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Niovi Kehayopulu and Michael Tsingelis

DECOMPOSITION OF COMMUTATIVE ORDERED SEMIGROUPS INTO ARCHIMEDEAN COMPONENTS

(submitted by M. M. Arslanov)

Abstract. The decomposition of a commutative semigroup (without order) into its archimedean components, by means of the division relation, has been studied by Clifford and Preston. Exactly as in semigroups, the complete semilattice congruence "$\mathcal{N}$" defined on ordered semigroups by means of filters, plays an important role in the structure of ordered semigroups. In the present paper we introduce the relation "η" by means of the division relation (defined in an appropriate way for ordered case), and we prove that, for commutative ordered semigroups, we have $\eta = \mathcal{N}$. As a consequence, for commutative ordered semigroups, one can also use that relation η which has been also proved to be useful for studying the structure of such semigroups. We first prove that in commutative ordered semigroups, the relation η is a complete semilattice congruence on S. Then, since $\mathcal{N}$ is the least complete semilattice congruence on S, we have $\eta = \mathcal{N}$. Using the relation η, we prove that the commutative ordered semigroups are, uniquely, complete semilattices of archimedean semigroups which means that they are decomposable, in a unique way, into their archimedean components.

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1. Introduction-prerequisites

The relation "$\mathcal{N}$" defined on semigroups (without order) by means of filters, plays an important role in the structure, especially in the decomposition of semigroups. In ordered semigroups the filters are naturally defined with the help of order as well. Exactly as in semigroups, the relation "$\mathcal{N}$" defined on ordered semigroups by means of filters, plays a basic role in the structure of ordered semigroups. In particular, it plays an important role in the decompositions of such semigroups. An important role in the structure of ordered semigroups is played by the pseudoorder as well. For and ordered semigroup, the relation "$\mathcal{N}$" is actually a complete semilattice congruence on S, in particular, it is the least complete semilattice congruence on S. In this paper we first introduce the division relation for ordered semigroups. Then we prove that in commutative ordered semigroups the relation "$\mathcal{N}$" can be defined in terms of the division relation as well. We prove that in commutative ordered semigroups, "$\mathcal{N}$" is equal to the relation "η" defined by $a\eta\, b$ if and only if there exist natural numbers m, n such that $a|b^m$ and $b|a^n$, where $a|b$ is defined as follows: $a|b$ if there exists $x \in S^1$ such that $b \le ax$. We first prove that in commutative ordered semigroups, the relation η defined above is a complete semilattice congruence on S. Then, since $\mathcal{N}$ is the least complete semilattice congruence on S, we have $\eta = \mathcal{N}$. As a consequence, in studying the structure of commutative ordered semigroups, we can also use that relation η (instead of $\mathcal{N}$) which has been also proved to be useful for studying the structure of commutative ordered semigroups. Using this relation η, we prove that the commutative ordered semigroups are decomposable into their archimedean components, and the decomposition is unique. The analogous problem in case of semigroups without order has been studied by Clifford and Preston in [1]. They proved that each semigroup can be decomposable into its archimedean components, and the decomposition is uniquely defined. This has been proved in [1] by means of the division relation of semigroups.

Let $(S, ., \le)$ be an ordered semigroup. A subsemigroup F if S is called a *filter* of S [2] if the following assertions are satisfied:

(1) If $a, b \in S$ and $ab \in F$, then $a \in F$ and $b \in F$.

(2) If $a \in F$ and $c \in S$ such that $c \ge a$, then $c \in F$.

We denote by $N(a)$ the filter of S generated by a ($a \in S$), and by $\mathcal{N}$ the equivalence relation on S defined as follows:

$$\mathcal{N} := \{(a, b) \mid N(a) = N(b)\} \ \ [3].$$

Let $(S, ., \leq)$ be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a, a^2) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$ [3]. A congruence σ on S is called *complete semilattice congruence* [5] if the following conditions are satisfied:

(1) $(ab, ba) \in \sigma$ for each $a, b \in S$ and

(2) If $a \leq b$, then $(a, ab) \in \sigma$.

A relation σ on S is called *pseudoorder* [7] if we have the following:

(1) $\leq \subseteq \sigma$.

(2) If $(a, b) \in \sigma$ and $(b, c) \in \sigma$, then $(a, c) \in \sigma$.

(3) If $(a, b) \in \sigma$, then $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$.

An ordered semigroup S is called a *semilattice of archimedean semigroups* (resp. *complete semilattice of archimedean semigroups*) if there exists a semilattice congruence (resp. complete semilattice congruence) σ on S such that the σ-class $(x)_\sigma$ is an archimedean subsemigroup of S for every $x \in S$ (cf. also [4]).

