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Zhi-Gang Wang

CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH RESPECT TO K-SYMMETRIC POINTS

(submitted by F. G. Avkhadiev)

Abstract. In the present paper, the author introduce two new subclasses $\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ of close-to-convex functions and $\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$ of quasi-convex functions with respect to k-symmetric points. The sufficient conditions and integral representations for functions belonging to these classes are provided, the inclusion relationships and convolution conditions for these classes are also provided.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in C : |z| < 1\}$. Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\mathcal{U}$.

We denote by $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{Q}C$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex, close-to-convex and

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quasi-convex in $\mathcal{U}$. Thus, by definition, we have (see, for details, [4, 8]; see also [11, 12])

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \ (z \in \mathcal{U}) \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \ \ (z \in \mathcal{U}) \right\},$$

$$\mathcal{C} = \left\{ f : f \in \mathcal{A}, \ \exists g \in \mathcal{S}^*, \ \text{such that } \Re\left\{\frac{zf'(z)}{g(z)}\right\} > 0 \ \ (z \in \mathcal{U}) \right\},$$

and

$$\mathcal{Q}C = \left\{ f : f \in \mathcal{A}, \ \exists g \in \mathcal{K}, \ \text{such that } \Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0 \ \ (z \in \mathcal{U}) \right\}.$$

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [1]).

Sakaguchi [9] once introduced a class $\mathcal{S}_s^*$ of functions starlike with respect to symmetric points, it consists of functions $f(z) \in \mathcal{S}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \ \ (z \in \mathcal{U}).$$

Following him, many authors discussed this class and its subclasses. And let $\mathcal{C}_s^*$ denote the class of functions in $\mathcal{S}$ convex with respect to symmetric points, which satisfy the following inequality

$$\Re\left\{\frac{(zf'(z))'}{f'(z) + f'(-z)}\right\} > 0 \ \ (z \in \mathcal{U}).$$

Let $\mathcal{S}_s^{(k)}(\alpha)$ denote the class of functions in $\mathcal{S}$ satisfying the following inequality

$$\Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > \alpha \ \ (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1, \ k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^{\nu} z) \ \ (\varepsilon^k = 1; \ z \in \mathcal{U}). \tag{1.2}$$

And let $\mathcal{C}_s^{(k)}(\alpha)$ denote the class of functions in $\mathcal{S}$ satisfying the following inequality

$$\Re\left\{\frac{(zf'(z))'}{f_k'(z)}\right\} > \alpha \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1,\ k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2). The class $\mathcal{S}_s^{(k)}(\alpha)$ of functions starlike with respect to k-symmetric points of order α was studied by Sokół [5, 6] and Stankiewicz [7].

Motivated by $\mathcal{S}_s^{(k)}(\alpha)$ and $\mathcal{C}_s^{(k)}(\alpha)$, we introduce and investigate the following two more generalized subclasses $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ and $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$ of $\mathcal{S}$ with respect to k-symmetric points, and obtain some interesting results.

Definition 1. Let $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ denote the class of functions in $\mathcal{S}$ satisfying the following inequality

$$\left|\frac{\frac{zf'(z)}{f_k(z)} - 1}{\beta\frac{zf'(z)}{f_k(z)} + (1-\gamma)}\right| < 1 - \alpha \quad (z \in \mathcal{U}), \tag{1.3}$$

where $0 \leq \alpha < 1,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma < 1,\ k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2).

If $\beta = 0$ and $\gamma = 0$, then the class $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ reduces to the class $\mathcal{S}_s^{(k)}(\alpha)$ [5, 6, 7]. If $\alpha = 0,\ \beta = 0,\ \gamma = 0$ and $k = 2$, then the class $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ reduces to the class $\mathcal{S}_s^*$ [9].

Definition 2. Let $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$ denote the class of functions in $\mathcal{S}$ satisfying the following inequality

$$\left|\frac{\frac{(zf'(z))'}{f_k'(z)} - 1}{\beta\frac{(zf'(z))'}{f_k'(z)} + (1-\gamma)}\right| < 1 - \alpha \quad (z \in \mathcal{U}),$$

where $0 \leq \alpha < 1,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma < 1,\ k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.2).

