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DIFFERENTIAL-ALGEBRAIC EQUATIONS IN THE THEORY OF INVARIANT MANIFOLDS FOR SINGULAR EQUATIONS

(submitted by A. M. Elizarov)

Abstract. Analogs of Grobman-Hartman theorem on stable and unstable manifolds solutions for differential equations in Banach spaces with degenerate Fredholm operator at the derivative are proved. Jordan chains tools and the implicit operator theorem are used. In contrast to the usual evolution equation here the central manifold appears even for the case of spectrum absence on the imaginary axis. If on the imaginary axis there is only a finite number of spectrum points, then the original nonlinear equation is reduced to two differential–algebraic systems on the center manifold.

1. Introduction

Branching theory of solutions of nonlinear equations has various applications in scientific computing [5, 7, 8]. This is one of the areas in applied mathematics which is intensively developing in last fifty years. The goals of this theory are the qualitative theory of dynamical systems [7], computation of their solutions [4] without assumptions ensuring the uniqueness of solutions. The classical Lyapounov-Schmidt method, even in the modern form [19], is often insufficient for computation of complicated dynamics, like bifurcation to invariant tori. Therefore in the last

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two decades the center manifold theory [2, 7, 10, 14, 16] and methods are developed. However, no results of this theory concerning evolution equations with degenerate operator at the derivative are known, though these equations have numerous applications in filtration theory [1], nonlinear waves theory (the Boussinesq-Love equation) [22] and motion theory of non-Newtonian fluids [15].

The present work, as an introduction to center manifold methods for evolution equations with Fredholm operator at the derivative, considers invariant manifolds technique on the base of the resolving systems theory [13] developed by authors. It has found some applications to investigation of the bifurcating solutions stability [11].

The second section of this article contains the necessary tools of generalized Jordan chains [19], the third, forth, and fifth ones; some aspects of invariant manifolds theory, and Grobman–Hartman theorem analogs for such equations. Here the nontrivial center manifold arises even in the case when the operator B has no A-spectrum $\sigma_A(B)$ on the imaginary axis.

For the computation of center manifold, in section 3 successive approximation method is suggested. It is considered also the sufficiently general case of $\sigma_A(B)$ presence on imaginary axis (section 4) that will be the subject of our future investigations. Only for representation of the nonlinear equation in the form of two equations system in the direct sum of Banach spaces complete results are obtained. Here, if the spectrum on imaginary axis $\sigma_A^0(B)$ is non-empty and it is separated on the other parts of spectrum, then the original nonlinear equation is reduced to two differential-algebraic systems on the center manifold, for solving of which the authors suppose to develop numerical methods.

2. Generalized Jordan chains and sets for Fredholm operators

Let E_1 and E_2 be Banach spaces, $A : E_1 \supset D_A \to E_2$, $B : E_1 \supset D_B \to E_2$ be densely defined closed linear Fredholm operators, where $D_B \subset D_A$ and A is subordinated to B (i.e. $\|Ax\| \le \|Bx\| + \|x\|$ on D_B), or $D_A \subset D_B$ and B is subordinated to A (i.e. $\|Bx\| \le \|Ax\| + \|x\|$ on D_A). The differential equation

$$A\frac{dx}{dt} = Bx - R(x), \quad R(0) = 0, \quad R_x(0) = 0 \tag{1}$$

with sufficiently smooth operator R is considered.

It is supposed the nontriviality of the the zero-subspaces $\mathcal{N}(A) = \mathrm{span}\{\phi_1, \ldots, \phi_m\}, \mathcal{N}(B) = \mathrm{span}\{\varphi_1, \ldots, \varphi_n\}$ with non-degeneracy condition $\mathcal{N}(A) \cap \mathcal{N}(B) = \{0\}$ and the defect-subspaces $\mathcal{N}^*(A) = \mathrm{span}\{\hat{\psi}_1, \ldots, \hat{\psi}_m\}$, $\mathcal{N}^*(B) = \mathrm{span}\{\psi_1, \ldots, \psi_n\}$. The corresponding biorthogonal systems $\{\vartheta_j\}_1^m$, $\langle \phi_i, \vartheta_j \rangle = \delta_{ij}; \{\zeta_j\}_1^m$, $\left\langle \zeta_i, \hat{\psi}_j \right\rangle = \delta_{ij}$, $\{\gamma_j\}_1^n, \langle \varphi_i, \gamma_j \rangle = \delta_{ij}; \{z_j\}_1^n$, $\langle z_i, \psi_j \rangle = \delta_{ij}$ are introduced in [19]. For the reader convenience here some auxiliary results from [11, 12, 17, 19] are given.

