank} A_0^{(1)}$. For the iterations (1) one has

$$\begin{aligned} \deg\det A^{(i)}(\lambda) = kr + S + n - r + n - (r + s_1) + \cdots \\ + n - (r + s_1 + s_2 + \cdots + s_{i-1}) \\ \geq k(r + s_1 + s_2 + \cdots + s_i), \end{aligned} \tag{2}$$

where $r + s_1 + s_2 + \cdots + s_i = \mathrm{rank} A_0^{(i)}$. Subtraction the $(i-1)$-th inequality from the i-th one leads to the estimate

$$s_i \leq (n - (s_1 + s_2 + \cdots + s_{i-1}))/k \leq (n - r)/k. \tag{3}$$

Now, substituting $i = k$ and $s_i \leq (n - r)/k$ into the right-hand-side of (3), we get

$$\deg\det A^{(k)}(\lambda) \geq k(r + s_1 + s_2 + \cdots + s_k) \geq kn. \tag{4}$$

Since $A^{(k)}(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i^{(k)}$, from the estimate (4) it follows that $\det A_0^{(k)} \neq 0$. $\square$

The following properties of λ-matrices take place.

Property 1. *If $A(\lambda)$ is a regular matrix, which does not possess the DP, then there exists a positive integer m such that the matrix $\lambda^m A(\lambda)$ possesses the DP.*

Property 2. *If the matrix $A(\lambda)$ possesses the DP, then the matrix $(E+\lambda(E-A_0A_0^-))A(\lambda)$ also possesses the DP.*

3. Integral equations

Consider the following system of integral equations

$$Ax(t)+\int_0^t K(t-\tau)x(\tau)d\tau = f(t), \qquad t\in[0,1], \tag{5}$$

where A is an $(n\times n)-$matrix, $\det A=0$, $K(t)$ is an $(n\times n)-$matrix with real-analytical coefficients, $f(t)$ is a given sufficiently smooth vector-function and $x(t)$ is the unknown continuous n-dimensional vector-function.

Since the elements of the matrix $K(t)$ are real-analytical functions, in case $A=0$ there exists at least one value of i such that $K^{(i)}(0)\neq 0$. Differentiating this equation $i+1$ times and setting $K^{(i)}(0)=A$, we obtain the equation of the form (5).

Henceforth we will assume that (5) has at least one solution. A rather simple criterion of the uniqueness of the solution for the problem (5) will be proposed below.

Let us give an example. The system

$$\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} x(t)+\int_0^t \begin{pmatrix} 0 & 1\\ 1 & c(t-\tau)\end{pmatrix} x(\tau)d\tau = \begin{pmatrix} f_1(t)\\ f_2(t)\end{pmatrix}, \tag{6}$$

where c is a scalar, $t\in[0,1]$, has the following unique solution

$$x_1(t)=f_2'(t)-c(f_1(t)-f_2'(t))/(1-c),$$
$$x_2(t)=(f_1'(t)-(f_2''(t))/(1-c)$$

for any $f_1\in C^1$, $f_2\in C^2$, $f_2(0)=0$, $f_1(0)=f_2'(0)$.

If, otherwise, $c=1$ then any pair $x_1=v(t)$, $x_2=f_1'-v'$, with arbitrary continuous function $v(t)$, $v(0)=f_1(0)=0$, is a solution of the system (6) if and only if $f_1(0)=0$, $f_1(t)\equiv f_2'(t)$.

If the system (5) has a unique solution there are several ways to find it out. In particular, applying the integral Laplace transformation [16] to (5), it is possible to reduce it to the system on finite-dimensional equations in the space of transforms and then to answer the question of existence and uniqueness of solution for the obtained system.

In [2], [19] other investigations can be found allowing to conclude on existence of the unique solution for the integral equation (5).

The method of reduction of the problem (5) to the system of integral equations of second kind has been proposed in [11]. For such reduction realization it is necessary to investigate an "l-extended system" of the dimension $(ln \times (l+1)n)$.

