Lobachevskii Journal of Mathematics
http://ljm.ksu.ru
ISSN 1818-9962
Vol. 22, 2006, 35–46

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INDEX VECTOR-FUNCTION AND MINIMAL CYCLES

(submitted by B. N. Shapukov)

Abstract. Let P be a closed triangulated manifold, $\dim P = n$. We consider the group of simplicial 1-chains $C_1(P) = C_1(P, \mathbb{Z}_2)$ and the homology group $H_1(P) = H_1(P, \mathbb{Z}_2)$. We also use some nonnegative weighting function $L : C_1(P) \to \mathbb{R}$. For any homological class $[x] \in H_1(P)$ the method proposed in article builds a cycle $z \in [x]$ with minimal weight $L(z)$. The main idea is in using a simplicial scheme of space of the regular covering $p : \hat{P} \to P$ with automorphism group $G \cong H_1(P)$. We construct this covering applying the index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to any basis of group $H_{n-1}(P)$, $r = \operatorname{rank} H_{n-1}(P)$.

1. Index Vector-Function

Consider a triangulated closed manifold P, $\dim P = n$, and a basis $[z_1^{n-1}], \dots, [z_r^{n-1}]$ of homology group $H_{n-1}(P) = H_{n-1}(P, \mathbb{Z}_2)$. Let $\operatorname{Ind} : H_1(P) \times H_{n-1}(P) \to \mathbb{Z}_2$ be the intersection index.

Definition 1. Define the homomorphism $J_0 : Z_1(P) \to \mathbb{Z}_2^r$ by $J_0^k(y) = \operatorname{Ind}([y], [z_k^{n-1}])$, $k = 1, \dots, r$, $J_0 = (J_0^1, \dots, J_0^r)$. We call its arbitrary extension $J : C_1(P) \to \mathbb{Z}_2^r$ the index vector-function. For any chain $x \in C_1(P)$ the value $J(x)$ is called its index relative to the basis $[z_1^{n-1}]$, $\dots$, $[z_r^{n-1}]$.

2000 Mathematical Subject Classification. 57Q15, 57N65, 65D18.
Key words and phrases. Triangulated manifold, homology group, minimal cycle, intersection index, regular covering.

Remark 1. The index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ is not uniquely defined, however we can use this function for solve our problems.

Proposition 1. *If $J : C_1(P) \to \mathbb{Z}_2^r$ is the index vector-function relative to the basis $\{[z_1^{n-1}], \dots, [z_r^{n-1}]\}$ of group $H_{n-1}(P)$, $x, y \in C_1(P)$ and $\partial x = \partial y$, then $J(x) = J(y)$ if and only if $x \sim y$.*

Proof. Let $\{[z_1^1], \dots, [z_r^1]\}$ be a basis of group $H_1(P) = H_1(P, \mathbb{Z}_2)$, that is dual to the given basis $\{[z_1^{n-1}], \dots, [z_r^{n-1}]\}$. Assume now that $z = x + y$. Then $z \in Z_1(P)$ and $[z] = \sum_{i=1}^r l^i [z_i^1]$, where $l_i \in \mathbb{Z}_2$. This implies that $J^k(z) = \mathrm{Ind}([z], [z_k^{n-1}]) = l^k$ for all $k = 1, \dots, r$. So $J(x) = J(y)$ if and only if $l^1 = \dots = l^r = 0$. And this latter expression is equivalent to the equality $[z] = 0$. □

ALGORITHM 1. Construction of index vector-function relative to the basis of group $H_{n-1}(P)$.

Input:

1) simple basis cycles $z_1^{n-1}, z_2^{n-1}, \dots, z_r^{n-1}$ which are lists of $(n-1)$-dimensional simplices;

2) list $K^1(P)$ of edges for polyhedron P;

3) lists $K_1^n(P, z_1^{n-1}), \dots, K_r^n(P, z_r^{n-1})$ consisting of n-dimensional simplices from neighborhoods of cycles $z_1^{n-1}, \dots, z_r^{n-1}$ respectively;

Output:

1) vectors $J(a) = (J^1(a), \dots, J^r(a)) \in \mathbb{Z}_2^r$ for all edges $a \in K^1(P)$;

2) chains $M_1, \dots, M_r$ of edges indexed relative to cycles $z_1^{n-1}, \dots, z_r^{n-1}$ respectively;

3) lists $M_k(u)$, $k = 1, \dots, r$ of edges, that we add to M_k when considering vertex u of cycle z_k^{n-1};

4) sets $\Sigma_k(u)$, $k = 1, \dots, r$ of n-simplices incident to edges from $M_k(u)$.

Algorithm Description.

Step 0. For all $k = 1, \dots, r$ execute steps 1 – 3.

