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EKELAND'S PRINCIPLE FOR VECTOR-VALUED MAPS BASED ON THE CHARACTERIZATION OF UNIFORM SPACES VIA FAMILIES OF GENERALIZED QUASI-METRICS

(submitted by A. Lapin)

Abstract. Using a new characterization of uniform spaces via Families of generalized quasi-metrics, we present a variant of Ekeland's variational principle for vector-valued maps being a consequence of minimal point theorem.

1 Introduction

Ekeland's variational principle [9] is an important tool in nonlinear analysis. In the last decades various theorems had been presented which turned out to be equivalent to Ekeland's principle. One of them, a lemma due to R. R. Phelps (see [30] and especially the version of ([31] from 1989) can be considered as the first minimal point theorem. Phelp's lemma yields the existence of minimal point with respect to a partial ordering in a subset of $X \times R$, where X is a Banach space and R denotes the reals.

Minimal point theorems in a product space $X \times Y$ were established by Gopfert and Tammer [13], 1995 and generalized by Gopfert, Tammer and Zalinescu in [15], 2000 and in [14], 1999. In the latest version, X is a complete metric space and Y is a separated locally convex space. These

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theorems are useful tools in vector optimization. In [15], [14] a variational principle for vector-valued functions $f : X \to Y$ was presented to be an easy consequence of the minimal point theorem.

A generalization of Ekeland's variational principle with respect to the space X was given by Fang [10], 1996. He introduced the concept of "F-type topological spaces" generating the topology by families of quasi-metrics. Andreas Hamel and Andreas Lohne proved in [1] that the class of Fang's F-type spaces coincides with the class of separated uniform spaces introduced by Weil [34], 1937.

In this paper we present a variant of Ekeland's variational principle for vector-valued maps as a consequence of minimal point theorem. The proof of the result is based on the Characterization of Uniform Spaces via Families of generalized quasi-metrics.

2 Family of generalized quasi-metrics and Uniform Spaces

In this section we present a characterization of uniform spaces via families of generalized quasi-metrics.

Initially, we shall recall the concept of uniform space. For further details see Kelly [22] or Kothe [23].

Let X be a nonempty set. We consider a system $\mathcal{R}$ of subsets N of $X \times X$. For $N \subset X \times X$ we denote $N^{-1} := \{(y,x) : \ (x,y) \in N\}$ and $N \circ N := \{(x,y) \in X \times X : \ \exists z \in X \ (x,z),(z,y) \in N\}$. The set $\Delta := \{(x,x) : \ x \in X\}$ is called the diagonal. The set X is said to be an uniform space if and only if there exists a filter $\mathcal{R}$ on $X \times X$ satisfying

$$\begin{array}{lll} (N1) & \forall N \in \mathcal{R} & \Delta \subset N; \\ (N2) & \forall N \in \mathcal{R} & N^{-1} \in \mathcal{R}; \\ (N3) & \forall N \in \mathcal{R} & \exists M \in \mathcal{R} \ M \circ M \subset N. \end{array}$$

The system $\mathcal{R}$ is called a uniformity on X. By the sets

$$\theta(x) := \{U_N(x) : \ N \in \mathcal{R}\},$$

where $U_N(x) = \{y \in X : \ (x,y) \in N\}$, a topology is given, which is called the uniform topology on X. Of course, an uniform space is already well-defined by a base of its uniformity $\mathcal{R}$, i.e a filter base $\mathcal{B}$ of the uniformity $\mathcal{R}$. The topology of an uniform space is separated if and only if

$$(N4) \ \bigcap_{N \in \mathcal{R}} N = \Delta.$$

For a proof see ([23],p. 32).

We recall a well-established result, the characterization of uniform spaces using families of pseudo-metrics see [22].