Definition 1. [19] *The elements* $\phi_i^{(s)}$*,* $s = 1, \ldots, q_i$*,* $\phi_i^{(1)} = \phi_i$*,* $i = 1, \ldots, m$ *(*$\varphi_i^{(s)}$ *,* $s = 1, \ldots, p_i$ *,* $\varphi_i^{(1)} = \varphi_i$ *,* $i = 1, \ldots, n$*) form the complete canonical generalized Jordan set (GJS* $\equiv$ *B-JS) relative to the operator-function* $A - \lambda B$ *(*$B - \mu A$*, respectively) if*

$$
\begin{aligned}
&A\phi_i^{(s)} = B\phi_i^{(s-1)}, \left\langle \phi_i^{(s)}, \vartheta_j \right\rangle = 0, s = 2, \ldots q_i, i, j = 1, \ldots, m;\\
&(B\varphi_i^{(s)} = A\varphi_i^{(s-1)}, \left\langle \varphi_i^{(s)}, \gamma_j \right\rangle = 0, s = 2, \ldots, p_i, i, j = 1, \ldots, n)\\
&D_q \equiv \det\left[\left\langle B\phi_i^{(q_i)}, \widehat{\psi}_j \right\rangle\right] \neq 0,\ (D_p \equiv \det\left[\left\langle A\varphi_i^{(p_i)}, \psi_j \right\rangle\right] \neq 0).
\end{aligned}
$$

This GJS is called bicanonical if the corresponding B^*-JS (A^*-JS) of the adjoint operator A^* (B^*) is also canonical.

The conditions in definition 1 determine the B-JS (A-JS) uniquely. Its elements are linearly independent and form a basis for the root-subspace $K(A; B)$ ($K(B; A)$) of the Fredholm point $\lambda = 0 \in \sigma_B(A)$ ($\mu = 0 \in \sigma_A(B)$) of the operator-function $A - \lambda B$ ($B - \mu A$), where $k_A = \dim K(A; B) = \sum\limits_{i=1}^{m} q_i$ ($k_B = \dim K(B; A) = \sum\limits_{i=1}^{n} p_i$) is called the root-number of the Fredholm point.

Elements of B and B^*-Jordan sets (A-and A^*-Jordan sets) of the operator-functions $A - \lambda B$ and $A^* - \lambda B^*$ ($B - \mu A$ and $B^* - \mu A^*$) can be chosen so that the following biorthogonality conditions hold true:

$$
\begin{gathered}
\left\langle \phi_i^{(j)}, \vartheta_k^{(l)} \right\rangle = \delta_{ik}\delta_{jl},\ \left\langle \zeta_i^{(j)}, \widehat{\psi}_k^{(l)} \right\rangle = \delta_{ik}\delta_{jl},\ j(l) = 1, \ldots, q_i(q_k),\\
\vartheta_k^{(l)} = B^*\widehat{\psi}_k^{(q_k+1-l)},\ \zeta_i^{(j)} = B\phi_i^{(q_i+1-j)},\ i, k = 1, \ldots, m
\end{gathered} \tag{2}
$$

$$
\begin{gathered}
\left\langle \varphi_i^{(j)}, \gamma_k^{(l)} \right\rangle = \delta_{ik}\delta_{jl},\ \left\langle z_i^{(j)}, \psi_k^{(l)} \right\rangle = \delta_{ik}\delta_{jl},\ j(l) = 1, \ldots, p_i(p_k),\\
\gamma_k^{(l)} = A^*\psi_k^{(p_k+1-l)},\ z_i^{(j)} = A\varphi_i^{(p_i+1-j)},\ i, k = 1, \ldots, n.
\end{gathered} \tag{3}
$$

The relations (2)–(3) allow to introduce the projectors [19]

$$\begin{aligned}
\mathbf{p} &= \sum_{i=1}^{m}\sum_{j=1}^{q_i}\left\langle \cdot, \vartheta_i^{(j)}\right\rangle \phi_i^{(j)} = \langle \cdot, \vartheta\rangle\,\phi : E_1 \to E_1^{k_A} = K(A,B),\\
\mathbf{q} &= \sum_{i=1}^{m}\sum_{j=1}^{q_i}\left\langle \cdot, \widehat{\psi}_i^{(j)}\right\rangle \zeta_i^{(j)} = \left\langle \cdot, \widehat{\psi}\right\rangle \zeta : E_2 \to E_{2,k_A} = \mathrm{span}\{\zeta_i^{(j)}\},\\
\mathbf{P} &= \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, \gamma_i^{(j)}\right\rangle \varphi_i^{(j)} = \langle \cdot, \gamma\rangle\,\varphi : E_1 \to E_1^{k_B} = K(B;A),\\
\mathbf{Q} &= \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, \psi_i^{(j)}\right\rangle z_i^{(j)} = \langle \cdot, \psi\rangle\, z : E_2 \to E_{2,k_B} = \mathrm{span}\{z_i^{(j)}\}
\end{aligned} \tag{4}$$

(where $\phi = (\phi_1^{(1)}, \cdots, \phi_1^{(q_1)}, \cdots, \phi_m^{(1)}, \cdots, \phi_m^{(q_m)})$, and the vectors ϑ, $\widehat{\psi}$, ζ, φ, γ, ψ, z are defined in the same way) generating the following direct sums expansions

$$\begin{aligned}
E_1 = E_1^{k_A}\dot{+}E_1^{\infty-k_A},\ E_2 = E_{2,k_A}\dot{+}E_{\infty-k_A},\\
E_1 = E_1^{k_B}\dot{+}E_1^{\infty-k_B},\ E_2 = E_{2,k_B}\dot{+}E_{2,\infty-k_B}.
\end{aligned} \tag{5}$$