Step 1. Start operations. Assume $M_k = \emptyset$, $J^k(a) := 0$ for all $a \in K^1(P)$. We denote z_k^{n-1} by X and $K_k^n(P, z_k^{n-1})$ by $K^n(P, X)$. We create then lists of vertices and edges for all simplices of cycle X, $K^0(X)$ and $K^1(X)$ respectively.

Step 2. Indexing edges that do not belong to the cycle. For each vertex $u \in K^0(X)$ execute steps 2.1 – 2.4.

Step 2.1. Initializing vertex neighborhood. Let us create a list $K^n(P, u) \subset K^n(P, X)$ of n-dimensional simplices of the polyhedron P, that contain u, and a list $K^{n-1}(P, u)$ of all $(n-1)$-dimensional faces of simplices from $K^n(P,u)$. At the same time, for each simplex $\sigma^{n-1} \in K^{n-1}(P,u)$ we get a list $\partial^{n-1}(\sigma^{n-1},u)$ of n-dimensional simplices from $K^n(P,u)$ those are incident to σ^{n-1}, and assume $\mu(\sigma^{n-1}) := 0$. Then we create empty lists $M_k(u) := \emptyset$ and $\Sigma_k(u) := \emptyset$.

Step 2.2. Creating the queue to keep n-simplices. We chose a simplex $\sigma_0^n \in K^n(P,u)$, create a queue $R := \{\sigma_0^n\}$ and remove σ_0^n from $K^n(P,u)$.

Step 2.3. Main procedure of Algorithm. While the queue R is not empty we will do the following actions. Take the first simplex $\sigma^n \in R$ and remove it from the queue R. For each $(n-1)$-dimensional face σ^{n-1} of the simplex σ^n we check the following: whether it belongs to the cycle X, whether $\mu(\sigma^{n-1})$ is equal to zero, whether the list $\partial^{n-1}(\sigma^{n-1},u)$ contains any simplices different from σ^n. If all above conditions are satisfied we will execute steps 2.3.1 – 2.3.2.

Step 2.3.1. Take the simplex $\sigma_*^n \in \partial^{n-1}(\sigma^{n-1},u) \setminus \{\sigma^n\}$, remove it from $K^n(P,u)$ and enqueue to R; set $\mu(\sigma^{n-1}) := 1$ and $\Sigma_k(u) = \Sigma_k(u) \cup \{\sigma_*^n\}$.

Step 2.3.2. For all vertices $w \neq u$ of the simplex σ^{n-1} we check whether the edge $a = [uw]$ is in the list $K^1(X)$; having $a \notin K^1(X)$ set $J^k(a) := J^k(a)+1 \mod 2$, $M_k(u) = M_k(u) \cup \{a\}$, $M_k := M_k + a \mod 2$.

Step 2.4. Main procedure repeated. If the list $M_k(u)$ is empty then go back to step 2.2.

Step 3. Indexing the edges of cycle. For each edge $a = [uv] \in K^1(X)$ we search any edges $b \in M_k(u)$ and $c \in M_k(v)$ such that $b \cap c \neq \emptyset$ and that a, b and c are sides of some triangle of polyhedron P. If the edges b and c do not exist then we set $J^k(a) := 1$ and $M_k = M_k + a \mod 2$.

End of algorithm.

Theorem 1. *If P is a closed n-dimensional manifold, $z_1^{n-1}, \dots, z_r^{n-1}$ are simple cycles, $x = a_1 + \dots + a_l \in C_1(P)$ and $J(x) = \sum_{i=1}^{l} J(a_i)$, then the vector $J(x) = (J^1(x), \dots, J^r(x)) \in \mathbb{Z}_2^r$ is the index of the chain $x \in C_1(P)$ relative to the basis $[z_1^{n-1}], \dots, [z_r^{n-1}]$ of group $H_{n-1}(P)$.*

Proof. Let $x \in Z_1(P)$. We will prove that $J^k(x) = \mathrm{Ind}([x],[z_k^{n-1}])$ for all $k = 1, \dots, r$. Set $z_0^* = z_k^{n-1}$. For all $p = 1, \dots, N$ we will make the following constructions; here N is the power of the set $K^0(z_k^{n-1})$. Consider a vertex $u_p \in K^0(z_k^{n-1})$ and its barycentric star $\mathrm{bst}(u_p, P)$.

Let $\Sigma_k^*(u_p)$ be the set of all n-simplices from the barycentric subdivision of $\Sigma_k(u_p)$. Construct the chain $c(u_p)$ of simplices $\sigma_1 \in \mathrm{bst}(u_p,P) \cap \Sigma_k^*(u_p)$.