Definition 1. Let X be a nonempty set. A function $p : X \times X \to [0, +\infty[$ is called pseudo-metric on X if and only if for all $x, y, z \in X$ the following conditions are satisfied:

$$\begin{array}{ll} (P1) & p(x,x) = 0; \\ (P2) & p(x,y) = p(y,x); \\ (P3) & p(x,z) \leq p(x,y) + p(y,z). \end{array}$$

Let $(\ell, \prec)$ be a directed set. A system $\{p_\lambda\}_{\lambda \in \ell}$ of pseudo-metrics $p_\lambda : X \times X \to [0, +\infty[$ satisfying

$$(P4)\ \lambda \prec \mu \Rightarrow \forall x, y \in X\ p_\lambda(x,y) \leq p_\mu(x,y)$$

is called a family of pseudo-metrics. If additionally the condition

$$p_\lambda(x,y) = 0\ \forall \lambda \in \ell \Rightarrow x = y$$

holds, the family of pseudo-metrics is said to be separating.

Proposition 1. A topological space (X, τ) is a separated uniform space if and only if its topology τ can be generated by a separating family of pseudo-metrics.

Proof. See ([22],p. 188, Theorem 15).

Fang [10] introduced so-called $F-$ type topological using families of quasi-metrics.

Definition 2. Let X be a nonempty set and let $(\ell, \prec)$ be a directed set. A system $\{q_\lambda\}_{\lambda \in \ell}$ of functions $q_\lambda : X \times X \to [0, +\infty[$ satisfying

$$\begin{array}{ll} (Q1) & \forall \lambda \in \ell \quad \forall x \in X \quad q_\lambda(x,x) = 0; \\ (Q2) & \forall \lambda \in \ell\ \forall x, y \in X \quad q_\lambda(x,y) = q_\lambda(y,x); \\ (Q3) & \forall \lambda \in \ell\ \exists \mu \in \ell \text{ such that } \lambda \prec \mu \text{ and} \\ & \forall x, y, z \in X q_\lambda(x,z) \leq q_\mu(x,y) + q_\mu(y,z); \\ (Q4) & \lambda \prec \mu \Rightarrow \forall x, y \in X q_\lambda(x,y) \leq q_\mu(x,y) \end{array}$$

is called a family of quasi-metrics. If in addition the condition

$$(q_\lambda(x,y) = 0\ \forall \lambda \in \ell) \Rightarrow x = y$$

is satisfied, the family of quasi-metrics is said to be separating.

Proposition 2. A topological space (X, τ) is a separated uniform space if and only if its topology τ can be generated by a separating family of quasi-metrics.

Proof. See ([1],p. 3, Theorem 4).

Definition 3. Let X be a nonempty set. A system $\{q_\lambda\}_{\lambda\in\ell}$ of functions $q_\lambda : X \times X \to [0, +\infty[$ satisfying

$(Q'1)$ $\forall\lambda \in \ell\ \forall x \in X \quad q_\lambda(x, x) = 0;$
$(Q'2)$ $\forall\lambda \in \ell\ \forall x, y \in X \quad q_\lambda(x, y) = q_\lambda(y, x);$
$(Q'3)$ $\exists\{\mu_1(\lambda)\}_{\lambda\in\ell}, \{\mu_2(\lambda)\}_{\lambda\in\ell} \subset \ell$ such that $\forall\lambda \in \ell\ \forall x, y, z \in X$
$z \neq y,\ y \neq x \quad q_\lambda(x, z) \leq q_{\mu_1(\lambda)}(x, y) + q_{\mu_2(\lambda)}(y, z)$

is called a family of generalized quasi-metrics. In this case, $(X, \{q_\lambda\}_{\lambda\in\ell})$ is called a generalized quasi-metric space. If in addition the condition

$$(q_\lambda(x, y) = 0\ \forall\lambda \in \ell) \Rightarrow x = y$$

is satisfied, the family of generalized quasi-metrics is said to be separating.

Our definition is slightly more general because on the one hand ℓ has not to be directed set in our case and on the other hand, the assumption (Q3) in Definition 2 is an optional condition, not automatically satisfied in our Definition.

Our first result clarifies the relation between separated uniform spaces and topological spaces generated by separating families of generalized quasi-metrics.

Proposition 3. A topological space (X, τ) is a separated uniform space if and only if its topology τ can be generated by a separating family of generalized quasi-metrics.