The intertwining relations are realized

$$\begin{aligned}
&A\mathbf{p} = \mathbf{q}A \ \text{ on } \ D_A, \quad B\mathbf{p} = \mathbf{q}B \ \text{ on } \ D_B,\\
&(B\mathbf{P} = \mathbf{Q}B \ \text{ on } \ D_B, \quad A\mathbf{P} = \mathbf{Q}A \ \text{ on } \ D_A),\\
&A\phi = \mathfrak{A}_A\zeta, \ \ B\phi = \mathfrak{A}_B\zeta, \ \ B^*\widehat{\psi} = \mathfrak{A}_B\vartheta,\\
&(B\varphi = \mathcal{A}_B z, \ \ A\varphi = \mathcal{A}_A z, \ \ A^*\psi = \mathcal{A}_A\gamma),
\end{aligned} \tag{6}$$

with cell-diagonal matrices $\mathfrak{A}_A = (A_1, \dots, A_m)$, $\mathfrak{A}_B = (B_1, \dots, B_m)$ ($\mathcal{A}_B = (B^1, \dots, B^n)$, $\mathcal{A}_A = (A^1, \dots, A^n)$), where the $q_i \times q_i$-cells ($p_i \times p_i$-cells) have the forms

$$A_i = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 1 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1\\ 0 & 0 & 0 & \dots & 1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 1 & 0 & \dots & 0 & 0\\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

(B^i's have the same form as the A_i's, correspondingly A^i's have also the same form as the B_i's). The following relations for the operators A and B hold:

$$\begin{aligned}
\mathcal{N}(A) \subset E_1^{k_A},\ AE_1^{k_A} \subset E_{2,k_A},\ A(E_1^{\infty-k_A} \cap D_A) \subset E_{2,\infty-k_A},\\
\mathcal{N}(B) \subset E_1^{\infty-k_A},\ BE_1^{k_A} \subset E_{2,k_A},\ B(E_1^{\infty-k_A} \cap D_B) \subset E_{2,\infty-k_A}.
\end{aligned} \tag{7}$$

$\overset{\sqcap}{A} = A|_{E_1^{\infty-k_A}\cap D_A}$, $\overset{\sqcap}{B} = B|_{E_1^{\infty-k_A}\cap D_B}$, and the mappings $B : E_1^{k_A} \to E_{2,k_A}$, $\overset{\sqcap}{A}$: $E_1^{\infty-k_A} \cap D_A \to E_{2,\infty-k_A}$ are one-to-one. In the same way, the operators B and A act in invariant pairs of the subspaces $E_1^{k_B}$, E_{2,k_B} and

$E_1^{\infty-k_B}$, $E_{2,\infty-k_B}$ and also $\overset{\sqcup}{B}= B|_{E_1^{\infty-k_B}\cap D_B} : E_1^{\infty-k_B}\cap D_B \to E_{2,\infty-k_B}$, $A : E_1^{k_B} \to E_{2,k_B}$ are isomorphisms.

3. Grobman–Hartman theorem analogs when $\sigma_A^0(B) = \emptyset$

We suppose that, for the A-spectrum $\sigma_A(B)$ of the operator B, Re $\sigma_A(B) \neq 0$ and the spectral sets $\sigma_A^-(B) = \{\mu \in \sigma_A(B)|\ \mathrm{Re}\ \mu < 0\}$ and $\sigma_A^+(B) = \{\mu \in \sigma_A(B)|\ \mathrm{Re}\ \mu > 0\}$ are distant from the imaginary axis on some distance $0 < d < \infty$.

All solutions of the corresponding to (1) linear Cauchy problem

$$A\frac{dx}{dt} = Bx, \qquad x(0) = x_0 \tag{8}$$

belong to $E_1^{\infty-k_A}$ and (8) is solvable if and only if $x_0 \in E_1^{\infty-k_A}$. In fact, one sets $x(t) = x_0 + v(t) + w(t)$, $v(t) = \sum\limits_{i=1}^{m}\sum\limits_{s=1}^{q_i} \xi_{is}(t)\phi_i^{(s)} \in E_1^{k_A}$, $w(t) \in E_1^{\infty-k_A}$. Then (8) splits into the system

$$\begin{array}{c} \overset{\sqcap}{A}\,\frac{dw}{dt} = \overset{\sqcap}{B}\, w + (I-\mathbf{q})Bx_0, \quad \frac{d\xi_{is}(t)}{dt} = \xi_{i,s-1}, \\ s = 2,\dots,q_i, \quad i = 1,\dots,m, \quad \xi_{iq_i} = 0. \end{array} \tag{9}$$

Consequently $\xi_{is}(t) \equiv 0$, $x_0 \in E_1^{\infty-k_A}$, $Bx_0 = \overset{\sqcap}{B}\, x_0 \in E_{2,\infty-k_A}$ and the solution of (8) takes the form

$$x(t) = x_0 + \int_0^t [\exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\,(t-s)]\,\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\,x_0 ds = \exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\,t)x_0, \tag{10}$$

Thus one has $\sigma_A(B) = \sigma(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B})$. Here the function $\exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\,t)$ has the form of the contour integral $\dfrac{1}{2\pi i}\displaystyle\int_\gamma (\mu I - \overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B})^{-1}e^{\mu t}\,dt$ at the assumption about sectorial property [7] of the operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ (or, that is the same, about A-sectorial property of the operator B [18]) with some special contour γ belonging to sector $S_{\alpha,\theta}(B)$ in A-resolvent set of the operator B [18]. Moreover, this is true when the operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ is bounded.