Then we write the chain boundary $c(u_p)$ as a sum $Y_1 + Y_2$, where Y_1 is the sum of all its $(n-1)$-dimensional simplices, that belong to the cycle z_k^{n-1} and Y_2 is the sum of all remaining simplices from the chain $\partial c(u_p)$. Set $z_p^* = z_{p-1}^* + Y_1 + Y_2 \mod 2$.

By construction $z_p^* \sim z_{p-1}^*$ for all $p = 1, \ldots, N$. Hence, the cycle $z^* = z_N^*$ is homologous to the cycle $z_k^{n-1} = z_0^*$.

Let now prove that for any edge $a = [uv] \in K^1(P)$ and $\sigma_b \in \mathrm{bst}(a)$ the simplex σ_b belongs to z^* if and only if $a \in M_k$.

Let view all possible positions of the edge a. At the same time we also agree to think that $M_k(u) = \emptyset$ and that $\Sigma_k(u) = \emptyset$ for all $u \notin K^0(z_k^{n-1})$.

0. If $a \notin M_k(u) \cup M_k(v)$ and $a \notin K^1(z_k^{n-1})$, then, according to the algorithm, $a \notin M_k$. On the other hand, the edge a can not be incident to simplices from the lists $\Sigma_k(u)$ and $\Sigma_k(v)$ and hence $\sigma_b \notin z^*$.

1. Let $u \in K^0(z_k^{n-1})$, $a \in M_k(u)$ and $v \notin K^0(z_k^{n-1})$. Then the edge a will be still in the chain M_k when algorithm 1 is completed. At the same time the barycentric star $\mathrm{bst}(a)$ belongs to the boundary of the chain $c(u)$ and does not belong to the cycle z_k^{n-1}. Thus in this case $a \in M_k$ and the chain $\mathrm{bst}(a)$ belongs to the cycle z^*.

2. Further, assume that $u, v \in K^0(z_k^{n-1})$ and $a \in M_k(u)$. At that, $a \notin K^1(z_k^{n-1})$.

2.1. If $a \in M_k(v)$, then $a \notin M_k$, and simplices of its barycentric star will be added twice to the initial cycle z_k^{n-1} and will not be in the resulting cycle z^*.

2.2. If $a \notin M_k(v)$, then $a \in M_k$ and any simplex $\sigma_b \in \mathrm{bst}(a)$ is added to the cycle z^* exactly once. So $\sigma_b \in z^*$.

3. Finally, let $a \in K^1(z_k^{n-1})$.

3.1. Let assume that the condition from step 3 of algorithm 1 is satisfied, i.e.:

($*$) there exist edges $b \in M_k(u)$ and $c \in M_k(v)$ such that $b \cap c \neq \emptyset$ and that a, b and c are sides of some triangle $\sigma' \in K^2(P)$.

In this case, according to the algorithm $a \notin M_k$.

Let view all triangles σ' from ($*$), and all n-dimensional simplices incident to them. The such n-simplices belong both to $\Sigma_k(u)$ and $\Sigma_k(v)$. Consider an n-dimensional simplex σ, $\sigma_b \in \sigma$. If $\sigma_b \in \mathrm{bst}(a)$, then σ either belongs to the both sets $\Sigma_k(u)$ and $\Sigma_k(v)$ or does not belong to them. Hence, the simplex σ_b either is not added to the cycle z^* or is added twice. Therefore $\sigma_b \notin z^*$.

3.2. Assume now that condition ($*$) is not satisfied. Then according to step 3 of algorithm 1, $a \in M_k$.

Barycentric star $\mathrm{bst}(a)$ of the edge $a = [uv]$ belongs to the union $D(a)$ of all n-simplices that contain the edge a. We will prove that the subpolyhedron $D(a)$ belongs to the union of simplices from the sets $\Sigma_k(u)$ and $\Sigma_k(v)$.

The cycle z_k^{n-1} divides $D(a)$ into two components of strong connectivity $D^+(a)$ and $D^-(a)$.

By construction the set $\Sigma_k(u)$ can not be empty. Moreover, if the simplex $\sigma \in z_k^{n-1}$ is incident to the vertex u, then σ is a face of some n-simplex from $\Sigma_k(u)$. So there exists a simplex $\sigma^n \in \Sigma_k(u)$ that contains the edge a.

Let the simplex σ^n belong to $D^+(a)$. Then under the strong connectivity $D^+(a)$ and according to algorithm 1, all n-simplices from $D^+(a)$ also belong to $\Sigma_k(u)$.

This implies, in accordance with our assumption, that no n-simplex from $D^+(a)$ can belong to the set $\Sigma_k(v)$.

The set $\Sigma_k(v)$ cannot be empty also. Since each simplex of z_k^{n-1} incident to the vertex v is a face of some n-simplex from $\Sigma_k(v)$, it follows that there exists a simplex $\sigma_*^n \in \Sigma_k(v)$ that contains the edge a. By the above proof, σ_*^n belongs to $D^-(a)$. Then all n-simplices from $D^-(a)$ belong to the set $\Sigma_k(v)$, too. Consequently all n-simplices of the polyhedron $D(a) = D^+(a) \cup D^-(a)$ belong either to the set $\Sigma_k(u)$ or to $\Sigma_k(v)$.