If $\beta = 0$ and $\gamma = 0$, then the class $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$ reduces to the class $\mathcal{C}_s^{(k)}(\alpha)$. If $\alpha = 0,\ \beta = 0,\ \gamma = 0$ and $k = 2$, then the class $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$ reduces to the class $\mathcal{C}_s^*$.

In our investigation of the classes $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ and $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$, we shall also make use of the following definition and lemmas.

Definition 3 (*Hadamard Product or Convolution*)**.** Given two functions $f,\ g \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Lemma 1 [2]. *Let $H(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ be analytic in $\mathcal{U}$, $0 \leq \alpha < 1,\ 0 \leq \beta \leq 1$ and $0 \leq \gamma < 1$, then the condition*

$$\left| \frac{H(z) - 1}{\beta H(z) + (1-\gamma)} \right| < 1 - \alpha \quad (z \in \mathcal{U})$$

is equivalent to

$$H(z) \prec \frac{1 + (1-\alpha)(1-\gamma)z}{1 - (1-\alpha)\beta z} \quad (z \in \mathcal{U}).$$

Lemma 2. *Let $f(z) \in \mathcal{C}^{(k)}(0,0,0)$, then we have $f_k(z) \in \mathcal{S}^* \subset \mathcal{S}$.*

This lemma is a special case of Theorem 2 in [3].

Lemma 3. *Let $f(z) \in \mathcal{QC}^{(k)}(0,0,0)$, then we have $f_k(z) \in \mathcal{K} \subset \mathcal{S}^*$.*

This lemma is a special case of Theorem 2 in [13].

Lemma 4 [10]. *Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, then we have*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

In the present paper, we shall provide the sufficient conditions and integral representations for functions belonging to the classes $\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{QC}^{(k)}(\alpha,\beta,\gamma)$, we shall also provide the inclusion relationships and convolution conditions for these classes.

2. Inclusion Relationships

First we give some inclusion relationships for the classes $\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{QC}^{(k)}(\alpha,\beta,\gamma)$.

Theorem 1. *Let $0 \leq \alpha < 1,\ 0 \leq \beta \leq 1$ and $0 \leq \gamma < 1$, then we have*

$$\mathcal{C}^{(k)}(\alpha,\beta,\gamma) \subset \mathcal{C}^{(k)}(0,0,0) \subset \mathcal{C} \subset \mathcal{S}.$$

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\alpha,\beta,\gamma)$, note that $H(z) = zf'(z)/f_k(z)$ satisfying the condition of Lemma 1, from this we know that the condition (1.3) can be written as

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + (1-\alpha)(1-\gamma)z}{1 - (1-\alpha)\beta z} \quad (z \in \mathcal{U}). \tag{2.1}$$

Note that

$$\Re\left\{\frac{zf'(z)}{f_k(z)}\right\} = \Re\left\{\frac{1 + (1-\alpha)(1-\gamma)\omega(z)}{1 - (1-\alpha)\beta\omega(z)}\right\} > \frac{1 + (1-\alpha)(1-\gamma)}{1 - (1-\alpha)\beta} \geq 0,$$

thus, by Lemma 2, we have $\mathcal{C}^{(k)}(\alpha,\beta,\gamma) \subset \mathcal{C}^{(k)}(0,0,0) \subset \mathcal{C} \subset \mathcal{S}$.

By means of Lemma 3, using the similar method as in Theorem 1, we have

Corollary 1. *Let* $0 \le \alpha < 1,\ 0 \le \beta \le 1$ *and* $0 \le \gamma < 1$*, then we have*

$$\mathcal{QC}^{(k)}(\alpha, \beta, \gamma) \subset \mathcal{QC}^{(k)}(0,0,0) \subset \mathcal{QC} \subset \mathcal{C}.$$

Theorem 2. *Let* $0 \le \alpha_1 \le \alpha_2 < 1,\ 0 \le \beta_2 \le \beta_1 \le 1,$ *and* $0 \le \gamma_1 \le \gamma_2 < 1$*, then we have*

$$\mathcal{C}^{(k)}(\alpha_2, \beta_2, \gamma_2) \subset \mathcal{C}^{(k)}(\alpha_1, \beta_1, \gamma_1).$$

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\alpha_2, \beta_2, \gamma_2)$, by (2.1), we have