For the case when the criterion of solution uniqueness is satisfied, it is suggested the transformation of the problem (5) to the system of integral equations of the second kind

$$x(t) + \int_0^t \bar{K}(t-\tau)x(\tau)d\tau = \bar{f}(t), \qquad t \in [0,1], \tag{7}$$

where $\bar{K}(t)$ is a real-analytic $(n \times n)-$matrix, $\bar{f}(t)$ is a continuous vector-function. This equation (7) has a unique solution [16].

Introduce notations

$$A_0 = A, \qquad A_i = K^{(i-1)}(0), \qquad i = 1, 2, \dots, k, \tag{8}$$

where A and $K(t)$ are the same matrices as in the original equation (5), and equation sequence

$$A^{(i)}x(t) + \int_0^t K_i(t-\tau)x(\tau)d\tau = f_i(t), \qquad t \in [0,1], \tag{9}$$

where

$$\begin{gathered} A^{(i)} = A^{(i-1)} + (E - A^{(i-1)}A^{(i-1)-})K_{i-1}(0), \\ K_i(t) = K_{i-1}(t) + (E - A^{(i-1)}A^{(i-1)-})K'_{i-1}(t), \\ f_i(t) = f_{i-1}(t) + (E - A^{(i-1)}A^{(i-1)-})f'_{i-1}(t), \\ A^{(0)} = A, \qquad K_0(t) = K(t), \qquad f_0(t) = f(t). \end{gathered}$$

Theorem 2. *Let*

1) for the equation (5) *there exists a number k such that the matrix $A(\lambda)$ possesses the DP and $A^{(i)}$ are determined by (9);*

2) rank$A^{(i)}$ = rank$\{A^{(i)}|f_i(0)\}$, $\qquad i = 0, 1, \dots, k$.

3) the functions $K(t-\tau)$, $f(t)$ belong to the class C^k

Then

1) the system (5) *has a unique continuous solution;*

2) the problem (5) *is equivalent to any system from (9);*

3) in the system (9) $\det A^{(k)} \neq 0$.

Proof. At the assumption that $t = 0$ in the equations (9), one has the following system of linear algebraic equations

$$A^{(i)}x(0) = f_i(0),$$

solvability of which follows from the condition 2 of the theorem. Let us prove that the solution of the $(i-1)-$th system in (9) is a solution of the $i-$th system, and vice versa. To this end it is sufficient to note that the $i-$th system of (9) is the result of the operator $(E + (E - A^{(i-1)}A^{(i-1)-})d/dt)$ application to the previous $(i-1)-$th system.

Then it follows [12] that problems

$$(E + (E - A^{(i-1)}A^{(i-1)-})d/dt)x(t),\ x(0) = 0,\ t \in [0, 1],$$

have only trivial solution, and hence, the solution of the $(i-1)-$th system in (9) is also the solution of the $i-$th system. For the proof that the solution of the original problem is unique replace in Theorem 1 the scalar λ with the differential operator d/dt. Due to the first condition of the Theorem one has

$$\det A^{(k)} \neq 0,$$

and hence the system (9) for $i = k$, is the system of integral equations of second kind, having for continuous $K_k(t-\tau),\ f_k(t)$ a unique continuous solution [16]. The continuity of the components $K_k(t-\tau),\ f_k(t)$ follows from the condition 3 of the theorem. □

4. Integro-differential equations

Consider the following system of integro-differential equations

$$B_0 x^{(p)}(t) + B_1 x^{(p-1)}(t) + \cdots + B_p x(t) + \int_0^t D(t-\tau)x(\tau)d\tau = g(t), \quad (10)$$

$$x^{(j)}(0) = a_j, \qquad j = 0, 1, \cdots, p-1, \quad (11)$$

where B_j are $(n \times n)$-matrices with constant coefficients, $t \in [0, 1]$, $D(t)$ and $g(t)$ are sufficiently smooth matrix and vector-function, respectively, and also $\operatorname{rank} B_0 = r < n$.

Now take the matrix

$$B(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} B_i, \quad (12)$$

where the matrices $B_i, \quad i = 0, 1, \ldots, p$ are the same ones as in (10), and $B_{p+j} = D^{(j-1)}(t)|_{t=0}, \quad j = 1, 2, \ldots, k-p$. Apply to the system (10) the operator

$$P_k = \prod_{i=1}^{k} (E + d/dt(E - B_0^{(k-i)} B_0^{(k-i)-})), \quad (13)$$

where $B_0^{(j)} = B_0^{(j-1)} + (E - B_0^{(j-1)} B_0^{(j-1)-}) B_1^{(j-1)}$, $B_0^{(0)} = B_0, j = 1, 2, \ldots, k-1$.