Consider $\sigma_b \in \mathrm{bst}(a)$. If there exists a simplex $\sigma \in \Sigma_k(u)$ containing σ_b, then $\sigma \notin \Sigma_k(v)$. Otherwise, in accordance to the above proof, there is a simplex $\tilde{\sigma} \in \Sigma_k(v)$ such that $\sigma_b \subset \tilde{\sigma}$. It follows that σ_b is involved in the cycle z^* exactly once, so $\sigma_b \in z^*$.

Thus we have proved that the cycle $z^* \sim z_k^{n-1}$ consists of barycentric stars of the edges from chain M_k. That means that this cycle intersects transversally only the edges of the cycle x, that are in the list M_k. According to algorithm 1 $J^k(a) = 1$ for all $a \in M_k$ and $J^k(b) = 0$ for all edges $b \notin M_k$. So

$$\mathrm{Ind}([x], [z_k^{n-1}]) = \mathrm{Ind}([x], [z^*]) = \sum_{a \in x} J^k(a) \mod 2 = J^k(x).$$

$\square$

Remark 2. The fact that $[z_1^{n-1}], \ldots, [z_r^{n-1}]$ is a basis of group $H_{n-1}(P)$ has no impact on the behavior of algorithm 1. So we can apply this algorithm to an arbitrary set of simple $(n-1)$-dimensional cycles of the manifold P. In particular this set may consist of only one cycle z^{n-1}. Then we will get a function $J : C_1(P) \to \mathbb{Z}_2$ such that $\sum_{i=1}^{l} J(a_i) = \mathrm{Ind}([x], [z^{n-1}])$ for $x = a_1 + \cdots + a_l \in Z_1(P)$. So we can use algorithm 1 to compute the intersection index of a given $(n-1)$-cycle $z^{n-1} \in Z_{n-1}(P)$ with any one-dimensional cycle of the manifold P.

Remark 3. We can find any basis $[z_1^{n-1}], \ldots, [z_r^{n-1}]$ of group $H_{n-1}(P)$ using standard matrix algorithm (see, for example, [1], chapter III, section 21). If $n = 2$, we also can apply algorithms that do not use incidence matrices (see [2, 3]).

2. Regular Covering with the Automorphism Group $H_1(P)$

Let P be an n-dimensional triangulated closed manifold and $S = (V, K)$ be its simplicial scheme. We will construct an abstract simplicial scheme $\hat{S} = (\hat{V}, \hat{K})$ as follows.

Set $\hat{V} = V \times G$, where $G = \mathbb{Z}_2^r$. Let $\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m \in \hat{V}$, where $\hat{v}_i = (v_i, b_i)$ for all $i = 0, 1, \ldots, m$. We will think that $\{\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_m\} \in \hat{K}$ if the below conditions are satisfied:

(U1) $\{v_0, v_1, \ldots, v_m\} \in K$;
(U2) $g_0 + g_i = J([v_0 v_i])$ for any $i = 1, \ldots, m$; here $J([v_0 v_i])$ is the index of the edge $[v_0 v_i]$.

Remark 4. When the conditions (U1) and (U2) are satisfied the equalities $g_i + g_j = J([v_i v_j])$ are also true for all $i, j = 1, \ldots, m$. In fact, according to (U1), the cycle $z = [v_j v_i] + [v_i v_0] + [v_0 v_j]$ is homologous to zero. So $J([v_i v_j]) = J([v_i v_0]) + J([v_0 v_j])$. By invoking (U2) we can have these equalities $J([v_i v_j]) = g_i + g_0 + g_0 + g_j = g_i + g_j$.

Let define now a mapping $p^0 : \hat{V} \to V$ and a left action $\lambda^0 : G \times \hat{V} \to \hat{V}$ of group G on $\hat{V}$, assuming

$$p^0((v, g)) = v \quad \text{and} \quad \lambda^0(g', (v, g)) = g' \cdot (v, g) = (v, g' + g) \tag{1}$$

for all $(v, g) \in \hat{V}$ and $g' \in G$.

Let $\hat{P}$ define some realization of the scheme $\hat{S} = (\hat{V}, \hat{K})$. At that we identify the set of vertices of the polyhedron $\hat{P}$ with $\hat{V}$.