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + (1-\alpha_2)(1-\gamma_2)z}{1-(1-\alpha_2)\beta_2 z}.$$

Since $0 \le \alpha_1 \le \alpha_2 < 1,\ 0 \le \beta_2 \le \beta_1 \le 1$ and $0 \le \gamma_1 \le \gamma_2 < 1$, then we have

$$-1 \le -(1-\alpha_1)\beta_1 \le -(1-\alpha_2)\beta_2 < (1-\alpha_2)(1-\gamma_2) \le (1-\alpha_1)(1-\gamma_1) \le 1.$$

Thus, by Lemma 4, we have

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + (1-\alpha_2)(1-\gamma_2)z}{1-(1-\alpha_2)\beta_2 z} \prec \frac{1 + (1-\alpha_1)(1-\gamma_1)z}{1-(1-\alpha_1)\beta_1 z},$$

that is $f(z) \in \mathcal{C}^{(k)}(\alpha_1, \beta_1, \gamma_1)$. This means that

$$\mathcal{C}^{(k)}(\alpha_2, \beta_2, \gamma_2) \subset \mathcal{C}^{(k)}(\alpha_1, \beta_1, \gamma_1),$$

and hence the proof is complete.

For the class $\mathcal{QC}^{(k)}(\alpha, \beta, \gamma)$, we have

Corollary 2. *Let* $0 \le \alpha_1 \le \alpha_2 < 1,\ 0 \le \beta_2 \le \beta_1 \le 1,$ *and* $0 \le \gamma_1 \le \gamma_2 < 1$*, then we have*

$$\mathcal{QC}^{(k)}(\alpha_2, \beta_2, \gamma_2) \subset \mathcal{QC}^{(k)}(\alpha_1, \beta_1, \gamma_1).$$

3. Sufficient Conditions

In this section, we give the sufficient conditions for functions belonging to the classes $\mathcal{C}^{(k)}(\alpha, \beta, \gamma)$ and $\mathcal{QC}^{(k)}(\alpha, \beta, \gamma)$.

Theorem 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathcal{U}$*, if for* $0 \le \alpha < 1,\ 0 \le \beta \le 1$ *and* $0 \le \gamma < 1$*, we have*

$$\begin{aligned}\sum_{n=1}^{\infty} \{nk + (1-\alpha)[(nk+1)\beta + (1-\gamma)]\} \, |a_{nk+1}| \\ + \sum_{\substack{n=2 \\ n \ne lk+1}}^{\infty} n[1 + (1-\alpha)\beta] \, |a_n| \le (1-\alpha)(1+\beta-\gamma),\end{aligned} \tag{3.1}$$

then $f(z) \in \mathcal{C}^{(k)}(\alpha, \beta, \gamma)$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $f_k(z)$ is defined by equality (1.2). Then for $z \in \mathcal{U}$, we have

$$\begin{aligned} M &= |zf'(z) - f_k(z)| - (1-\alpha)\,|\beta z f'(z) + (1-\gamma) f_k(z)| \\ &= \left| z + \sum_{n=2}^{\infty} n a_n z^n - z - \sum_{n=2}^{\infty} a_n c_n z^n \right| \\ &\qquad -(1-\alpha)\left| \beta\left(z + \sum_{n=2}^{\infty} n a_n z^n \right) + (1-\gamma)\left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right) \right|, \end{aligned}$$

where

$$c_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu} \quad (\varepsilon^k = 1).$$

Thus, for $|z| = r < 1$, we have

$$\begin{aligned} M &\leq \sum_{n=2}^{\infty} (n - c_n)\,|a_n|\, r^n - (1-\alpha)\Big[(1+\beta-\gamma) r \\ &\qquad - \sum_{n=2}^{\infty} [n\beta + (1-\gamma) c_n]\,|a_n|\, r^n \Big] \\ &< \Big(\sum_{n=2}^{\infty} \{(n - c_n) + (1-\alpha)[n\beta \\ &\qquad + (1-\gamma) c_n]\}\,|a_n| - (1-\alpha)(1+\beta-\gamma) \Big) r. \end{aligned} \tag{3.2}$$

From the definition of c_n we know

$$c_n = \begin{cases} 1, & n = lk+1, \\ 0, & n \neq lk+1. \end{cases} \tag{3.3}$$

