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ON CLOSED INVERSE IMAGES OF BASE-PARACOMPACT SPACES

(submitted by M. A. Malakhaltsev)

Abstract. In this paper, we prove that every base-paracompact mapping $f : X \longrightarrow Y$ inversely preserves base-paracompactness if $w(X) \geq w(Y)$, where $w(X)$ and $w(Y)$ denote the weight of X and the weight of Y respectively. As an application of this result, we prove that every closed Lindelöf mapping $f : X \longrightarrow Y$ inversely preserves base-paracompactness if X is a regular space and $w(X)$ is a regular cardinality, where "X is a regular space" cannot be relaxed to "X is a Hausdorff space", which give some answers for a question on inverse images of base-paracompact spaces posed by L.Wu.

1. **Introduction**

In his paper [7], J.E.Porter introduced base-paracompactness, and obtained some analogous results of base-paracompactness to paracompactness. In particular, he proved that perfect mappings inversely preserve base-paracompactness [7, Theorem 3.6]. It is known that closed Lindelöf mappings with regular domain inversely preserve paracompactness [2, Theorem 7.1], which is obtained by a "nice characterization" of paracompactness: a regular space is paracompact if and only if its every open

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cover has a σ-locally finite open refinement [6]. Naturally, it is interesting to investigate closed Lindelöf inverse images of base-paracompact spaces. Contrary to what one might hope or expect, we do not know whether the analogous "nice characterization" of base-paracompactness is true. Thus our investigation from paracompactness to base-paracompactness case is not straightforward. Based on the discussion mentioned above, L.Wu [9] raised the following question.

Question 1.1. *Is base-paracompactness inversely preserved under closed Lindelöf mappings?*

In this paper, we investigate Question 1.1 for domains are Hausdorff spaces and regular spaces, respectively. We denote Axiom of Choice and Generalized Continuum Hypothesis by AC and GCH respectively. The weight $w(X)$ of a space X is the minimal cardinality of bases for X. Let κ be a cardinality. We denote the cofinality of κ by $cf(\kappa)$. A cardinality κ is called regular if $cf(\kappa) = \kappa$. We prove that every base-paracompact mapping $f : X \longrightarrow Y$ inversely preserve base-paracompactness if $w(X) \geq w(Y)$, which improves [7, Theorem 3.6]. As an application of this result, we prove that (AC+GCH) every closed Lindelöf mapping $f : X \longrightarrow Y$ inversely preserves base-paracompactness if X is a regular space and $w(X)$ is a regular cardinality, where "X is a regular space" can not be relaxed to "X is a Hausdorff space". By these results, we give some answers for Question 1.1.

Throughout this paper, all spaces are assumed to be Hausdorff and all mappings are continuous and onto. N denotes the set of all natural numbers, ω_0 denotes the first infinite cardinality. The cardinality of a set A is denoted $|A|$. Without loss of generality, in this paper we can assume that $w(X) \geq \omega_0$. If $f : X \longrightarrow Y$ is a mapping, $\mathcal{U}$ and $\mathcal{V}$ are families of subsets of X and Y respectively, then $f(\mathcal{U}) = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$. For a set A, families $\mathcal{U}$ and $\mathcal{V}$ of sets, $\bigcup\mathcal{U}$=$\bigcup\{U : U \in \mathcal{U}\}$, $\mathcal{U} \bigwedge A = \{U \bigcap A : U \in \mathcal{U}\}$, and $\mathcal{U} \bigwedge \mathcal{V} = \{U \bigcap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$. We say that $\mathcal{V}$ is a partial refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$; moreover, we say that $\mathcal{V}$ is a refinement of $\mathcal{U}$, if in addition $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ is also satisfied. One may refer to [2], [3] and [5] for undefined notations and terminology.