An ordered semigroup S is a semilattice of archimedean semigroups if and only if there exists a semilattice Y and a family $\{S_\alpha \mid \alpha \in Y\}$ of archimedean subsemigroups of S such that

(1) $S_\alpha \cap S_\beta = \emptyset$ for each $\alpha, \beta \in Y$, $\alpha \neq \beta$.

(2) $S = \bigcup \{S_\alpha \mid \alpha \in Y\}$.

(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$ (cf. also [4]).

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(3) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$.

(4) If $\alpha, \beta \in Y$ such that $S_\alpha \cap (S_\beta] \neq \emptyset$, then $\alpha = \alpha\beta (= \beta\alpha)$ [8].

For convenience, we use the notation $S^1 := S \cup \{1\}$, where $1 \notin S$, $1x := x1 := x$ for every $x \in S$, and $11 := 1$. For each $x \in S$, we define $x^0 = 1$.

2. In commutative ordered semigroups, $\eta = \mathcal{N}$

In this section we introduce the relation η by means of the division relation, and we prove that for commutative ordered semigroups the relation η coincides with the usual relation $\mathcal{N}$.

Remark 2.1. Each complete semilattice congruence σ defined on an ordered semigroup S, is a semilattice congruence on S. Indeed, if $a \in S$ then, since $a \le a$, we have $(a, a^2) \in \sigma$.

Definition 2.2. Let $(S, ., \le)$ be an ordered semigroup. For two elements a, b of S we say that a divides b and write $a|b$ if there exists $x \in S^1$ such that $b \le ax$.

Proposition 2.3. *Let $(S, ., \le)$ be an ordered semigroup. Then we have the following:*
(1) *$a|a$ for every $a \in S$.*
(2) *If $a|b$ and $b|c$, then $a|c$.*
(3) *If $a|b$, then $ca|cb$ for every $c \in S$.*
In particular, if S is commutative, then
(4) *If $a|b$, then $ac|bc$ for every $c \in S$.*

Proof. (1) Let $a \in S$. Since $a \le a = a1$, where $1 \in S^1$, we have $a|a$.
(2) Let $a|b$ and $b|c$. Then there exist $x, y \in S^1$ such that $b \le ax$ and $c \le by$. Since $c \le (ax)y = a(xy)$, where $xy \in S^1$, we have $a|c$.
(3) Let $a|b$ and $c \in S$. Let $x \in S^1$ such that $b \le ax$. Since $cb \le (ca)x$, where $x \in S^1$, we have $ca|cb$.
If S is commutative then, by (3), condition (4) also holds. □

Remark 2.4. If S is an ordered semigroup, then for each $a, b \in S$, we have $a|ab$. So $a|a^2$ for each $a \in S$. Moreover, for each $a \in S$ and $b \in S^1$, we have $a|ab$.

Proposition 2.5. *Let $(S, ., \le)$ be an ordered semigroup. If $a \le b$, then $b|a$.*

Proof. Let $a \le b$. Since $a \le b = b1$, where $1 \in S^1$, we have $b|a$.

Notation 2.6. We write $a\delta b$ if and only if $b|a$.

By Proposition 2.5 and conditions (2)–(4) of Proposition 2.3, we have the following:

Proposition 2.7. *If $(S, ., \le)$ is a commutative ordered semigroup, then the relation δ is a pseudoorder on S.*

Definition 2.8. Let $(S, ., \le)$ be an ordered semigroup. Define a relation η on S as follows:
$a\eta\, b$ if and only if there exist $m, n \in N$ such that $a|b^m$ and $b|a^n$.
(N = {1,2,3, ... } is the set of natural numbers).

Proposition 2.9. *Let $(S, ., \leq)$ be an ordered semigroup. For the relation η on S, we have the following:*
(1) *η is reflexive.*
(2) *η is symmetric.*
(3) *If $a \leq b$, then $(a, ab) \in \eta$.*

Proof. (1) Let $a \in S$. By Proposition 2.3(1), we have $a|a := a^1$, so $(a, a) \in \eta$.
(2) This is clear.
(3) Let $a \leq b$. Since $a^2 \leq (ab)1$, where $1 \in S^1$, we have $ab|a^2$. On the other hand, $a|ab$. Thus we have $(a, ab) \in \eta$. □

Proposition 2.10. *Let $(S, ., \leq)$ be a commutative ordered semigroup. Then we have the following:*
(1) *If $a|b$, then $a^m|b^m$ for every $m \in N$.*
(2) *$ab^m|(ab)^m$ for every $a, b \in S$ and every $m \in N$.*