Substituting (3.3) into inequality (3.2), we get

$$\begin{aligned} M &< \sum_{n=1}^{\infty} \{nk + (1-\alpha)[(nk+1)\beta + (1-\gamma)]\}\,|a_{nk+1}| \\ &\qquad + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n[1 + (1-\alpha)\beta]\,|a_n| - (1-\alpha)(1+\beta-\gamma). \end{aligned}$$

From inequality (3.1) we know that $M < 0$, thus we can get inequality (1.3), that is $f(z) \in \mathcal{C}^{(k)}(\alpha, \beta, \gamma)$. This completes the proof of Theorem 3.

For the class $\mathcal{Q}C^{(k)}(\alpha, \beta, \gamma)$, we have

Corollary 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic in* $\mathcal{U}$*, if for* $0 \leq \alpha < 1,\ 0 \leq \beta \leq 1$ *and* $0 \leq \gamma < 1$*, we have*

$$\sum_{n=1}^{\infty}(nk+1)\{nk+(1-\alpha)[(nk+1)\beta+(1-\gamma)]\}\,|a_{nk+1}| + \sum_{\substack{n=2 \\ n \neq lk+1}}^{\infty} n^2[1+(1-\alpha)\beta]\,|a_n| \leq (1-\alpha)(1+\beta-\gamma),$$

then $f(z) \in \mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$.

4. Integral Representations

In this section, we give the integral representations of functions belonging to the classes $\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$.

Theorem 4. *Let* $f(z) \in \mathcal{C}^{(k)}(\alpha,\beta,\gamma)$*, then we have*

$$f_k(z) = z \cdot \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z}\frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt\right\}, \tag{4.1}$$

where $f_k(z)$ *is defined by equality (1.2),* $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof. Suppose that $f(z) \in \mathcal{C}^{(k)}(\alpha,\beta,\gamma)$, form (2.1) we can get

$$\frac{zf'(z)}{f_k(z)} = \frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)}, \tag{4.2}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting z by $\varepsilon^\mu z$ in (4.2) respectively ($\mu = 0,1,2,\ldots,k-1;\ \varepsilon^k = 1$), we have

$$\frac{\varepsilon^\mu z f'(\varepsilon^\mu z)}{f_k(\varepsilon^\mu z)} = \frac{1+(1-\alpha)(1-\gamma)\omega(\varepsilon^\mu z)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu z)} \quad (\mu = 0,1,2,\ldots,k-1). \tag{4.3}$$

Note that $f_k(\varepsilon^\mu z) = \varepsilon^\mu f_k(z)$, summing (4.3), we can get

$$\frac{zf_k'(z)}{f_k(z)} = \frac{1}{k}\sum_{\mu=0}^{k-1}\frac{1+(1-\alpha)(1-\gamma)\omega(\varepsilon^\mu z)}{1-(1-\alpha)\beta\omega(\varepsilon^\mu z)} \quad (\mu = 0,1,2,\ldots,k-1), \tag{4.4}$$

from equality (4.4) we get

$$\frac{f_k'(z)}{f_k(z)} - \frac{1}{z} = \frac{1}{k}\sum_{\mu=0}^{k-1}\frac{(1-\alpha)(1+\beta-\gamma)\omega(\varepsilon^\mu z)}{z[1-(1-\alpha)\beta\omega(\varepsilon^\mu z)]}. \tag{4.5}$$

Integrating equality (4.5) we have

$$\log\left\{\frac{f_k(z)}{z}\right\} = \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^z \frac{(1-\alpha)(1+\beta-\gamma)\omega(\varepsilon^\mu\zeta)}{\zeta[1-(1-\alpha)\beta\omega(\varepsilon^\mu\zeta)]}d\zeta$$
$$= \frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt,$$

from the above equality, we can get equality (4.1) easily. Hence the proof is complete.