Proposition 2. *For the mapping $p^0 : \hat{V} \to V$ there exists the unique continuation $p : \hat{P} \to P$ that is simplicial regular covering with a group of covering transformations $G \cong H_1(P)$.*

Proof. Simplicial and surjective properties of the mapping p^0 follow directly from its definition and from the construction of the complex $\hat{K}$. If $\hat{s} = \{(v_0, g_0), (v_1, g_1), \ldots, (v_m, g_m)\} \in \hat{K}$, then $\{v_0, v_1, \ldots, v_m\} \in K$ and $g_0 + g_i = J([v_0 v_i])$ for all $i = 1, \ldots, m$. On the other hand, $g \cdot \hat{s} = \{(v_0, g + g_0), (v_1, g + g_1), \ldots, (v_m, g + g_m)\}$ for an arbitrary $g \in G$. Since

$g + g_0 + g + g_i = g_0 + g_i = J([v_0 v_i])$, then $g \cdot \hat{s} \in \hat{K}$. So the action λ^0 is also simplicial.

Let $s = \{v_0, v_1, \dots, v_m\} \in K$ and $\hat{v}_0 \in (p^0)^{-1}(v_0)$. Then $\hat{v}_0 = (v_0, g_0)$, where $g_0 \in G$. Set $g_i = g_0 + J([v_0 v_i])$ and $\hat{v}_i = (v_i, g_i)$ for all $i = 1, \dots, m$. At that $\hat{s} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_m\} \in \hat{K}$, $\hat{v}_0 \in \hat{s}$ and $p^0(\hat{s}) = s$. Hence, the mapping p^0 has the following property:

(C1) for each abstract simplex $s \in K$ and for any vertex $\hat{v} \in (p^0)^{-1}(s)$ there is the unique abstract simplex $\hat{s} \in \hat{K}$ containing the vertex $\hat{v}$ and satisfying the equality $p^0(\hat{s}) = s$.

Let choose an abstract simplex $\hat{s} = \{\hat{v}_0, \hat{v}_1, \dots, \hat{v}_m\} \in \hat{K}$, and an element g of group G and assume that $g \cdot \hat{s} = \hat{s}$. Then $\hat{v}_i = (v_i, g_i)$ and $g \cdot \hat{v}_i = (v_i, g + g_i)$ for all $i = 1, \dots, m$. At the same time it follows from the equality $g \cdot \hat{s} = \hat{s}$ that $(v_0, g + g_0) = (v_k, g_k)$ for some $k \in \{0, 1, \dots, m\}$. The latter is possible only if $k = 0$ and $g = 0$. Thus the action λ^0 has the following property:

(C2) if $g \cdot \hat{s} = \hat{s}$ for at least one non-empty simplex $\hat{s} \in \hat{K}$, then g is the neutral element of the group G.

Let now consider the simplices $\hat{s} = \{(v_0, g_0), (v_1, g_1), \dots, (v_m, g_m)\}$ and $\hat{s}'$ of the complex $\hat{K}$.

First, if $g \in G$ and $\hat{s}' = g \cdot \hat{s}$, then $\hat{s}' = \{(v_0, g + g_0), (v_1, g + g_1), \dots, (v_m, g + g_m)\}$. At that $p^0(\hat{s}') = \{v_0, v_1, \dots, v_m\} = p^0(\hat{s})$.

Further, assume that $p^0(\hat{s}') = p^0(\hat{s}) = \{v_0, v_1, \dots, v_m\}$. Then according to (1), $\hat{s}' = \{(v_0, g'_0), (v_1, g'_1), \dots, (v_m, g'_m)\}$, where $g'_0, g'_1, \dots, g'_m$ are some elements of group G, and $g_0 + g_i = J([v_0 v_i]) = g'_0 + g'_i$ for $i = 1, \dots, m$. Set $g = g'_0 + g_0$. Then according to the above equalities $g'_i = g + g_i$ for all $i = 0, 1, \dots, m$ and hence $\hat{s}' = g \cdot \hat{s}$.

This proves that p^0 and λ^0 have the following property:

(C3) for arbitrary abstract simplices $\hat{s}, \hat{s}' \in \hat{K}$ the equality $p^0(\hat{s}) = p^0(\hat{s}\,')$ is equivalent to the existence of an element $g \in G$ such that $g \cdot \hat{s} = \hat{s}\,'$.

