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A NOTE ON KRULL DIMENSION OF SKEW POLYNOMIAL RINGS

(submitted by M. M. Arslanov)

Abstract. Let A be a commutative Noetherian ring such that Krull dimension of A is α. Let M be a finitely generated critical module over $A[x, \sigma]$, (where σ is an automorphism of A) and Krull dimension of M is $\alpha + 1$. Then M has a prime annihilator.

1. Introduction

All rings are with identity, and all modules unitary. If a module M over a ring R has a Krull dimension α, we denote it by $\mid M \mid = \alpha$. Classical Krull dimension of a ring R is denoted by $cl.K(R)$. For a module M over a ring R with $\mid M \mid = \alpha$, we say M is α-homogeneous if M contains no non-zero submodule of Krull dimension less than α. For more details and some concerning results on Krull dimension, the reader is referred to [3]. Let B be a right Noetherian ring, $T_N(B)$ denotes the torsion submodule of B at the prime radical $N(B)$ of B. We use a similar notation for $T_N(B[x, \sigma])$, which is the torsion submodule of $B[x, \sigma]$, where σ is an automorphism of B and $B[x, \sigma]$ is the usual skew polynomial ring of B in which coefficients of polynomials are taken on the right, and therefore $B[x, \sigma] = \{\sum x^i a_i,\ a_i \in B,\ 0 \leq i \leq n,$ for some positive integer n$\}$,

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subject to the relation $ax = x\sigma(a)$. $C(N(B))$ denotes the set of elements of B regular modulo $N(B)$.

Let now A be a commutative Noetherian ring with $\mid A \mid = \alpha$. Then $\mid A[x,\sigma] \mid = \alpha + 1$, where σ is an automorphism of A. We show that any finitely generated critical module M over $A[x,\sigma]$ with $\mid M \mid = \alpha + 1$ has a prime annihilator.

2. Critical modules over $R[x,\sigma]$

We begin this section with the following Proposition:

Proposition 2.1. *Let B be a right Noetherian ring and σ be an automorphism of B. Then $\sigma(T_N(B)) = T_N(B)$.*

Proof. Using the fact that $\sigma(B) = B$ and if $c \in C(N)$, then $\sigma(c) \in C(N)$, it can be easily proved that $\sigma(T_N(B)) = T_N(B)$. $\square$

Proposition 2.2. *Let B be a right Noetherian ring and σ be an automorphism of B. If N is the prime radical of B, then $\sigma(N) = N$.*

Proof. $\sigma(N) \subseteq N$ because $\sigma(N)$ is a nilpotent ideal of B. Let $n \in N$. Then $n = \sigma(a)$ for some $a \in B$. Therefore $\sigma^{-1}(n) = a$, and $\sigma^{-1}(N) = \{a \in B$, such that $a = \sigma^{-1}(n)$, some $n \in N\} =$ I (say) is an ideal of B. Now $\sigma(I) \subseteq \sigma(N)$. Hence $\sigma(N) = N$. $\square$

Proposition 2.3. *Let B be a semiprime Noetherian ring. Let $f \in B[x,\sigma]$ be regular in $B[x,\sigma]$. Then there exists $g \in B[x,\sigma]$ such that gf has leading coefficient regular in B.*

Proof. Note that $ax = x\sigma(a)$ for all $a \in B$. Let $S = \{x^m a_m + ... + a_0 \in B[x,\sigma]f$, some m$\} \cup \{0\}$. Let $a_m \in S$ and $c \in R$. Then we have some $g = \sum x^i a_i \in B[x,\sigma]$, (i = 1, 2, ..., n) and $f = \sum x^j d_j$, (j = 1, 2, ..., t) regular in $B[x,\sigma]$ such that leading coefficient of gf is a_m; i.e. $\sigma^t(a_n)d_t = a_m$. Now $h = \sum x^j \sigma^{-t}(c)a_j \in B[x,\sigma]$, (j = 1, 2, ..., t), and therefore hf has leading coefficient in S; i.e. $c\sigma^t(a_n)d_t \in S$. Thus S is a left ideal. We now show that S is essential. Let $0 \neq I \subseteq B$ be a left ideal of B. Then it is easy to see that $I[x,\sigma]$ is a left ideal of $B[x,\sigma]$. Now $f \in B[x,\sigma]$ is regular, therefore $B[x,\sigma]f$ is an essential left ideal of $B[x,\sigma]$; i.e. $I[x,\sigma] \cap B[x,\sigma]f \neq (0)$. Let $\sum x^i a_i \in I[x,\sigma] \cap B[x,\sigma]f$, (i = 1, 2, ..., k). Then $a_k \in I$ and $a_k \in S$, and therefore S is essential as a left ideal. So S contains a left regular element by Goldie‘s Theorem, see for example [1, Theorem (1.37)]. Now B is semiprime Noetherian implies that S contains a regular element. Hence there exists $g \in B[x,\sigma]$ such that gf has leading coefficient regular in B. $\square$