Definition 1.2. *A space X is called base-paracompact* [7] *if there exists a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$ such that every open cover of X has a locally finite refinement $\mathcal{B}' \subset \mathcal{B}$.*

Remark 1.3. *Base-paracompact $\Longrightarrow$ paracompact $\Longrightarrow$ normal.*

Let A be a subset of a space X and let $\mathcal{U}$ be a family of subsets of X. We call that $\mathcal{U}$ is locally finite at A in X [8], if for every $x \in A$ there exists an open (in X) neighborhood of x that intersects at most finite members of $\mathcal{U}$.

Definition 1.4. *A mapping $f : X \longrightarrow Y$ is called base-paracompact, if there exists a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$ such that for every $y \in Y$ and every family $\mathcal{U}$ of open subsets of X which covers $f^{-1}(y)$, there exist an open neighborhood O_y of y and a partial refinement $\mathcal{B}_y$ of $\mathcal{U}$, where $\mathcal{B}_y \subset \mathcal{B}$, such that $f^{-1}(O_y) \subset \bigcup \mathcal{B}_y$ and $\mathcal{B}_y$ is locally finite at $f^{-1}(O_y)$ in X.*

Recall a subset F of a space X is called a Lindelöf subset if every open cover of F has a countable subcover.

Definition 1.5. *A closed mapping $f : X \longrightarrow Y$ is called perfect (closed Lindelöf), if $f^{-1}(y)$ is a compact subset (Lindelöf subset) of X for every $y \in Y$.*

2. Inverse Images of Base-paracompact Spaces for Hausdorff Domains

Throughout section, all domains need not to be regular.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a base-paracompact mapping and $w(X) \geq w(Y)$. If Y is base-paracompact, then X is base-paracompact.*

Proof. Let $\mathcal{B}_Y$ be a base for Y which witnesses base-paracompactness for Y and let $\mathcal{B}_X$ be a base for X with $|\mathcal{B}_X| = w(X)$ which witnesses base-paracompactness for f. Put $\mathcal{B} = \mathcal{B}_X \bigwedge f^{-1}(\mathcal{B}_Y)$. Since $w(X) \geq w(Y)$, $|\mathcal{B}| = |\mathcal{B}_X| = w(X)$ and $\mathcal{B}$ is a base for X. We prove that $\mathcal{B}$ witnesses base-paracompactness for X as follows.

Let $\mathcal{U}$ be an open cover of X. Since $\mathcal{B}_X$ witnesses base-paracompactness for f, for every $y \in Y$ there exist an open neighborhood O_y of y and a partial refinement $\mathcal{B}_y$ of $\mathcal{U}$, where $\mathcal{B}_y \subset \mathcal{B}_X$, such that $f^{-1}(O_y) \subset \bigcup \mathcal{B}_y$ and $\mathcal{B}_y$ is locally finite at $f^{-1}(O_y)$ in X. Notice that Y is regular from Remark 1.3. There exists an open neighborhood G_y of y such that $\overline{G}_y \subset O_y$, thus $f^{-1}(y) \subset f^{-1}(G_y) \subset f^{-1}(\overline{G}_y) \subset f^{-1}(O_y) \subset \bigcup \mathcal{B}_y$. Put $\mathcal{V} = \{G_y : y \in Y\}$. By base-paracompactness for Y, $\mathcal{V}$ has a locally finite refinement $\mathcal{B}'_Y \subset \mathcal{B}_Y$. We write $\mathcal{B}'_Y = \{B'_\alpha : \alpha \in \Gamma\}$. For every $\alpha \in \Gamma$, pick $y_\alpha \in Y$ such that $B'_\alpha \subset G_{y_\alpha}$. Put $\mathcal{B}' = \bigcup\{\mathcal{B}_{y_\alpha} \bigwedge f^{-1}(B'_\alpha) : \alpha \in \Gamma\}$. Then $\mathcal{B}'$ is a refinement of $\mathcal{U}$ and $\mathcal{B}' \subset \mathcal{B}$. To complete the proof, it suffices to show that $\mathcal{B}'$ is locally finite.