It is known that p^0 and λ^0 may have the unique continuation to the simplicial mapping $p : \hat{P} \to P$ and the simplicial action $\lambda : G \times \hat{P} \to \hat{P}$ of group G on $\hat{P}$. It also follows from (C1) – (C3) that p is a regular covering, and G is a corresponding group of covering transformations (see, for example, [4], chapter 2, section 6, theorem 7). □

Proposition 3. *Let* $x = [v_0 v_1] + [v_1 v_2] + \dots + [v_{s-1} v_s]$ *and* $y = [u_0 u_1] + [u_1 u_2] + \dots + [u_{t-1} u_t]$ *be edge paths of the polyhedron* P*, that run from the vertex* $v_0 = u_0$ *to the vertex* $v_s = u_t$*,* $\hat{x} = [\hat{v}_0 \hat{v}_1] + [\hat{v}_1 \hat{v}_2] + \dots + [\hat{v}_{s-1} \hat{v}_s]$

and $\hat{y} = [\hat{u}_0\hat{u}_1] + [\hat{u}_1\hat{u}_2] + \cdots + [\hat{u}_{t-1}\hat{u}_t]$ *paths of* $\hat{P}$*, that cover the paths* x *and* y *respectively and have the same beginning* $\hat{v}_0 = \hat{u}_0$*. Then* $\hat{v}_s = \hat{u}_t$ *if and only if* $x \sim y$*.*

Proof. Let $z = [w_0w_1]+[w_1w_2]+\cdots+[w_{s-1}w_s]$ be a path in the polyhedron P and $g_0 \in G = \mathbb{Z}_2^r$. Then the unique path $\hat{z}$ of the polyhedron $\hat{P}$, starting in the vertex $\hat{w}_0 = (w_0, g_0)$ and covering the path z, is defined by the formulas

$$\hat{w}_i = (w_i, g_0 + J(z_i))\,, \quad i = 1, \ldots, s, \tag{2}$$

where $z_i = [w_0w_1] + [w_1w_2] + \cdots + [w_{i-1}w_i]$, and

$$\hat{z} = [\hat{w}_0\hat{w}_1] + [\hat{w}_1\hat{w}_2] + \cdots + [\hat{w}_{s-1}\hat{w}_s]. \tag{3}$$

Set $g_i = g_0 + J(z_i)$ for $i = 1, \ldots, s$ and $z_0 = 0$. Then $J(z_i) = J(z_{i-1}) + J([w_{i-1}w_i])$ for all $i = 1, \ldots, s$. At the same time $g_i = g_{i-1} + J([w_{i-1}w_i])$ and the vertices $\hat{w}_{i-1}$ and $\hat{w}_i$ from $\hat{V}$, defined by the formula (2), are connected by the edge $[\hat{w}_{i-1}\hat{w}_i] \in \hat{K}$. Then in the polyhedron $\hat{P}$ there is defined a path (3) starting at the vertex $\hat{w}_0 = (w_0, b_0)$. As $p(\hat{w}_i) = p((w_i, g_0 + J(z_i))) = w_i$ for all $i = 0, 1, \ldots, s$, then $\hat{z}$ covers the path z. Since p is a covering then the path $\hat{z}$ is unique.

Assume now that $\hat{v}_0 = (v_0, g_0)$, where $g_0 \in G$. By the above proof, the equalities $p(\hat{x}) = x$, $p(\hat{y}) = y$ and $\hat{v}_0 = \hat{u}_0$ imply that $\hat{v}_s = (v_s, g_0 + J(x))$ and $\hat{u}_t = (u_t, g_0 + J(y))$. So $\hat{v}_s = \hat{u}_t$ if and only if $J(x) = J(y)$. According to proposition 1, the last equality is equivalent to the homology of the chains x and y. □

Remark 5. The definition of the simplicial scheme $\hat{S} = (\hat{V}, \hat{K})$ gives the algorithm for constructing the covering polyhedron $\hat{P}$.

3. Minimal Cycles Searching

Let $E(P) = K^1(P)$ be the set of edges of the polyhedron P, and $L : E(P) \to \mathbb{R}$ be a non-negative function. Using the formulas

$$L(0) = 0 \text{ and } L(\{a_1, \ldots, a_s\}) = \sum_{i=1}^{s} L(a_i). \tag{4}$$

we can extend L to the function $L : C_1(R) \to (R)$. This function is often called weight function. And for an arbitrary chain $x \in C_1(P)$ the value $L(x)$ is called its weight (see, for example, [5], chapter 25).

Let define a weight function $\hat{L} : C_1(\hat{P}) \to \mathbb{R}$ assuming that

$$\hat{L}(\hat{x}) = L(p(\hat{x})) \tag{5}$$

for an arbitrary chain $\hat{x} \in C_1(\hat{P})$.

ALGORITHM 2. Searching for the minimal path homologous to the given 1-chain.

Input:

1) list $V(P)$ of vertices for polyhedron P;
2) lists $U(v, P)$ of vertices incident to v for all vertices $v \in V(P)$;
3) index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to some basis of group $H_{n-1}(P)$;
4) weight function $L : C_1(P) \to \mathbb{R}$;
5) vertices $u_1 \in V(P)$ and $u_2 \in V(P)$;
6) 1-chain $x \in C_1(P)$, $\partial x = \{u_1, u_2\}$.

Output:

1-chain $z \in C_1(P)$.

Algorithm Description.