Proof. (1) Let $a|b$ and $m \in N$. Suppose $x \in S^1$ such that $b \leq ax$. Then, since S is commutative, we have $b^m \leq (ax)^m = a^m x^m$. Since $x \in S^1$, we have $x^m \in S^1$. Since $b^m \leq a^m x^m$, where $x^m \in S^1$, we have $a^m|b^m$.
(2) Let $a, b \in S$ and $m \in N$. Since S is commutative, we have

$$(ab)^m = a^m b^m = ab^m a^{m-1} \text{ (where } a^{m-1} := 1, \text{ if } m = 1).$$

Then, since $a^{m-1} \in S^1$, we have $ab^m|(ab)^m$. □

Proposition 2.11. *Let $(S, ., \leq)$ be a commutative ordered semigroup. Then, for the relation η on S, we have the following:*
(1) *η is transitive.*
(2) *If $(a, b) \in \eta$, then $(ca, cb) \in \eta$ for every $c \in S$.*
(3) *If $(a, b) \in \eta$, then $(ac, bc) \in \eta$ for every $c \in S$.*
(4) *$(ab, ba) \in \eta$ for all $a, b \in S$.*

Proof. (1) Let $(a, b) \in \eta$ and $(b, c) \in \eta$. Since $(a, b) \in \eta$, there exist $m, n \in N$ such that $a|b^m$, $b|a^n$. Since $(b, c) \in \eta$, there exist $t, h \in N$ such that $b|c^t$, $c|b^h$. Since S is commutative, $b|c^t$ and $m \in N$, by Proposition 2.10(1), we have $b^m|c^{tm}$. Since $a|b^m$ and $b^m|c^{tm}$, by Proposition 2.3(2), we have $a|c^{tm}$, where $tm \in N$. In a similar way we prove that $c|a^{nh}$, where $nh \in N$. Thus we get $(a, c) \in \eta$, and η is transitive.
(2) Let $(a, b) \in \eta$ and $c \in S$. Then $(ca, cb) \in \eta$. Indeed:
Since $(a, b) \in \eta$, there exist $m, n \in N$ such that $a|b^m$, $b|a^n$. Since $a|b^m$ and $c \in S$, by Proposition 2.3(3), we have $ca|cb^m$. Since S is commutative, $c, b \in S$ and $m \in N$, by Proposition 2.10(2), we get $cb^m|(cb)^m$. Since $ca|cb^m$ and $cb^m|(cb)^m$, by Proposition 3(2), we have $ca|(cb)^m$, where $m \in$

N. In a similar way we prove that $cb|(ca)^n$, where $n \in N$. Since $ca|(cb)^m$ and $cb|(ca)^n$, where $m, n \in N$, we have $(ca, cb) \in \eta$.
Condition (3) follows from (2), and (4) by Proposition 2.9(1). □

By Propositions 2.9 and 2.11 we have the following:

Theorem 2.12. *If S is a commutative ordered semigroup, then the relation "η" is a complete semilattice congruence on S.*

Lemma 2.13. [5] *For an ordered semigroup S, the relation "$\mathcal{N}$" is the least complete semilattice congruence on S.*

Theorem 2.14. *Let S be a commutative ordered semigroup. Then $\eta = \mathcal{N}$.*

Proof. Let $(a, b) \in \eta$. Then there exist $m, n \in N$ such that $a|b^m$ and $b|a^n$. Since $a|b^m$, there exists $x \in S$ such that $b^m \le ax$. Since $b \in N(b)$, we have $b^m \in N(b)$. Since $N(b) \ni b^m \le ax$, we have $ax \in N(b)$, then $a \in N(b)$, and $N(a) \subseteq N(b)$. By $b|a^n$, by symmetry, we get $N(b) \subseteq N(a)$. Thus we have $N(a) = N(b)$, and $(a, b) \in \mathcal{N}$. So $\eta \subseteq \mathcal{N}$. On the other hand, by Theorem 2.12 and Lemma 2.13, we have $\mathcal{N} \subseteq \eta$. Hence we have $\eta = \mathcal{N}$, and the proof is complete. □

Proposition 2.15. *Let S be an ordered semigroup and $a|b$. Then $N(a) \subseteq N(b)$.*

Proof. Suppose $x \in S^1$ such that $b \le ax$. Since $N(b) \ni b \le ax$, we have $ax \in N(b)$, and $a \in N(b)$. So $N(a) \subseteq N(b)$. □

Proposition 2.16. *If S is an ordered semigroup, then $\delta \cap \delta^{-1} \subseteq \mathcal{N}$.*