Proof. Let (X, τ) be a topological space where τ is generated by a separating family $\{q_\lambda\}_{\lambda\in\ell}$ of generalized quasi-metrics, i.e. τ is given by

$$\wp(x) = \{U(x, n, \lambda_1, .., \lambda_n, t) :\ t > 0,\ n \in N,\ \lambda_1, .., \lambda_n \in \ell\}$$

where

$$U(x, n, \lambda_1, .., \lambda_n, t) = \{y \in X :\ q_{\lambda_i}(x, y) < t,\ 1 \leq i \leq n\}.$$

We claim that a base of a uniformity is given by the system

$$\aleph := \{S(n, \lambda_1, .., \lambda_n, t) :\ t > 0,\ n \in N,\ \lambda_1, .., \lambda_n \in \ell\}$$

where

$$S(n, \lambda_1, .., \lambda_n, t) := \{(x, y) \in X \times X :\ q_{\lambda_i}(x, y) < t,\ 1 \leq i \leq n\}.$$

To show that $\aleph$ is a filter base let $t_1 > 0,\ n \in N,\ \lambda_1, .., \lambda_n \in \ell$ and $t_2 > 0, p \in N, \beta_1, .., \beta_p \in \ell$ be arbitrarily given. Set $t_3 := \min(t_1, t_2)$,$r = n + p$. Consider the sequence $\lambda_1, .., \lambda_n, \beta_1 .., \beta_p$. Then $S(r, \lambda_1, .., \lambda_n, \beta_1 .., \beta_p, t_3) \subset S(n, \lambda_1, .., \lambda_n, t_1) \bigcap S(p, \beta_1, .., \beta_p, t_2)$.

Furthermore, $\emptyset \notin \aleph$ since each $S(n, \lambda_1, .., \lambda_n, t)$ contains the diagonal.

Let $\Re$ be the filter generated by $\aleph$. We shall show that $\aleph$ is a base of uniformity on X. The axioms (N_1) and (N_2) are satisfied for $\aleph$. To verify (N_3) let $t > 0,\ n \in N,\ \lambda_1, .., \lambda_n \in \ell$ be arbitrarily given. Taking $(\mu_1 = \mu_1(\lambda_1), \mu_2 = \mu_2(\lambda_1)), .., (\mu_{2n-1} = \mu_1(\lambda_n), \mu_{2n} = \mu_2(\lambda_n))$ from (Q'3) we set $M := S(3n, \lambda_1, .., \lambda_n, \mu_1, \mu_2, \mu_3, \mu_4, .., \mu_{2n-1}, \mu_{2n}, \frac{t}{2})$.

Then we have $M \circ M \subset S(n, \lambda_1.., \lambda_n, t)$. Indeed, let $(x, y) \in M \circ M$, i.e.

$$\exists z \in X : \ (x, z), (z, y) \in M.$$

If $z = y$ or $z = x$, then $(x, y) \in M$. Therefore,

$$q_{\lambda_i}(x, y) < \frac{t}{2} < t \text{ for all } i \text{ such that } 1 \leq i \leq n.$$

Hence, $(x, y) \in S(n, \lambda_1, .., \lambda_n, t)$.

If now $z \neq y$ and $z \neq x$, then $q_{\mu_i}(x, z) < \frac{t}{2}$, $q_{\mu_i}(z, y) < \frac{t}{2}$, $1 \leq i \leq 2n$. Hence

$$q_{\lambda_i}(x, y) \leq q_{\mu_{2i-1}}(x, z) + q_{\mu_{2i}}(z, y) < t,\ 1 \leq i \leq n.$$

Therefore, $\Re$ is a uniformity generating the topology τ. If additionally the family $\{q_\lambda\}_{\lambda \in \ell}$ of generalized quasi-metrics is separating, then the uniform space (X, τ) is separated.

The opposite assertion follows by Proposition 1 taking into account that a family of pseudo-metrics is in particular a family of generalized quasi-metrics.

An important class of uniform spaces is the class of topological vector spaces. Indeed, For topological vector space the topology can be generated by a family of quasi-norms. This result is due to Hyers [19], 1939 who used the term "pseudo-norms" instead of "quasi-norms".

3. Main Tools

For the convenience of the reader we present the main tools for the proof of our minimal point theorem. The first one is the Brézis-Browder principle.