According to Definition 1 we obtain the system

$$\sum_{i=0}^{p} B_i^{(k)} x^{(p-i)}(t) + \int_0^t \bar{D}(t-\tau)x(\tau)d\tau = \bar{g}(t), \qquad t \in [0,1]. \tag{14}$$

Theorem 3. *Let for some value of k the matrix $B(\lambda)$ (12) has the DP. Then, in the system (14) $\det B_0^{(k)} \neq 0$.*

The proof of this theorem is similar to the proof of Theorem 2.

Definition 6. *The minimal possible value of k for which the matrix $B(\lambda)$ (12) possesses the DP, will be called the nonsolvability index of the system (10).*

Consider the following partial case of the system (10):

$$\sum_{i=0}^{p} B_i x^{(p-i)}(t) = q(t), \qquad t \in [0,1]. \tag{15}$$

If for some $k, 1 \leq k \leq p$, the matrix $B(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} B_i$ possesses the DP, then applying the operator $P_k = \prod_{i=1}^{k}(E + d/dt(E - B_0^{(k-i)} B_0^{(k-i)-}))$ to the system (15) we obtain the following system

$$\sum_{i=0}^{p} B_i^{(k)} x^{(p-i)}(t) = q_k(t), \qquad t \in [0,1], \tag{16}$$

with nonsingular matrix at the derivative of the highest order.

In this case,

$$B_j^{(i)} = B_j^{(i-1)} + (E - B_0^{(i-1)} B_0^{(i-1)-}) B_{(j+1)}^{(i-1)},$$

$$B_j^{(0)} = B_j,\ j = 0,1,\ldots,p,\ i = 1,2,\ldots,k.$$

Find the value of k for the case when $B(\lambda)$ does not possess the DP, while $\det B(\lambda) \neq 0$.

Omitting details of transformations and arguments, we derive

$$k = \begin{cases} (np-s)/(n-r), & np-s \text{ is multiple to } n-r, \\ [(np-s)/(n-r)]+1, & \text{in the opposite case,} \end{cases} \tag{17}$$

Here $[\alpha]$ is the integer part of the number α, $s = \deg\det B(\lambda)$, $r = \mathrm{rank} B_0$.

For the investigation of the first kind systems, note the important role of the pencil $\lambda B_0 + B_1$ index, i.e. the minimal nonnegative integer k for which the following equality holds: [3]

$$\mathrm{rank}((\lambda B_0 + B_1)^{-1} B_0)^{k+1} = \mathrm{rank}((\lambda B_0 + B_1)^{-1} B_0)^{k}.$$

Formula (17) gives more simple way to the index of the pencil λB_0+B_1 computation, namely

$$k = \begin{cases} (n-s)/(n-r) & n-s \text{ is multiple to } n-r, \\ [(n-s)/(n-r)]+1 & \text{in the opposite case,} \end{cases} \tag{17a}$$

where $s = \deg\det(\lambda B_0 + B_1)$, $r = \mathrm{rank} B_0$.

5. Finite-dimensional nonlinear equations

Consider the following system of nonlinear equations

$$F(x) = 0, \tag{18}$$

where $F : D \to R^n,\ D \subset R^n$.

In this section, for the system (18) the solution multiplicity notion is formulated and the reduction to the system with isolated solution is suggested.

Let x^* be a solution of (18), i.e. $F(x^*) = 0$. Further we assume that the vector-function $F(x)$ is sufficiently smooth in a neighborhood of the point x^* for the consequent transformations be correct.

Introduce the notation

$$A_0(x) = \partial F(x)/\partial x. \tag{19}$$

If $\det A_0(x^*) \neq 0$, then x^* is usually called an isolated solution of (18). In [15], [18] a sufficiently complete theory has been developed for the original problem. Here we consider the case

$$\det A_0(x^*) = 0. \tag{20}$$

Such solutions are usually called multiple, or sometimes singular. To find them is an essentially difficult problem (see, for example, [1], [4], [5], [13], [14]).