Let $x \in X$. Notice that $f^{-1}(\mathcal{B}'_Y) = \{f^{-1}(B'_\alpha) : \alpha \in \Gamma\}$ is locally finite in X. There exist an open neighborhood U of x and a finite subset Γ_x of Γ such that $U \bigcap f^{-1}(B'_\alpha) = \emptyset$ for every $\alpha \in \Gamma - \Gamma_x$.

Let $\alpha \in \Gamma_x$. If $x \notin f^{-1}(\overline{B'_\alpha})$, then there exists an open neighborhood V_α of x such that $V_\alpha \bigcap f^{-1}(\overline{B'_\alpha}) = \emptyset$. Thus $V_\alpha \bigcap f^{-1}(B'_\alpha) = \emptyset$. If $x \in f^{-1}(\overline{B'_\alpha})$, then $x \in f^{-1}(\overline{G}_{y_\alpha}) \subset f^{-1}(O_{y_\alpha})$. Since $\mathcal{B}_{y_\alpha}$ is locally finite at $f^{-1}(O_{y_\alpha})$ in X, there exists an open neighborhood V_α of x that intersects at most finite members of $\mathcal{B}_{y_\alpha}$. Thus we can obtain V_α for every $\alpha \in \Gamma_x$ as above.

Put $W = U \bigcap(\bigcap\{V_\alpha : \alpha \in \Gamma_x\})$. Then W is an open neighborhood of x. It is not difficult to check that W intersects at most finite members of $\mathcal{B}'$. □

Remark 2.2. *We do not know if the condition "$w(X) \geq w(Y)$" in Theorem 2.1 can be omitted.*

Lemma 2.3. [3]. *A mapping $f : X \longrightarrow Y$ is closed if and only if for every $y \in Y$ and every open subset U in X which contains $f^{-1}(y)$, there exists an open neighborhood V of y such that $f^{-1}(V) \subset U$.*

Remark 2.4. *Base-paracompact mappings are closed mappings from Lemma 2.3.*

Proposition 2.5. *Let $f : X \longrightarrow Y$ be a perfect mapping. Then f is base-paracompact.*

Proof. Let $\mathcal{B}$ be a base for X with $|\mathcal{B}| = w(X)$. Let $y \in Y$ and let $\mathcal{U}$ be a family of open subsets of X which covers $f^{-1}(y)$. For every $x \in f^{-1}(y)$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ for some $U \in \mathcal{U}$. Since $f^{-1}(y)$ is compact, the family $\{B_x : x \in f^{-1}(y)\}$ has a finite subfamily $\mathcal{B}_y \subset \mathcal{B}$ such that $f^{-1}(y) \subset \bigcup \mathcal{B}_y$. By Lemma 2.3, there exists an open neighborhood O_y of y such that $f^{-1}(O_y) \subset \bigcup \mathcal{B}_y$. Notice that $\mathcal{B}_y$ is finite, so $\mathcal{B}_y$ is locally finite at $f^{-1}(O_y)$ in X. Thus f is base-paracompact. □

Corollary 2.6. [7]. *Let $f : X \longrightarrow Y$ be a perfect mapping. If Y is base-paracompact, then X is base-paracompact.*

Proof. It is straight from Proposition 2.5 , [3, Theorem 3.7.19] and Theorem 2.1. □

Now we use [4, The Counterexample] to answer Question 1.1 negatively.

Example 2.7. *Closed Lindelöf inverse images of Base-paracompact spaces need not to be base-paracompact.*

Proof. Let X, Q and I be the set of all real numbers, the set of all rational numbers and the set of all irrational numbers respectively. Define a base $\mathcal{B}$ of X as follows.

$\mathcal{B} = \{\{x\} : x \in I\} \bigcup \{G(x,n) : x \in Q, n \in N\}$, here $G(x,n) = \{y \in I : -1/n < y - x < 1/n\} \bigcup \{x\}$.

That is, X is a Bennett and Lutzer's space [1]. Define an equivalence relation R on X as follows: xRy if and only if either $x, y \in Q$ or $x = y$. Put Y is the quotient space X/R and put $f : X \longrightarrow Y$ is a natural mapping.