Theorem 1. Let $(W, \preceq)$ be a quasi-ordered set (i.e. $\preceq$ is a reflexive and transitive relation) and let $\phi : W \to R$ be a function satisfying

- $(A1)$ ϕ is bounded below ;
- $(A2)$ $w_1 \preceq w_2 \Rightarrow \phi(w_1) \leq \phi(w_2)$;
- $(A3)$ For every $\preceq-$decreasing sequence $\{w_n\}_{n\in N} \subset W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in N$.

Then, for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

$$(i)\ \bar{w} \preceq w_0; \qquad (ii)\ \hat{w} \preceq \bar{w} \Rightarrow \phi(\hat{w}) = \phi(\bar{w}).$$

In particular, if we strengthen (A2) to $(A'2)$ $(w_1 \preceq w_2,\ w_1 \neq w_2) \Rightarrow \phi(w_1) < \phi(w_2)$ it holds

$$(ii')\ \hat{w} \preceq \bar{w} \Rightarrow \hat{w} = \bar{w},\ i.e.\ \bar{w} \text{ is } \preceq\!-\text{minimal in } W.$$

Proof. See [[3], Corollary 1].

Note that A'2 implies the antisymmetry of the relation $\preceq$.

A second important tool is a scalarization method established by Gerstewitz (Tammer), Iwanow [12] and Gerth (Tammer), Weidner [11].

Theorem 2. Let Y be a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -cl\ K$. The functional $z : Y \to R \bigcup \{+\infty\}$, defined as $z(y) := \inf\{t \in R :\ y \in tk_0 - cl\ K\}$ has the following properties

$(H1)$ z is sublinear;
$(H2)$ $y_1 \preceq_K y_2 \Rightarrow z(y_1) \leq z(y_2)$;
$(H3)$ $\forall y \in Y\ \forall \alpha \in R\ z(\alpha k_0 + y) = z(y) + \alpha$;
$(H4)$ If $Y_0 \subset Y$ is $\preceq_K$ bounded below, then z is bounded below on Y_0.

Proof. See [[15], Lemma 7] taking into account that Y has not to be separated for the proof. Moreover, in the definition of the functional the closed cone can be replaced by the closure of a not necessarily closed cone (since $y_1 \preceq_K y_2$ implies $y_1 \preceq_{clK} y_2$). Then, if Y is not separated, we have to choose $k_0 \in K \setminus -cl\ K$ to avoid $k_0 \in cl\{0\}$. If Y is separated it suffices to suppose $k_0 \in K \setminus -K$.

Let Y be a topological vector space and $K \subset Y$ a convex cone. We use the following assumption to derive strong (in [15] called "authentic") variants of the minimal point theorem.

(C) There exists a proper cone convex $C \subset Y$ satisfying $K \setminus \{0\} \subset int\ C$.

Theorem 3. Let Y be a topological vector space, $K \subset Y$ a convex cone satisfying assumption (C). Let $k_0 \in K \setminus \{0\}$. The functional $z_C : Y \to R$, defined as $z_C(y) := \inf\{t \in R :\ y \in tk_0 - cl\ C\}$ has the following properties

$(H'1)$ z_C is sublinear;
$(H'2)$ $(y_1 \preceq_K y_2, y_1 \neq y_2) \Rightarrow z_C(y_1) < z_C(y_2)$;
$(H'3)$ $\forall y \in Y\ \forall \alpha \in R\ z_C(\alpha k_0 + y) = z_C(y) + \alpha$;
$(H'4)$ For $Y_0 \subset Y, \tilde{y} \in Y$ the condition $Y_0 \bigcap (\tilde{y} - int\ C) = \emptyset$ implies that z_C is bounded below on Y_0.

Proof. See [[15], Lemma 7] taking into account that $z_C(y) = \infty$ is not possible under our assumptions. Note that we have $k_0 \in int\ C \setminus -cl\ C$. Therefore, as above, Y has not to be separated.

4. Minimal Point Theorem

Minimal point theorems in product spaces $X \times Y$ presented by Gopfert and Tammer [13], by Gopfert, Tammer and Zalinescu [15], [14], and by Andreas Hamel and Andreas Lohne [1] give useful generalizations of Ekeland's variational principle. We wish to generalize some of theorems in [1] with respect to the space X. Instead of family of quasi-metrics we consider a family of generalized quasi-metrics.