Numerical methods for finding the multiple solutions for (18), with information on the rank (corank) of the matrix $A_0(x^*)$ have been proposed in [4], [5]. Note that this information does not allow to answer the main question: what is the multiplicity of the solution x^* for the system (18)? For example, both the systems

$$\begin{cases} x^2 + y^2 &= 0, \\ x^2 - y^2 &= 0, \end{cases} \tag{21}$$

$$\begin{cases} x^4 + y^4 &= 0, \\ x^4 - y^4 &= 0, \end{cases} \tag{22}$$

have the solution $x^* = (x = 0,\ y = 0)$ with $\mathrm{rank} A_0(x^*) = 0$. But for the first system this solution has multiplicity 2, while for the second one the multiplicity is 4.

Moreover, even in the case when $F(x)$ is a quadratic polynomial, $\mathrm{rank} A_0(x^*)$ does not give the information about the solution multiplicity. For example, the Jacobi matrices of two similar systems

$$\left\{ \begin{array}{ll} x - y^2 & = 0, \\ \quad\; y^2 & = 0, \end{array} \right. \tag{23}$$

$$\left\{ \begin{array}{ll} x - y^2 & = 0, \\ x^2 & = 0, \end{array} \right. \tag{24}$$

have the same rank for $x = y = 0$, but the multiplicities of this solution are different. Choosing the initial approximation x^0 such that $\det A_0(x^0) \neq 0$ and applying to these systems Newton method, we obtain $x^{i+1} = 0,\ y^{i+1} = y^i/2$ for (23), and $x^{i+1} = x^i/2,\ y^{i+1} = y^i/2 + x^i/4y^i$ for (24). Consequently, Newton method for (23) converges at the rate of geometric progression, but for (24), when $x^0 = \varepsilon,\ y^0 = \varepsilon^2$, the convergence is "very slow".

Note, the definition of the multiple solution for (18) must also imply the classical definition of the multiple solution for the scalar equation, i.e. for $n = 1$ x^* is the solution of multiplicity k if $F^{(i)}(x)|_{x=x^*} = 0, i = 1, 2, ..., k-1,\ \ F^{(k)}(x)|_{x=x^*} \neq 0$.

Let us construct the matrices $A_i(x)$ according to the rule

$$A_i(x) = \sum_{j=0}^{n} \alpha_j \partial A_{i-1}(x)/\partial x_j, \tag{25}$$

where α_j are scalar values and $A_0(x)$ is defined by (19).

Introduce the matrix

$$A(\lambda) = \sum_{i=0}^{k} \lambda^{k-i} A_i(x^*), \tag{26}$$

where the matrices A_i are defined by the rule (25).

Definition 7. *The minimal possible number $k+1$, for which the $\lambda-$matrix defined by* (25), (26) *possesses the DP, is called the multiplicity of the solution x^* for the system* (18).

As an illustration consider the examples given above.

For the system (21) we have

$$A_0(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ A_1(x^*) = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix},$$

$$\lambda A_0(x^*) + A_1(x^*) = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}.$$

Here $\text{rank} A_0(x^*) = \deg\det\{\lambda A_0(x^*) + A_1(x^*)\} = \deg(-8) = 0$, and hence the multiplicity of the solution is 2 ($\alpha_1 = \alpha_2 = 1$).

For the system (22),

$$A_0(x^*) = A_1(x^*) = A_2(x^*) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_3(x^*) = \begin{pmatrix} 24 & 24 \\ 24 & -24 \end{pmatrix}.$$

The matrices $\lambda A_0(x^*) + A_1(x^*)$ and $\lambda^2 A_0(x^*) + \lambda A_1(x^*) + A_2(x^*)$ do not possess the DP (by virtue of the fact that the operation $\deg(0)$ has not been defined), and the matrix

$$\lambda^3 A_0(x^*) + \lambda A_1(x^*) + \lambda A_2(x^*) + A_3(x^*) = \begin{pmatrix} 24 & 24 \\ 24 & -24 \end{pmatrix},$$

has $\text{rank} A_0(x^*) = \deg\det\{\lambda^3 A_0(x^*) + \lambda A_1(x^*) + \lambda A_2(x^*) + A_3(x^*)\} = \deg(-1152) = 0$. Hence the multiplicity of the solution is 4. Here $\alpha_1 = \alpha_2 = 1$.