Fact 1. f is a closed Lindelöf mapping.

Fact 2. X is Hausdorff, but X is not regular.

Fact 3. X is not paracompact, and so X is not base-paracompact.

Fact 4. Y is normal.

Fact 5. Y is base-paracompact.

We only need to prove Fact 5, other facts hold from [4, The Counterexample].

Let $\mathcal{B}$ be a base for Y with $|\mathcal{B}| = w(Y)$. Pick $x_0 \in Q$ and put $y_0 = f(x_0)$. Note that $\{y\}$ is open in Y for every $y \in Y - \{y_0\}$. So $\{y\} \in \mathcal{B}$ for every $y \in Y - \{y_0\}$. Let $\mathcal{U}$ be any open cover of Y. There exists $B_0 \in \mathcal{B}$ such that $y_0 \in B_0 \subset U$ for some $U \in \mathcal{U}$. Put $\mathcal{B}' = \{B_0\} \bigcup \{\{y\} : y \in Y - B_0\}$. Then $\mathcal{B}' \subset \mathcal{B}$. It is clear that elements of $\mathcal{B}'$ are mutually disjoint. So $\mathcal{B}'$ is a locally finite refinement of $\mathcal{U}$. Consequently, Y is base-paracompact. □

3. Inverse Images of Base-paracompact Spaces for Regular Domains

Throughout this section, all domains assume to be regular.

Proposition 3.1. *If $f : X \longrightarrow Y$ is a closed Lindelöf mapping, then f is base-paracompact.*

Proof. Let $\mathcal{B}$ be a base for X with $|\mathcal{B}| = w(X)$. We can assume that $\mathcal{B}$ is closed under finite unions, finite intersections and complements of closures from [7, Theorem 3.4]. Let $y \in Y$ and let $\mathcal{U}$ be a family of open subsets of X, which covers $f^{-1}(y)$. For every $x \in f^{-1}(y)$, there exist B'_x, $B''_x \in \mathcal{B}$ such that $x \in B'_x \subset \overline{B'_x} \subset B''_x \subset U$ for some $U \in \mathcal{U}$. Since $f^{-1}(y)$ is Lindelöf, the family $\{B'_x : x \in f^{-1}(y)\}$ has a countable subfamily $\{B'_{x_n} : n \in N\}$ covering $f^{-1}(y)$. By Lemma 2.3, there exists an open neighborhood O_y of y such that $f^{-1}(O_y) \subset \bigcup \{B'_{x_n} : n \in N\}$.

Put $B_1 = B''_{x_1}$ and $B_n = B''_{x_n} - \bigcup \{\overline{B'_{x_i}} : i < n\}$ for every $n \geq 2$. Put $\mathcal{B}_y = \{B_n : n \in N\}$.

Claim 1. $\mathcal{B}_y \subset \mathcal{B}$: It follows from that $\mathcal{B}$ is closed under finite unions, finite intersections and complements of closures.

Claim 2. $\mathcal{B}_y$ is a partial refinement of $\mathcal{U}$: It is clear.

Claim 3. $f^{-1}(O_y) \subset \bigcup \mathcal{B}_y$: Let $x \in f^{-1}(O_y)$. Put $n_x = min\{i \in N : x \in \overline{B'_{x_i}}\}$. Then $x \in B_{n_x}$.

Claim 4. $\mathcal{B}_y$ is locally finite at $f^{-1}(O_y)$ in X: Let $x \in f^{-1}(O_y)$. Then there exists $i \in N$ such that $x \in B'_{x_i}$, thus B'_{x_i} is an open neighborhood of x which misses B_n for every $n > i$.