Step 1. Determination of index of chain x. Calculate the vector $i = J(x)$.

Step 2. Initializing chain z. Set $z := \emptyset$.

Step 3. Initializing sets $\hat{T} \subset V(P) \times G$, $\hat{P}^* \subset V(P) \times G$ and a mapping $\hat{D} : V(P) \times G \to \mathbb{R}$. Let $\hat{T} := \{(u_1, 0)\}$, where 0 – null vector of space $G = \mathbb{Z}_2^r$, $\hat{P}^* := \emptyset$ and $\hat{D}(u_1, 0) := 0$.

Step 4. First extension of $\hat{P}^*$ and $\hat{D}$. For each vertex $v \in U(u_1, P)$ set $j := J([u_1 v])$ and add the pair (v, j) into the list $\hat{P}^*$. At the same time set $\hat{D}(v, j) := L([u_1 v])$, $F(v, j) := (u_1, 0)$.

Step 5. Choosing a next element to add to $\hat{T}$. Find the pair $(w, k) \in (\hat{P}^* \setminus \hat{T})$ such that $\hat{D}(w, k) = \min\limits_{(v,j) \in (\hat{P}^* \setminus \hat{T})} \hat{D}(v, j)$.

Step 6. Stop criterion of $\hat{T}$, $\hat{P}^*$, $\hat{D}$ construction. If $w = u_2$ $k = i$, then go to step 10.

Step 7. Extension of the set $\hat{T}$. Add the pair (w, k) into the list $\hat{T}$.

Step 8. Next extension of $\hat{P}^*$ and $\hat{D}$. For each vertex $v \in U(w, P)$ set $j := k + J([wv])$. If the pair $(v, j) \notin \hat{P}^*$, then set $\hat{D}(v, j) := \hat{D}(w, k) + L([wv])$, $F(v, j) := (w, k)$ and add the pair (v, j) into $\hat{P}^*$. If $(v, j) \in (\hat{P}^* \setminus \hat{T})$ and $\hat{D}(w, k) + L([wv]) < \hat{D}(v, j)$, then set $\hat{D}(v, j) = \hat{D}(w, k) + L([wv])$ and $F(v, j) := (w, k)$.

Step 9. Continuation of $\hat{T}$, $\hat{P}^*$, $\hat{D}$ construction. Go back to step 5.

Step 10. Construction of chain z.

Step 10.1. Take a pair $(v, j) = F(w, k)$ and set $z := z + [vw]$.

Step 10.2. If $(v, j) \neq (u_1, 0)$, then set (w, k) equal to (v, j) and go back to step 10.1.

End of algorithm.

Theorem 2. *The chain $z \in C_1(P)$ computed by algorithm 2 has the following properties:*

- $\partial z = \{u_1, u_2\}$*;*
- $z \sim x$*;*
- $L(z) \leq L(y)$ *for all chains $y \in C_1(P)$ that satisfy conditions $\partial y = \{u_1, u_2\}$ and $y \sim x$.*

Proof. By construction $J(z) = i = J(x)$, so, according to statement 1, $z \sim x$.

Let T^* be the result set of Dijkstra's algorithm for a one-dimensional skeleton $\hat{P}^1$ of the polyhedron $\hat{P}$ if we choose the pair $(u_1, 0)$ as the start point, and the pair (u_2, i) as the end point (see, for example, [6], chapter 6, section 6.3).

According to the definition of the complex $\hat{K}$, the pairs $\hat{v} = (v, j)$ and $\hat{u_1} = (u_1, 0)$ in step 4, as well as the pairs $\hat{v} = (v, j)$ and $\hat{w} = (w, k)$ in step 8 are connected by the edges $[\hat{v}\hat{u}_1] \in E(\hat{P})$ and $[\hat{v}\hat{w}] \in E(\hat{P})$ respectively. Also, according to (5), we have the equalities $\hat{L}([\hat{v}\hat{u}_1]) = L([vu_1])$ in step 4 and $\hat{L}([\hat{v}\hat{w}]) = L([vw])$ in step 8. This implies that the set $\hat{T}$ constructed by step 10 is the same that T^*.

Let note that step 10 is not limited to compute $z = [v_0v_1]+\cdots+[v_{q-1}v_q]$ starting at $v_0 = u_1$ and ending at $v_q = u_2$, but it also gives us the possibility to construct the vector sequence $j_0, j_1, \ldots, j_q \in \mathbb{Z}_2^r$, that will satisfy the equalities $j_0 = 0$, $j_q = i$ and $j_s = j_{s-1} + J([v_{s-1}v_s])$.