Similarly for the system (23) the multiplicity of the solution is 2.

For the system (24) one has

$$\lambda A_0(x^*) + A_1(x^*) =$$
$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Here $\text{rank} A_0(x*) = 1, \deg\det\{\lambda A_0(x^*) + A_1(x^*)\} = \deg(4) = 0$, i.e. the matrix $\lambda A_0(x^*) + A_1(x^*)$ does not possess the DP. Meanwhile, the matrix

$$\lambda^2 A_0(x^*) + \lambda A_1(x^* + A_3(x^*) =$$
$$\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

possesses the DP because of $\deg\det\{\lambda^2 A_0(x^*) + \lambda A_1(x^*)\} = \deg(4\lambda^2) = 2$, and so, the multiplicity of the solution is 3. Here $\alpha_1 = \alpha_2 = 1$.

Consider the case when the matrix pencil $\lambda A_0(x^*) + A_1(x^*)$ possesses the DP, i.e. it satisfies the "rank-degree" criterion. The result for such pencil is well-known.

Theorem 4 [6], [8]. *If the matrix pencil $\lambda A_0 + A_1$ satisfies the "rank-degree" criterion then the matrices*

$$A_0 + (E - A_0 A_0^-)A_1, A_0 + A_1(E - A_0^- A_0)$$

are nonsingular.

This theorem allows one to reduce the original system (18) to the following systems with isolated solution

$$F_1(x) = F(x) + A_0(x)(E - A_0^- A_0)\Theta, \tag{27}$$

or

$$F_2(x) = F(x) + (E - A_0 A_0^-)A_0(x)\Theta, \tag{28}$$

where $\Theta = (\alpha_1, \alpha_2, \cdots, \alpha_n)^\top$. Here the solution of (18) is the solution of (27) and (28) with $\det \partial F_1/\partial x|_{x=x^*} \neq 0$, $\det \partial F_2/\partial x|_{x=x^*} \neq 0$.

The example (24) shows that even in the case when $F(x)$ represents a polynomial whose degree is less than 3, the multiplicity of the solution can be greater than 2. In this case, the multiplicity of the solution is defined only by the structure of the matrix pencil $\lambda A_0(x^*) + A_1$ (the matrix A_1 is a constant matrix) and the multiplicity of the solution can be determined (we omit discussions) by formula (17a).

Another notion of solution multiplicity is given in [10].

6. Example of numerical computation

Take the system of differential-algebraic equations

$$Ax' + f(x) = 0, \quad t \in [0, 0.9], \quad x(0) = x_0,$$

where $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $f(x) = \begin{pmatrix} x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$ and $x_1(0) = x_2(0) = 1$.

This problem has unique solution $x_1(t) = x_2(t) = 2/(2 - t)$. Let us apply the implicit Euler scheme for numerical solving of this problem

$$A(x_{i+1} - x_i) + hf(x_{i+1}) = 0, \quad i = 1, 2, \ldots, N, \quad h = 1/N.$$

As a result, we come to the system of nonlinear equations with respect to x_{i+1}, with singular Jacobi matrix on the solution. The reduction (27) was employed for the numerical solving of this system. After this reduction, one iteration of Newton method was made, and the value on the previous step, i.e. $x_{i+1}^0 = x_i$, was taken as the initial approximation.

The results of computations are given in the following table

h	0.1	0.05	0.025	0.0125
p	0.1214	0.0544	0.0259	0.0126

Here p is the modulus of the maximum error of the components at the mesh points.

Conclusion. The connection between systems of linear differential equations and matrix polynomials is well-known. In the present paper we studied this connection for integral and integro-differential equations with singular matrix at the main parts. We suggested to use reduction of such singular equations to regular ones. Also we have investigated systems of nonlinear algebraic, or transcendent, equations. We have proposed a general notion of solution multiplicity which is compatible with the relevant notion for one equation.

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