In what follows let $(X, \{q_\lambda\}_{\lambda \in \ell})$ be a separated uniform space generated by the separating family $\{q_\lambda\}_{\lambda \in \ell}$ of generalized quasi-metrics and let Y be a topological vector space. We write $w = (w_1, w_2) \in W$ to deal with the two components of an element w of the product space $W := X \times Y$.

It is well-known that a convex cone $K \subset Y$ generates a quasi-ordering on Y by

$$y_1 \preceq_K y_2 \Leftrightarrow y_2 - y_1 \in K.$$

If K is pointed, the relation is also antisymmetric, therefore a partial ordering. Using an element $k_0 \in K \setminus -cl\ K$ we introduce a relation $\preceq_{k_0}$ on W:

$$(x_1, y_1) \preceq_{k_0} (x_2, y_2) \Leftrightarrow \forall \lambda \in \ell \ y_1 + q_\lambda(x_1, x_2)k_0 \preceq_K y_2.$$

Lemma 1. If $K \subset Y$ is a convex cone, the relation $\preceq_{k_0}$ is reflexive and transitive on W. If additionally K is pointed, the relation $\preceq_{k_0}$ is antisymmetric and thus partial ordering on W.

Proof. Exemplary, we prove the transitivity. Let $w_i = (x_i, y_i) \in W$ $(i = 1, 2, 3)$ satisfying $(x_1, y_1) \preceq_{k_0} (x_2, y_2)$ and $(x_2, y_2) \preceq_{k_0} (x_3, y_3)$. Therefore,

$$\forall \lambda \in \ell \ y_1 + q_\lambda(x_1, x_2)k_0 \preceq_K y_2 \text{ and } y_2 + q_\lambda(x_2, x_3)k_0 \preceq_K y_3.$$

The transitivity of the relation $\preceq_K$ yields

$$\forall \lambda \in \ell \ y_1 + (q_\lambda(x_1, x_2) + q_\lambda(x_2, x_3))k_0 \preceq_K y_3.$$

If $x_2 = x_1$ or $x_2 = x_3$, then the assumption $w_1 \preceq_{k_0} w_3$ holds. Otherwise, using Q'3 we deduce that

$$\forall \lambda \in \ell \ y_1 + q_\lambda(x_1, x_3)k_0 \preceq_K y_3.$$

Hence, $w_1 \preceq_{k_0} w_3$.

We continue with our main result, the minimal point theorem in uniform spaces. Just as the Brézis-Browder principle (Theorem 5), the following theorem (as well as its equivalent formulations, Theorems [11], [13], [14]) consists of two parts. The "weak" assertion (ii) yields the existence of an element $\bar{w}$ of a certain set A such that some $\hat{w}$ which is dominated by $\bar{w}$ with respect to a quasi-ordering necessarily has the same $X-$ component. However, the $Y-$ component may be distinct. The "strong" (authentic) assertion (ii) yields the minimality of some $\bar{w} \in A$ in A with respect to a partial ordering. Note that assumption (C) of section 3 ensures that we deal in fact with a partial ordering. It plays the key role in establishing the strong assertion and can be traced back to the early work of Bishop and Phelps.

Theorem 4. (Minimal point theorem) Let $(X, \{q_\lambda\}_{\lambda\in\ell})$ be a separated uniform space generated by the separating family $\{q_\lambda\}_{\lambda\in\ell}$ of generalized quasi-metrics, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -cl\ K$. Let $A \subset W$ be a nonempty subset of the product space $W := X \times Y$ and let $w_0 \in A$ be given such that for the set $W_0 := \{w \in A :\ w \preceq_{k_0} w_0\}$ the following assumptions hold true

$(M1)$ The set $(W_0)_Y := \{y \in Y :$ there exists $x \in X$ such that $(x, y) \in W_0\}$ is $\preceq_K -$bounded below;

$(M2)$ For any $\preceq_{k_0}$- decreasing sequence $\{w_n\}_{n\in N} \subset W_0$ there exists some $w \in W_0$ such that$w \preceq w_n$ for all $n \in N$.