Corollary 2.4. *Let B be a right Noetherian ring and $f \in B[x,\sigma]$ regular modulo N^*, where N^* is the prime radical of $B[x,\sigma]$. Then there exists $g \in B[x,\sigma]$ such that gf has leading coefficient regular in B/N, where N is the prime radical of B.*

Proposition 2.5. *Let B be a right Noetherian ring, and σ an automorphism of B. Then $(T_N(B))[x,\sigma]$ is a right ideal of $B[x,\sigma]$, and $(T_N(B))[x,\sigma] = (T_N(B[x,\sigma])$.*

Proof. By 2.1 above $\sigma(T_N(B)) = T_N(B)$, therefore $(T_N(B))[x,\sigma]$ is a right ideal of $B[x,\sigma]$. Now on the same lines as in [6, Proposition (1.1)] with some manipulations on σ, it can be easily proved that $T_N(B[x,\sigma]) \subseteq (T_N(B))[x,\sigma]$. □

Theorem 2.6. *Let A be commutative Noetherian ring with $\mid A \mid = \alpha$. If M is a finitely generated critical right module over $A[x,\sigma]$ with $\mid M \mid = \alpha + 1$. Then M has a prime annihilator, where σ is an automorphism of A.*

Proof. Let F_α = Sum of all submodules of A of Krull dimension less than α. Then as in [6, Theorem(1.2)], $F_\alpha = \cap A_\alpha$, where A_α are ideals of A such that A/A_α is α-homogeneous. Also observe that $A/\sigma(F_\alpha)$ is isomorphic to A/F_α. Therefore $A/\sigma(F_\alpha)$ is α-homogeneous. So $\sigma(F_\alpha) \subseteq F_\alpha$. Now let F^*_α = Sum of submodules of $A[x,\sigma]$ having Krull dimension less than $\alpha + 1$. Then $F_\alpha[x,\sigma] \subseteq F^*_\alpha$ and $F^*_\alpha = \cap A^*_\alpha$, where $A[x,\sigma]/A$ is $\alpha + 1$-homogeneous. Let $Ann(M) = B$, where $Ann(M)$ denotes the annihilator of M. Then M is a critical $A[x,\sigma]/B$ module, which is also faithful since $\mid M \mid = \alpha + 1$, therefore $\mid A[x,\sigma]/B \mid = \alpha + 1$. Now M is critical with $\mid M \mid = \alpha + 1$, so by [2] $A[x,\sigma]/B$ is isomorphic to a submodule of direct sum of n copies of M. This easily yields that $A[x,\sigma]/B$ is $\alpha + 1$-homogeneous. Hence $F_\alpha[x,\sigma] \subseteq B$. Thus M is an $A[x,\sigma]/F_\alpha[x,\sigma]$ module. If $N(A/F_\alpha)$ is the prime radical of A/F_α, then since A/F_α is α-homogeneous, $(A/F_\alpha)/N(A/F_\alpha)$ is also α-homogeneous, because A is a commutative ring. Now using [6, Theorem (1.2)] and the fact that every critical module is compressible over A/F_α by [4, Theorem (2.5)], we get by [2, Proposition (3.6)] that $(A/F_\alpha)/T_N(A/F_\alpha)$ has an artinian quotient ring, therefore by [5, Theorem (3.1)] $(A/F_\alpha)[x,\sigma]/(T_N(A/F_\alpha))[x,\sigma]$ has an artinian quotient ring. Now 2.5 implies that

$$(A/F_\alpha)[x,\sigma]/T_N((A/F_\alpha)[x,\sigma]) = (A/F_\alpha)[x,\sigma]/(T_N(A/F_\alpha))[x,\sigma].$$

Now since

$$(A/F_\alpha)[x,\sigma]/(N(A/F_\alpha))[x,\sigma]$$

is $\alpha+1$-homogeneous as $(A/F_\alpha)/N(A/F_\alpha)$ is α-homogeneous, so again by an application of [2, Proposition (3.6)], we get that M as a module over $(A/F_\alpha)[x,\sigma]$ is compressible. Hence M has a prime annihilator. □

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