Set $\hat{v}_s = (v_s, j_s)$ for all $s = 0, 1, \ldots, q$. Then $[\hat{v}_{s-1}\hat{v}_s] \in E(\hat{P})$ for the same s and $\hat{z} = [\hat{v}_0\hat{v}_1] + [\hat{v}_1\hat{v}_2] + \cdots + [\hat{v}_{q-1}\hat{v}_q]$ is a path in the skeleton $\hat{P}^1$, starting at $(u_1, 0)$ and ending at (u_2, i). Since it can be computed by Dijkstra's algorithm, $\hat{L}(\hat{z})$ is not over than weight of any other path in $\hat{P}^1$, running from $(u_1, 0)$ to (u_2, i).

By the construction of the path $\hat{z}$ and according to (5), $L(z) = \hat{L}(\hat{z})$. Now, in the polyhedron P, let consider another path z' connecting the vertices u_1, u_2 and homologous to x. Since $p : \hat{P} \to P$ is a covering, there exists the unique path $\hat{z}'$ in $\hat{P}$, that covers z' and starts at the vertex $(u_1, 0)$. At the same time, by statement 3 the end points of these paths $\hat{z}$ and $\hat{z}'$ coincide. But then according to the above proof, $L(z) = \hat{L}(\hat{z}) \leq \hat{L}(\hat{z}') = L(z')$. $\square$

Remark 6. If $u_1 = u_2$, then algorithm 2 constructs containing u_1 cycle $z \in Z_1(P)$ homologous to the cycle x and having minimal weight $L(z)$ in set of all cycles with the same properties.

ALGORITHM 3. Searching for the minimal cycle from fixed gomology class.

Input:

1) list $V(P)$ of vertices for polyhedron P;
2) lists $U(v, P)$ of vertices incident to v for all vertices $v \in V(P)$;
3) simple basis cycles $z_1^{n-1}, z_2^{n-1}, \dots, z_r^{n-1}$ of homology group $H_{n-1}(P)$;
4) lists $V(z_1^{n-1}), \dots, V(z_r^{n-1})$ of vertices from cycles $z_1^{n-1}, \dots, z_r^{n-1}$ respectively;
5) index vector-function $J : C_1(P) \to \mathbb{Z}_2^r$ relative to basis $[z_1^{n-1}]$, …, $[z_r^{n-1}]$ of $H_{n-1}(P)$;
6) weight function $L : C_1(P) \to \mathbb{R}$;
7) cycle $x \in Z_1(P)$.

Output:

1-cycle $z \in Z_1(P)$.

Algorithm Description.

Step 1. Set $Z := \emptyset$.
Step 2. Determine the vector $i = J(x)$.
Step 3. If $i = 0$, then set $z = 0$ and go to step 7.
Step 4. Find a number $k \in \{1, \dots, r\}$ such that coordinate i^k of the vector i is equal 1.
Step 5. For each vertex $v \in V(z_k^{n-1})$ execute steps 5.1 – 5.3.
Step 5.1. Using algorithm 2 we find containing v cycle $z_v \in Z_1(P)$ homologous to the cycle x and having minimal weight $L(z_v)$ in set of all cycles with the same properties.
Step 5.2. Add the cycle z_v into the list Z.
Step 5.3. Take the next vertex $v \in V(z_k^{n-1})$.
Step 6. Choose the cycle $z \in Z$ such that $L(z) = \min\limits_{z' \in Z} L(z')$.
Step 7. Quit.

End of algorithm.

Theorem 3. *Let z be the the cycle found by the algorithm 3. Then*

- $z \sim x$;
- $L(z) = \min\limits_{y \in [x]} L(y)$.

Proof. First, if $i = 0$, then according to proposition 1, cycle x is homologous to zero. At the same time we assume in step 3 that $z = 0$. According to (4), $L(0) = 0$. Thus, in this case $z \sim x$ and $L(z) = \min_{y \in [x]} L(y)$.

Further, let $i \neq 0$. Then according to step 4 $i^k = 1$ for $k \in \{1, \dots, r\}$.

Let now consider an arbitrary element z_v in the list Z. It is chosen in step 5.1, and according to this step $z_v \sim x$. Since $z = z_v$ for some $v \in V(z_s^{n-1})$ then $z \sim x$ too.

Let assume that some one-dimensional cycle y of the polyhedron P belongs to the class $[x]$. Then $J(y) = J(x) = i$. Hence, $\mathrm{Ind}([y], [z_k^{n-1}]) = J^k(y) = 1$, and therefore the cycles y and z_k^{n-1} have at least one common vertex $u \in V(z_k^{n-1})$. In this case, according to the selection of cycle z_u in step 5.1 of algorithm 3, $L(z_u) \leq L(y))$. This implies according to step 6, that $L(z) \leq L(z_u) \leq L(y)$. $\square$

For all algorithms we created program realization of C++ classes. Then we found cycles of minimal length on several computer models of closed 2-dimensional manifolds.

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Received May 18, 2006