Theorem 5. *Let $f(z)\in\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$, then we have*

$$f(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu\zeta} \frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt\right\} \cdot\frac{1+(1-\alpha)(1-\gamma)\omega(\zeta)}{1-(1-\alpha)\beta\omega(\zeta)}d\zeta, \tag{4.6}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

Proof. Suppose that $f(z)\in\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$, from equalities (4.1) and (4.2) we can get

$$f'(z) = \frac{f_k(z)}{z}\cdot\frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)}$$
$$= \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu z} \frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt\right\} \cdot\frac{1+(1-\alpha)(1-\gamma)\omega(z)}{1-(1-\alpha)\beta\omega(z)}.$$

Integrating the above equality, we can get equality (4.6) easily. Hence the proof is complete.

For the class $\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$, we have

Corollary 4. *Let $f(z)\in\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$, then we have*

$$f_k(z) = \int_0^z \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu\zeta} \frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt\right\}d\zeta,$$

where $f_k(z)$ is defined by equality (1.2), $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0)=0$, $|\omega(z)|<1$.

Corollary 5. *Let $f(z)\in\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$, then we have*

$$f(z)=\int_0^z \frac{1}{\xi}\int_0^\xi \exp\left\{\frac{1}{k}\sum_{\mu=0}^{k-1}\int_0^{\varepsilon^\mu \zeta}\frac{(1-\alpha)(1+\beta-\gamma)\omega(t)}{t[1-(1-\alpha)\beta\omega(t)]}dt\right\} \cdot \frac{1+(1-\alpha)(1-\gamma)\omega(\zeta)}{1-(1-\alpha)\beta\omega(\zeta)}d\zeta d\xi,$$

where $\omega(z)$ *is analytic in* $\mathcal{U}$ *and* $\omega(0)=0$, $|\omega(z)|<1$.

5. Convolution Conditions

At last, we provide the convolution conditions for the classes $\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ and $\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$.

Theorem 6. *A function* $f(z)\in\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$ *if and only if*

$$\frac{1}{z}\left\{f*\left\{\frac{z}{(1-z)^2}[1-(1-\alpha)\beta e^{i\theta}] - [1+(1-\alpha)(1-\gamma)e^{i\theta}]h(z)\right\}\right\}\neq 0 \tag{5.1}$$

for all $z\in\mathcal{U}$ *and* $0\le\theta<2\pi$, *where* $h(z)$ *is given by (5.6).*

Proof. Suppose that $f(z)\in\mathcal{C}^{(k)}(\alpha,\beta,\gamma)$, by Theorem 1, we know that the condition (1.3) can be written as (2.1), since (2.1) is equivalent to

$$\frac{zf'(z)}{f_k(z)}\neq\frac{1+(1-\alpha)(1-\gamma)e^{i\theta}}{1-(1-\alpha)\beta e^{i\theta}} \tag{5.2}$$

for all $z\in\mathcal{U}$ and $0\le\theta<2\pi$. It is easy to know that the condition (5.2) can be written as

$$\frac{1}{z}\{zf'(z)[1-(1-\alpha)\beta e^{i\theta}]-f_k(z)[1+(1-\alpha)(1-\gamma)e^{i\theta}]\}\neq 0. \tag{5.3}$$

On the other hand, it is well known that

$$zf'(z)=f(z)*\frac{z}{(1-z)^2}. \tag{5.4}$$

And from the definition of $f_k(z)$ we know

$$f_k(z)=z+\sum_{n=2}^{\infty}a_nc_nz^n=(f*h)(z), \tag{5.5}$$

where

$$h(z)=z+\sum_{n=2}^{\infty}c_nz^n \tag{5.6}$$

for c_n satisfy equality (3.3). Substituting (5.4) and (5.5) into (5.3), we can get (5.1). This completes the proof of Theorem 6.

For the class $\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$, we have

Corollary 6. *A function* $f(z)\in\mathcal{Q}C^{(k)}(\alpha,\beta,\gamma)$ *if and only if*

$$\frac{1}{z}\Bigg\{f*\Big\{z\big\{\frac{z}{(1-z)^2}[1-(1-\alpha)\beta e^{i\theta}] - [1+(1-\alpha)(1-\gamma)e^{i\theta}]h(z)\big\}'\Big\}\Bigg\}\neq 0$$

for all $z\in\mathcal{U}$ *and* $0\le\theta<2\pi$, *where* $h(z)$ *is given by (5.6).*

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College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha, Hunan 410076, People's Republic of China

E-mail address: zhigwang@163.com

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