For the generalization of the Grobman-Hartman theorem we will follow the work [6]. Let us define the spaces D_k, $k = 1, 2$ with graphs norms:

(1) $D_1 = D_B \subset D_A$ with the norm $\|x\|_1 = \|x\|_{E_1} + \|Bx\|_{E_2}$, $x \in D_1$, if A is subordinated to B,
(2) $D_2 = D_A \subset D_B$ with the norm $\|x\|_2 = \|x\|_{E_1} + \|Ax\|_{E_2}$, $x \in D_2$, if B is subordinated to A,

and introduce the spaces $X_{k0}, X_{k1}, X_{k2}, Y_{k0}, Y_{k1}, Y_{k2}$ consisting of the bounded uniformly continuous functions $f(t)$ on $[0,\infty)$ with their values correspondingly in D_k, $D_k \cap E_1^{\infty-k_A}$, $E_1^{k_A}, E_2, E_{2,\infty-k_A}$, E_{2,k_A} with supremum norms on the relevant spaces, and the spaces

$$X^1_{ks} = \{f(t) \in X_{ks} | \dot f(t) \in X_{ks}\}, \|f(t)\|_{X^1_{ks}} = \max\{\|f(t)\|_{X_{ks}}, \|\dot f(t)\|_{X_{ks}}\}.$$

Everywhere below the operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ is supposed to be bounded in X_{k1} (for the case k=1 it is evident) and the operator R be sufficiently smooth in a small neighborhood of zero in D_k.

Then the operator

$$\mathbf{A}x = A\dot x - Bx \tag{11}$$

acting from X^1_{k0} to Y_{k0} is linear, continuous and vanishes on some set $\widetilde X_{k2} \subset \mathcal{N}(\mathbf{A})$ dense in X_{k2}.

Let be $D_k \supset S_k = \{$initial values of solutions of the equation (8), which are defined and remain in a small neighborhood of zero in D_k for $t \in [0,+\infty)\}$ and $U_k = \{$initial values of solutions of (8), which are defined and remain in a small neighborhood of zero in D_k for $t \in (-\infty,0]\}$. From (11) it follows that $S_k \dot{+} U_k = E_1^{\infty-k_A} \cap D_k$. Then the equality $\sigma_A(B) = \sigma(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B})$ allows to define the projectors $P^- u = \frac{1}{2\pi i}\displaystyle\int_{\gamma_-} (\mu I_{E_1^{\infty-k_A}} - \overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B})^{-1} u\, d\mu$ (γ_- is the contour in $\rho_A(B)$ surrounding the points $\mu \in \sigma_A(B)$ with Re $\mu < 0$), and $P^+ = I_{E_1^{\infty-k_A}} - P^-$. Whence $D_k = D_k^- \dot{+} D_k^0 \dot{+} D_k^+$, $D^0_k = E_1^{k_A}$, $D_k^\pm = P^\pm D_k$. Operator $\mathbf{A}$ is Noetherian [19] with $R(\mathbf{A}) = Y_{k1}$ and

$$\begin{aligned}
\mathcal{N}(\mathbf{A}) &= \{f(t) \in X^1_{k0} | f(t) = \exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\, t)P^- f(0) \in D_k^-\} \dot{+} \{f(t) \in D^0_k\} \\
&= \mathcal{N}_1(\mathbf{A}) \dot{+} \mathcal{N}_2(\mathbf{A}) \quad \text{for} \quad t \geq 0 \\
(\mathcal{N}(\mathbf{A}) &= \{f(t) \in X^1_{k0} | f(t) = \exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\, t)P^+ f(0) \in D_k^+\} \dot{+} \{f(t) \in D^0_k\} \\
&\quad\ \text{for} \ \ t \leq 0).
\end{aligned}$$

Now setting $x = y + z + v$, $z \in D_k^+$, $v \in D^0_k = E_1^{k_A}, y \in D_k^-$, one can write (1) in the form ($w = y + z$ in (9))

$$\mathbf{A}z = R(z + y + v) \quad (\mathbf{A}y = R(y + z + v)) \tag{12}$$

and apply the implicit operator theorem to (12) regarding y, v (z, v) as functional parameters (see the relevant theorems 22.1 and 22.2 in [19] for continuous and analytic operator R, respectively). It follows that (12) has a sufficiently smooth or analytic (according to the properties of the operator R) solution in some neighborhoods of zero values of parameters y, v (z, v):

$$z = z(y + v),\ z(0) = 0 = Dz(0)\ (y = y(z + v),\ y(0) = 0 = Dy(0)) \quad (13)$$

Thus we get the following Grobman–Hartman theorem [6] analog asserting that the local solutions behavior for nonlinear equation in hyperbolic equilibrium neighborhood is the same that for its linearization.