Proof. Let $(a, b) \in \delta \cap \delta^{-1}$. Since $(a, b) \in \delta$, we have $b|a$. Then, by Proposition 2.15, we have $N(b) \subseteq N(a)$. Since $(b, a) \in \delta$, by symmetry, we have $N(a) \subseteq N(b)$. Then $N(a) = N(b)$, so $(a, b) \in \mathcal{N}$. □

Proposition 2.17. *Let S be an ordered semigroup and $a, b \in S$. The following are equivalent:*

(1) *There exists $m \in N$ such that $a|b^m$.*
(2) *There exist $n \in N$ and $y \in S$ such that $b^n \le ay$.*

Proof. (1) $\Longrightarrow$ (2). Suppose $a|b^m$ for some $m \in N$. Then there exists $x \in S^1$ such that $b^m \le ax$. Then $b^{m+1} \le a(xb)$. Since $x \in S^1$, $b \in S$, we have $xb \in S$ ($\subseteq S^1$). So $a|b^{m+1}$, where $m + 1 \in N$.
(2) $\Longrightarrow$ (1). It is obvious. □

3. Main results

In this section, using the relation η defined above, we prove that the commutative ordered semigroups are, uniquely, complete semilattices of archimedean semigroups. That is, they are decomposable into archimedean semigroups and the decomposition is unique.

Definition 3.1. An ordered semigroup S is called *archimedean* of for every $a, b \in S$ there exist $m, n \in N$ such that $a|b^m$ and $b|a^n$.
Equivalent Definition: $S \times S = \eta$.

Proposition 3.2. *Let S be a commutative ordered semigroup. Then the η-class $(x)_\eta$ is an archimedean subsemigroup of S for every $x \in S$.*

Proof. Let $x \in S$. Since η is a semilattice congruence on S, $(x)_\eta$ is a subsemigroup of S (cf. also [6]). Let now $a, b \in (x)_\eta$. Then there exist $m, n \in N$ and $y, z \in (x)_\eta^1$ such that $b^m \leq ay$ and $a^n \leq bz$, which means that the η-class $(x)_\eta$ is archimedean. In fact:
Since $(a, b) \in \eta$, there exist $t, h \in N$ such that $a|b^t$ and $b|a^h$. Since $a|b^t$, by Proposition 2.17, there exist $u \in N$ and $s \in S$ such that $b^u \leq as$. Since $b|a^h$, there exist $v \in N$ and $k \in S$ such that $a^v \leq bk$. Since $b^u \leq as$, we have $b^{u+1} \leq asb = (bs)a$, from which $bs|b^{u+1}$. Besides, $b|bs = (bs)^1$. Since $bs|b^{u+1}$ and $b|(bs)^1$, we have $(bs, b) \in \eta$, then $bs \in (b)_\eta = (x)_\eta$. Thus we have $b^{u+1} \leq a(bs)$, where $u + 1 \in N$ and $bs \in (x)_\eta \subseteq (x)_\eta^1$. In a similar way we prove that there exist $n \in N$ and $z \in (x)_\eta^1$ such that $a^n \leq bz$ and the proof is complete. □

By Theorem 2.12 and Proposition 3.2, we have the following:

Theorem 3.3. *If S is a commutative ordered semigroup, then S is a complete semilattice of archimedean semigroups.*

Proposition 3.4. *Let $(S, ., \leq)$ be a commutative ordered semigroup and ρ a complete semilattice congruence on S such that the ρ-class $(x)_\rho$ is an archimedean subsemigroup of S for every $x \in S$. Then $\rho = \eta$.*

Proof. Let $(a, b) \in \rho$. Then, since $a, b \in (b)_\rho$ and $(b)_\rho$ is archimedean, there exist $m, n \in N$ and $y, z \in (b)_\rho^1$ such that $a^m \leq by$ and $b^n \leq az$. Then, since $y, z \in S^1$, we have $b|a^m$ and $a|b^n$. Thus we have $(a, b) \in \eta$. So $\rho \subseteq \eta$. By Lemma 2.13 and Theorem 2.14, η is the least semilattice congruence on S, so $\eta \subseteq \rho$. Therefore we have $\rho = \eta$. □

By Theorem 2.12 and Propositions 3.2 and 3.4, we have the following:

Theorem 3.5. *If S is a commutative ordered semigroup then S is, uniquely, a complete semilattice of archimedean semigroups.*

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University of Athens, Department of Mathematics, 157 84 Panepistimiopolis, Greece

E-mail address: nkehayop@math.uoa.gr

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