Then there exists some $\bar{w} \in A$ such that

$$(i)\ \bar{w} \preceq_{k_0} w_0; \quad (ii)\ (\hat{w} \in A, \hat{w} \preceq_{k_0} \bar{w}) \Rightarrow \hat{w} = \bar{w}.$$

Under the additional assumption (C) we can relax assumption (M1) to

$(M'1)$ There exists some $\tilde{y} \in Y$ such that $(W_0)_Y \bigcap (\tilde{y} - int\ C) = \emptyset$ and

$$(iii)\ \bar{w} \text{ is } \preceq_{k_0} \text{ minimal point in } A$$

Proof. By Lemma 1, the relation $\preceq_{k_0}$ is reflexive and transitive on W_0. We apply the Brézis-Browder principle (Theorem 1) on the quasi-ordered set $(W_0, \preceq_{k_0})$ using the functional $\phi : W_0 \to R$, $\phi(w) = z(w_Y - (w_0)_Y)$, where $z : Y \to R \bigcup \{+\infty\}$ is the scalarization functional of Theorem 2. First, we must have $\phi(w) \neq +\infty$. Indeed, for $w \in W_0$ it holds $w_Y \preceq_K (w_0)_Y$. Hence $w_Y - (w_0)_Y \in -K \subset -cl\ K$. By the definition of z we have $\phi(w) \leq 0$.

By (M1) and property (H4) of z (Theorem 2), ϕ is bounded below on W_0. Let be $w_1 \preceq_{k_0} w_2$, $w_1, w_2 \in W_0$, hence $(w_1)_Y \preceq_K (w_2)_Y$. Property (H2) of z implies assumption (A2) of Theorem 1. Of course, (M2) implies assumption (A3) of Theorem 1.

Theorem 1 yields the existence of some $\bar{w} \in W_0$ such that

$$\bar{w} \preceq_{k_0} w_0 \text{ and } \hat{w} \preceq_{k_0} \bar{w} \Rightarrow \phi(\hat{w}) = \phi(\bar{w}).$$

Let us show (ii). Take $\hat{w} \in A$ such that $\hat{w} \preceq_{k_0} \bar{w}$. The transitivity of $\preceq_{k_0}$ yields $\hat{w} \in W_0$. This implies

$$\forall \lambda \in \ell \ (\hat{w})_Y - (w_0)_Y + q_\lambda((\hat{w})_X, (\bar{w})_X)k_0 \preceq_K (\bar{w})_Y - (w_0)_Y.$$

Applying properties (H2) and (H3) of z we get

$$\forall \lambda \in \ell \ q_\lambda((\hat{w})_X, (\bar{w})_X) \leq \phi(\bar{w}) - \phi(\hat{w}) = 0.$$

Consequently,

$$\forall \lambda \in \ell \ q_\lambda((\hat{w})_X, (\bar{w})_X) = 0.$$

Since $(X, \{q_\lambda\}_{\lambda \in \ell})$ is separated, then $(\hat{w})_X = (\bar{w})_X$.

Now, let assumption (C) be satisfied. We can replace (M1) by (M'1) and proceed analogously, but using the functional z_C of Theorem 3 instead of z. In particular, the corresponding functional $\phi_C : W_0 \to R$, $\phi_C(w) = z_C(w_Y)$ (the functional can be chosen slightly simpler than before, because $z_C(y) \neq \infty \ \forall y \in Y$) is even strict $\preceq_{k_0}-$monotone, i.e. $w_1 \preceq_{k_0} w_2$, $w_1 \neq w_2$ implies $\phi_C(w_1) < \phi_C(w_2)$. Indeed, let $w_1 \preceq_{k_0} w_2$, and $w_1 \neq w_2$. If $(w_1)_X \neq (w_2)_X$ then, since $(X, \{q_\lambda\}_{\lambda \in \ell})$ is separated, there exists some $\lambda \in \ell$ satisfying $q_\lambda((w_1)_X, (w_2)_X) > 0$, thus,

$$(w_2)_Y - (w_1)_Y \in \{q_\lambda((w_1)_X, (w_2)_X)k_0\} + K \subset K \setminus -cl\ K + K \subset K \setminus \{0\}.$$

Otherwise, if $(w_1)_X = (w_2)_X$, we have $(w_1)_Y \neq (w_2)_Y$ and it also holds $(w_2)_Y - (w_1)_Y \in K \setminus \{0\}$. Property H'2 of z_C yields $\phi(w_1) < \phi(w_2)$. Therefore, assumption A'2 in Theorem 1 is satisfied too. The $\preceq_{k_0}-$minimality of $\bar{w}$ in A follows from (Theorem 1, (ii')) taking into account the transitivity of the relation $\preceq_{k_0}$.