This proves that f is base-paracompact. □

Corollary 3.2. *Let $f : X \longrightarrow Y$ be a closed Lindelöf mapping and $w(X) \geq w(Y)$. If Y is base-paracompact, then X is base-paracompact.*

Remark 3.3. *Let X and Y be spaces stated in Example 2.7. It is clear that $w(X) = w(Y) = c$, where c is the cardinal number of the continuum. So Proposition 3.1 and Corollary 3.2 do not hold for Hausdorff domains by Theorem 2.1 and Example 2.7.*

Lemma 3.4. [5]. *(AC+GCH) Let κ be a cardinality. If $cf(\kappa) > \omega_0$, then $\kappa^{\omega_0} = \kappa$.*

Lemma 3.5. *(AC+GCH) Let $f : X \longrightarrow Y$ be a closed* Lindelöf *mapping. If $cf(w(X)) > \omega_0$, then $w(X) \geq w(Y)$.*

Proof. Let $\mathcal{B}$ be a base for X such that $|\mathcal{B}| = w(X)$ and let $\mathcal{A} = \{\bigcup \mathcal{B}' : \mathcal{B}' \subset \mathcal{B} \text{ and } |\mathcal{B}'| \leq \omega_0\}$. Since $cf(w(X)) > \omega_0$, $|\mathcal{A}| = w(X)$ by Lemma 3.4. Put $\mathcal{C} = \{Y - f(X - A) : A \in \mathcal{A}\}$. Then $|\mathcal{C}| = |\mathcal{A}| = w(X)$. It suffices to show that $\mathcal{C}$ is a base for Y. It follows from the definition that every member of $\mathcal{C}$ is open. Let $y \in Y$ and W be a neighborhood (in Y) of y. Then $f^{-1}(y) \subset f^{-1}(W)$ and $f^{-1}(y)$ is a Lindelöf subset of X, thus there exists a $A \in \mathcal{A}$ such that $f^{-1}(y) \subset A \subset f^{-1}(W)$. It is not difficulty to prove that $y \in Y - f(X - A) \in \mathcal{C}$ and $Y - f(X - A) \subset W$. This proves that $\mathcal{C}$ is a base for Y. □

Theorem 3.6. *(AC+GCH) Let $f : X \longrightarrow Y$ be a closed Lindelöf mapping and $w(X)$ be a regular cardinality. If Y is base-paracompact, then X is base-paracompact.*

Proof. If $w(X) = \omega_0$, then X is metrizable. So X is base-paracompact from [7, Theorem 3.3].

If $w(X) > \omega_0$, then $cf(w(X)) = w(X) > \omega_0$ because $w(X)$ is a regular cardinality. Thus $w(X) \geq w(Y)$ from Lemma 3.5. So X is base-paracompact from Corollary 3.2. □

Remark 3.7. *Theorem 3.6 does not hold for Hausdorff domains by Example 2.7. But we do not know if the condition "$w(X)$ be a regular cardinality" in Theorem 3.6 can be omitted.*

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References

1. H.R.Bennett and D.J.Lutzer, A note on weak θ-refinability, Gen. Top. Appl., 2(1972), 49-54.
2. D.K.Burke, Covering properties, Handbook of Set-Theoretic Topology, (K.Kunen and J.Vanghan, eds.), Amsterdam: North-Holland Press, 1984, 347-422.
3. R.Engelking, General Topology, Sigma Series in Pure Mathematics 6, (Heldermann, Berlin, revised ed.), 1989.
4. Y.Ge, On closed inverse images of mesocompact spaces, Fasciculi Mathematici, 36(2005), 27-32.
5. K.Kunen, Set Theory. An introduction to independence proofs, Amsterdam: North-Holland Press, 1980.
6. E.A.Michael, A note on paracompact spaces, Proc. Amer. Math. Soc., 4(1953), 831-838.
7. J.E.Porter, Base-paracompact spaces, Top. Appl., 128(2003), 145-156.
8. Z.Qu and Y.Yasui, Relatively subparacompact spaces, Scientiae Mathematicae Japonicae, 54(2001), 281-287.
9. L.Wu, Surveys in inverse images of covering properties, 2005 International General Topology Symposium, Zhangzhou Teachers College, Zhangzhou, P.R.China.

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