Theorem 1. *There exist a neighborhood* $\omega^-(\omega^+)$ *of zero in* $D_k^0 \dot{+} D_k^-$ *(in* $D_k^0 \dot{+} D_k^+$*) and a sufficiently smooth mapping* $z_R = z_R(\xi, \eta) = z_R(\xi \cdot \phi + \eta) :$ $\omega^- \to D_k^+$*,* $\eta \in D_k^-$ $(y_R = y_R(\xi, \zeta) = y_R(\xi \cdot \phi + \zeta) : \omega^+ \to D_k^-,\ \zeta \in D_k^+)$*, such that* a) $z_R(0,0) = 0$*,* $D_\xi z_R(0,0) = 0$*,* $D_\eta z_R(0,0) = 0$ $(y_R(0,0) = 0$*,* $D_\xi y_R(0,0) = 0$*,* $D_\zeta y_R(0,0) = 0)$*,* b) *for any solution* $x(t)$ *of* (1) *with initial data* $x(0) = \xi\cdot\phi + \eta + z_R(\xi\cdot\phi+\eta)$ $(x(0) = \xi\cdot\phi + y_R(\xi\cdot\phi+\zeta)+\zeta)$ *one has* $z(t) = z_R(\xi(t)\cdot\phi + y(t)) \in D_k^+$ *for* $t \geq 0$ $(y(t) = y_R(\xi(t)\cdot\phi + z(t)) \in D_k^-$ *for* $t \leq 0)$*,* c) *any solution* $x(t)$ *of* (1) *with initial data from* b) *takes the form* $x(t) = \xi(t)\cdot\phi + y(t) + z_R(\xi(t)\cdot\phi + y(t))$ $(x(t) = \xi(t)\cdot\phi + y_R(\xi(t)\cdot\phi + z(t)) + z(t))$ *and tends to zero when* $t \to +\infty (t \to -\infty)$*, and belongs, consequently, to local stable manifold* $S_k(R)$ *(local unstable manifold* $U_k(R)$*).*

Proof. We give here the proof for the function z_R and the local stable manifold $S_k(R)$, the proof of the second part is analogous. Define the projector $\tilde{P}^-$of X_{k1}^1 onto $\mathcal{N}_1(\mathbf{A})$ by the equality $(\tilde{P}^- f)(t) = \exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\ t)P^- f(0)$, $t \geq 0$. If one sets $x(t) = v(t) + y(t) + z(t)$, $v(t) = \mathbf{p}x(t)$, $v(0) = \xi\cdot\phi = \sum_{i=1}^{m}\sum_{s=1}^{q_i} \xi_{is}\cdot\phi_i^{(s)}, y(t) = \tilde{P}^- x(t) = \exp(\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}\ t)\eta$, $\eta = y(0)$, $z(t) = (I_{X_{k1}^1} - \tilde{P}^-)\,x(t)$, then the Lyapounov–Schmidt method (theorem 27.1 [19] for Noetherian operators with d-characteristic $(n, 0)$ and the indicated above theorems (22.1, 22.2 [19]) implies that there is a unique solution of (12) $z = z_R(\xi(t)\cdot\phi + y(t)) \in X_{k1}^1$ such that $x(0) = \xi\cdot\phi + \eta + z_R(\xi\cdot\phi+\eta)$, i. e. the unique solution of (1) $x(t) = v(t) + y(t) + z_R(\xi(t)\cdot\phi + y(t))$, $v(t) = \xi(t)\cdot\phi$, in a sufficiently small semi-neighborhood of $t = 0$, where the function $z_R(\xi, \eta) = z_R(\xi\cdot\phi + \eta)$ is sufficiently smooth by ξ, η, and $z_R(0,0) = 0$, $D_\xi z_R(0,0) = 0$, $D_\eta z_R(0,0) = 0$.

Writing the equation (1) in $\mathbf{p}$, $\mathbf{q}$-projections one can get the system for the determination of $\xi_{is}(t)$ (so-named the resolving system (RS) for the equation (1) [11–13]). Here $x(t) = \xi(t)\cdot\phi + w(t)$, where $w(t) = y(t) + z_R(\xi(t)\cdot\phi + y(t))$ for $t \geq 0$ and $w(t) = y_R(\xi(t)\cdot\phi + z(t)) + z(t)$ for

$t \leq 0$

$$\overset{\sqcap}{A}\frac{dw}{dt} = \overset{\sqcap}{B} w - (I_{D_k} - \mathbf{q})R(\xi \cdot \phi + w) \tag{14}$$

$$\begin{aligned}
0 &= \xi_{iq_i}(t) - \left\langle R(\xi(t)\cdot\phi + w), \hat{\psi}_i^{(1)} \right\rangle, \\
\dot{\xi}_{iq_i}(t) &= \xi_{i,q_i-1}(t) - \left\langle R(\xi(t)\cdot\phi + w), \hat{\psi}_i^{(2)} \right\rangle, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dot{\xi}_{i2}(t) &= \xi_{i1}(t) - \left\langle R(\xi(t)\cdot\phi + w), \hat{\psi}_i^{(q_i)} \right\rangle, \\
\xi_{is}(0) &= \xi_{is}, \quad s = 1,\dots,q_i, \quad i = 1,\dots,m.
\end{aligned} \tag{15}$$