5 Ekeland's principle for vector-valued maps

In this section we present a variant of Ekeland's variational principle for vector-valued functions. As proposed in [15], [14], we extend the space Y by an element ∞ such that $y \preceq_K \infty$ for all $y \in Y$.

Theorem 5. (Variational Principle) Let $(X, \{q_\lambda\}_{\lambda \in \ell})$ be a separated uniform space generated by the separating family $\{q_\lambda\}_{\lambda \in \ell}$ of generalized quasi-metrics, Y a topological vector space, $K \subset Y$ a convex cone and $k_0 \in K \setminus -cl\ K$. Let $f : X \to Y \cup \{\infty\}$ be a proper function which is $\preceq_K$–bounded below and let for every $x \in dom\ f$ the set

$$S(x) := \{u \in X : \ \forall \lambda \in \ell \ f(u) + q_\lambda(u, x)k_0 \preceq_K f(x)\}$$

be sequentially closed.

Then, for each $x_0 \in dom\ f$ there exists $\bar{x} \in dom\ f$ such that

$$(i) \ \forall \lambda \in \ell \ f(\bar{x}) + q_\lambda(\bar{x}, x_0)k_0 \preceq_K f(x_0);$$

(ii) $\forall x \in X$ such that $x \neq \bar{x}$ $\exists \beta \in \ell$ $f(x) + q_\beta(\bar{x}, x)k_0 \not\preceq_K f(\bar{x})$.

Proof. We consider the set-valued mapping $F : X \to 2^Y$, $F(x) := \{f(x)\}$ if $f(x) \neq \infty$ and $F(x) := \emptyset$ otherwise. Let *dom* F denote the domain of F, i.e. *dom* $F := \{x \in X : F(x) \neq \emptyset\}$ and let *gr* F denote the graph of F, i.e. *gr* $F := \{(x, y) \in X \times Y : y \in F(x)\}$. Setting $A :=$ *gr* F, $w_0 = (x_0, f(x_0))$ all assumptions coincide with those of Theorem 4. Indeed, it remains to show that (M2) of Theorem 4 is satisfied. Let $\{(x_n, y_n)\}_{n\in N} \subset W_0$ be a $\preceq_{k_0}-$decreasing sequence and let $\varepsilon > 0$, $\lambda \in \ell$. Then for any $m, n \in N$ with $m \geq n$ we have

$$y_m - f(x_0) + q_\lambda(x_m, x_n)k_0 \preceq_K y_n - f(x_0).$$

The properties of z (Theorem 2) yield

$$\phi(y_m) + q_\lambda(x_m, x_n) \leq \phi(y_n),$$

where $\phi(y) = z(y - f(x_0))$. Thus $\{\phi(y_n)\}_{n\in N}$ is nonincreasing sequence. On the other hand, $\{\phi(y_n)\}_{n\in N}$ is bounded below, hence there exists some $n_0 \in N$ such that for all $n, m \geq n_0$ it holds

$$q_\lambda(x_m, x_n) \leq \phi(y_n) - \phi(y_m) < \varepsilon.$$

Hence, $q_\lambda(x_n, x_m) < \varepsilon$. This means, that $\{x_n\}$ is a Cauchy sequence in $(X, \{q_\lambda\}_{\lambda\in\ell})$ and by the sequentially completeness of X convergent to some $x \in X$.

On the other hand, we have $x_m \in S(x_n)$ for all $n, m \in N$ with $m \geq n$. Since $S(x_n)$ is sequentially closed, it follows that $x \in S(x_n)$ for all $n \in N$. Hence, $(x, f(x)) \preceq_{k_0} (x_n, y_n)$ for all $n \in N$. Theorem 4 implies all assertions.

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