Consequently, the manifold $S_k(R) = \{$ initial values of solutions of the equation (1), which are defined and remain in a small neighborhood of $0 \in D_k$ for $t \in [0,+\infty)\}$ (the manifold $U_k(R) = \{$ initial values of solutions (1), which are defined and remain in a small neighborhood of $0 \in D_k$ for $t \in (-\infty, 0]\}$) has the local presentation $x(0) = \xi \cdot \phi + \eta + z_R(\xi \cdot \phi + \eta)$ $(x(0) = \xi \cdot \phi + y_R(\xi \cdot \phi + \zeta) + \zeta)$, where $\eta \in D_k^-$ $(\zeta \in D_k^+)$ and ξ are small. □

Remark 1. *The invariant manifold* $\mathfrak{M}$ *determined by the function* $\xi\cdot\phi + \eta + z_R(\xi\cdot\phi+\eta)$ *for* $t \geq 0$ $(\xi\cdot\phi + y_R(\xi\cdot\phi+\zeta)+\zeta$ *for* $t \leq 0)$ *can be regarded as the center manifold* $(\xi \cdot \phi \in D_k^0)$, *that is nontrivial for the equation* (1) *even if* $\{\mu \in \sigma_A(B) | Re\, \mu = 0\} = \emptyset$. *Here* $\{\xi \cdot \phi\}$ *can be called the linear center manifold tangent to* $\mathfrak{M}$. *One can say that* $\mathfrak{M}$ *has an hyperbolic structure. Thus the RS* (15) *represents the differential-algebraic system on* $\mathfrak{M}$. *Of course, if the operator* A *is invertible,* $\mathfrak{M}$ *and the system* (15) *are absent, i.e. in the Grobman–Hartman theorem* $z_R = z_R(\eta)$ [6].

Theorem 2. *Let the operators* A, B *and* R *in* (1) *be intertwined by the group* G *representations* L_g *(acting in* E_1*) and* K_g *(acting in* E_2*) and the condition I (direct supplements* $E_1^{\infty-m}$ *to* $\mathcal{N}(A)$ *and* $E_1^{\infty-n}$ *to* $\mathcal{N}(B)$ *are invariant relative to* L_g*) holds true. Then the center manifold* $\mathfrak{M}$ *is invariant relative to the operators* L_g.

Proof. According to [13], projectors $\mathbf{p}, \mathbf{P}(\mathbf{q}, \mathbf{Q})$ commute with the operators $L_g(K_g)$ and invariant pairs of subspaces reduce the representations $L_g(K_g)$. □

In the article [11] it is proved that the stability (instability) of the trivial solution (even for non-autonomous) equation (1) at sufficiently general conditions is determined by the RS (15) with corollaries for the investigation of the stability (instability) of bifurcating solutions.

In applications, of interest is the case when $\sigma_A^+(B) = \emptyset$. Then $D_k = D_k^- \dot{+} D_k^0$, $x(t) = \xi(t) \cdot \phi + y(t)$ and the center manifold has the form $\xi(t) \cdot \phi + y(\xi(t) \cdot \phi)$. Here the equation (14) gives

$$\begin{gathered} \overset{\sqcap}{A}\, y'(\xi(t)\cdot\phi)(\tfrac{d\xi}{dt}\cdot\phi) = \overset{\sqcap}{B}\, y(\xi(t)\cdot\phi) + \\ (I-\mathbf{q})R(\xi(t)\cdot\phi + y(\xi(t)\cdot\phi)), \\ y(0)=0, \quad y'(0)=0 \end{gathered} \tag{16}$$

Combined with (15) this gives a possibility for the determination of center manifold $w(\xi(t) \cdot \phi) = \xi(t) \cdot \phi + y(\xi(t) \cdot \phi)$ by successive approximations in conditions of sufficiently smooth operator $y(\xi \cdot \phi)$. However on this way essential difficulties arise which are connected with the fact that the system (15) is differential-algebraic, i.e. the differential equations for the functions $\xi_{i1}(t)$, $i = 1, \ldots, m$, are absent. One can find $y(\xi \cdot \phi)$ iteratively at the differentiation of the first equations (15).

Remark 2. *Theorem* 1 *and all corollaries remain true for the parameter depending equation*

$$A\frac{dx}{dt} = Bx - R(x,\lambda), \quad R(0,\lambda) \equiv 0, \quad R_x(0,0) = 0, \tag{17}$$

($\lambda \in \Lambda$, Λ *is some Banach space) in a small neighborhood of* $\lambda = 0$, *when, as above, Re* $\sigma_A(B) \neq 0$, *i.e.* $\lambda = 0$ *is not a bifurcation point. However all functions* w, z_R *and* y_R *will depend on small parameter* λ.

4. The case of $\sigma_A^0(B) \neq \emptyset$

Here we consider the sufficiently general case when $\sigma_A^0(B)$ consists of a finite number of eigenvalues with finite multiplicities, but $\sigma_A^h(B) = \sigma_A^-(B) \cup \sigma_A^+(B)$ is separated from the imaginary axis by the lines Re $\mu = \pm d$, $0 < d < \infty$. As above, the main assumption consists of the operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ which is not bounded on X_{k1}. Then $\sigma_A(B) = \sigma_{\overset{\sqcap}{A}}(\overset{\sqcap}{B})$ and the Banach space $\mathcal{D}_k = D_k \cap E_1^{\infty - k_A}$ can be decomposed into the direct sum $\mathcal{D}_k = \mathcal{D}_k^0 \dot{+} \mathcal{D}_k^h$, $\mathcal{D}_k^h = \mathcal{D}_k^+ \dot{+} \mathcal{D}_k^-$. Now, to equation (14) one can apply the theorem on center manifold [7, 20] in order to prove the following statement:

Theorem 3. *Let the root number* k_A *be finite, the operator* R *l-time be differentiable, and the conditions of Section* 4 *hold true. Then, in a sufficiently small neighborhood* Ω *in* $\mathcal{D}_k$ *there exists the mapping* $\chi \in C^l(\mathcal{D}_k^0, \mathcal{D}_k^h)$ *such that* $\chi(0) = 0$, $D\chi(0) = 0$ *and the graph of* χ *is a manifold* $\mathcal{M}_c$ *having the following properties:*

(1) $\mathcal{M}_c$ *is locally invariant under the flow generated by the equation* (14) *in* $\mathcal{D}_k$,
(2) *if* $\sigma_A^+(B) = \emptyset$ $(\sigma_A^-(B) = \emptyset)$, *then* $\mathcal{M}_c$ *is locally exponentially attracting as* $t \to +\infty$ $(t \to -\infty)$.

Remark 3. *In applications the case* $\sigma_A^+(B) = \emptyset$ *is interesting. Then theorem* 3 *reduces the equation* (1) *to two differential-algebraic systems* (*resolving systems* [13]), *one of which is* (15) *on the manifold* $\mathfrak{M}$ *and the second one represents the system on the center manifold* $\mathcal{M}_c$.

Under assumptions of theorem 2 these differential-algebraic resolving systems inherit the group symmetry of the original equation (1). This follows from [13] according to projectors $\mathbf{p}, \mathbf{P}$ $(\mathbf{q}, \mathbf{Q})$-commutativity with the representation operators L_g (K_g) and their reducibility by invariant pairs of subspaces. The investigation of connections between $\mathcal{M}_c$ and $\mathfrak{M}$ and corresponding resolving systems is the subject of our future work.

These questions become clear for the corresponding to $\sigma_A^0(B) \neq \emptyset$ simple case [9] when $\sigma_A^+(B) = \emptyset$, but $\sigma_A^0(B) = \{\mu \in \sigma_A(B) | Re\ \mu = 0\} \neq \emptyset$ contains some finite number $2n = 2n_1 + \cdots + 2n_\ell$ A-eigenvalues $\pm i\alpha_s$ of multiplicities n_s, $s = 1, \ldots, \ell$, $\alpha_s = \kappa_s \alpha$, $\alpha \neq 0$ with coprime $\kappa_s > 0$ or (and) zero-eigenvalue. Without loss of generality we can suppose that the equation (1) is written in the form of the system

$$\begin{aligned} A_1 \dot{x} &= B_1 x - f(x,y) \\ A_2 \dot{y} &= B_2 y - R(x,y), \end{aligned} \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \ B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \tag{18}$$

where the linear operators $A_1, B_1 : E_1^{k_{B_1}} \to E_{2,k_{B_1}}$ $(k_{B_1} = 2n_1 p_1 + \cdots + 2n_\ell p_\ell$, p_s are A_1-Jordan chains lengths for $\pm i\alpha_s, s = 1, \ldots, \ell)$ act in the invariant pair of finite dimensional subspaces $E_1^{k_{B_1}}, E_{2,k_{B_1}}$ and A_2, B_2 act in the invariant pair of subspaces $E_1^{\infty - k_{B_1}}$, $E_{2,\infty - k_{B_1}}$. Thus, $\sigma_{A_1}(B_1) = \sigma_A^0(B)$ and $\sigma_{A_2}^0(B_2) = \emptyset$. Here f and R are C^2-functions vanishing together with their first derivatives at the origin.

In the simplest case the main assumption is

$$\mathcal{N}(A_1) = \{0\}, \quad \mathcal{N}(A_2) = \text{span}\,\{\phi_{(2)1}, \ldots, \phi_{(2)m_2}\} \tag{19}$$

Then, under conditions of section 3, there exists the function $y_R(\xi_2(t) \cdot \phi_{(2)}, x)$ vanishing together with its first derivatives at the origin, such that the second equation (18) reduces to the system

$$\overset{\sqcap}{A}_2 \frac{dy_R}{dt} = \overset{\sqcap}{B}_2\, y_R - (I - \mathbf{q}_{(2)}) R(x, \xi_2(t) \cdot \phi_{(2)} + y_R(\xi_2(t) \cdot \phi_{(2)}, x)) \tag{20}$$

$(\mathbf{q}_{(2)} = \sum\limits_{i=1}^{m_2}\sum\limits_{j=1}^{q_{2,i}} < \cdot, \hat{\psi}^{(j)}_{(2),i} > \zeta^{(j)}_{(2)} : E_{2,\infty-k_{B_1}} \to \operatorname{span}\{\zeta^{(j)}_{(2)i}\}$, $\overset{\sqcap}{A}_2, \overset{\sqcap}{B}_2$ act in invariant pair of subspaces $E_1^{\infty-k_{B_1}-k_{A_2}}$, $E_{2,\infty-k_{B_1}-k_{A_2}})$

$$\begin{gathered} 0 = \xi_{2iq_{2,i}}(t) - \left\langle R(x, \xi_2(t)\cdot\phi_{(2)} + y_R(\xi_2(t)\cdot\phi_{(2)}, x)), \hat{\psi}^{(1)}_{(2),i}\right\rangle, \\ \dot{\xi}_{2iq_{2,i}}(t) = \xi_{2i,q_{2,i}-1}(t) - \left\langle R(x, \xi_2(t)\cdot\phi_{(2)} + y_R(\xi_2(t)\cdot\phi_{(2)}, x)), \hat{\psi}^{(2)}_{(2),i}\right\rangle \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\xi}_{2i2}(t) = \xi_{2i1}(t) - \left\langle R(x, \xi_2(t)\cdot\phi_{(2)} + y_R(\xi_2(t)\cdot\phi_{(2)}, x)), \hat{\psi}^{(q_{2,i})}_{(2),i}\right\rangle, \\ \xi_{2i\sigma}(0) = \xi_{2i\sigma},\ \sigma = 1, \ldots, q_{2,i},\ \ i = 1, \ldots, m_2. \end{gathered} \tag{21}$$

If the system (18) is equipped with initial values $x(0)$, $y(0)$, then they must satisfy the equality

$$y(0) = \xi_2\cdot\phi_{(2)} + y_R(\xi_2\cdot\phi_{(2)}, x(0)). \tag{22}$$

Now one has to solve the problem

$$A_1\dot{x} = B_1x - f(x, \xi_2(t)\cdot\phi_{(2)} + y_R(\xi_2(t)\cdot\phi_{(2)}, x)) \tag{23}$$

at the initial data $x(0)$ satisfying (22).

Thus one has two systems (21) and (23) on the center manifold $y = y_R(\xi_2(t)\cdot\phi_{(2)}, x)$, where the differential–algebraic system (23) possesses the properties indicated in theorem 3.

5. Grobman–Hartman theorem analog for maps

According to section 3 the equation (14) can be written in the form

$$\frac{dw}{dt} = \overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}w - \overset{\sqcap}{A}{}^{-1}(I_{D_k} - \mathbf{q})R(\xi\cdot\phi + w) \tag{24}$$

in the space X^1_{k1}. Then the assumption about the boundedness of the operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ in X_{k1} allows to prove Grobman–Hartman theorem analog for maps [21]. In fact, then for small ξ there exists the resolving operator $U_\xi(t,\cdot) : X_{k1} \to X^1_{k1}, w_0 \mapsto w(t)$ for the problem (24) with the initial value $w(0) = w_0$ (at $\xi = 0, U_0(t)$ is linear). Thus the following assertion is true:

Theorem 4. *For small ξ at $\sigma^0_A(B) = \emptyset$ and operator $\overset{\sqcap}{A}{}^{-1}\overset{\sqcap}{B}$ boundedness assumption there exits the resolving operator $U_\xi(t, w_0)$ and a homeomorphism $\Phi_\xi : X^1_{k1} \to X^1_{k1}, ||\xi|| \ll 1$, such that for $t \in R$ and $w_0 \in X_{k1}$ the following relation*

$$U_0(t)\Phi_\xi(w_0) = \Phi_\xi(U_\xi(t, w_0)) = \Phi_\xi(w(t)) \tag{25}$$

is true, where the function $w(t)$ and the initial values w_0, ξ_0 satisfy the initial value problem for differential–algebraic system (15)*.*

Remark 4. *The case of $\sigma_A^0(B)$ presence on imaginary axis remains unstudied. See on this connection the work* [21]*.*

Conclusion and future work. The results of this article remain true for the more general operators subordinateness (A is subordinate to B if on D_B $\|Ax\| \leq \|Bx\| + \alpha\|x\|$, $\alpha \geq 0$).

At the usage of the work [3] one can extend our results on partial differential equations in Banach spaces with degenerate operator at the highest differential expression.

The obtained results can serve only as the first step in the center manifold theory and its methods for computation of bifurcation solution asymptotics and their stability investigation. Future work here is the development of qualitative and numerical methods for the investigation of these differential–algebraic systems.

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